Numerical Methods for Maxwell Equations

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SS05

Abstract

The Maxwell equations describe the interaction of electric and magnetic fields. Important applications are electric machines such as transformers or motors, or electromagnetic waves radiated from antennas or transmitted in optical fibres. To compute the solutions of real life problems on complicated geometries, numerical methods are required.

In this lecture we formulate the Maxwell equations, and discuss the finite element method to solve them. Involved topics are partial differential equations, variational formulations, edge elements, high order elements, preconditioning, a posteriori error estimates.

1 Maxwell Equations

In this chapter we formulate the Maxwell equations.

1.1 The equations of the magnetic fields

The involved field quantities are

B	$\frac{Vs}{m^2}$	magnetic flux density (germ: Induktion)
H	$\frac{A}{m}$	magnetic field intensity (germ: magn. Feldstärke)
j_{tot}	$\frac{A}{m^2}$	electric current density (germ: elektrische Stromdichte)

We state the magnetic equations in integral form. The magnetic flux density has no sources, i.e., for any volume V there holds

$$\int_{\partial V} B \cdot n \, ds = 0$$

Ampere's law gives a relations between the magnetic field and the electric current. A current through a wire generates a magnetic field around it. For any surface S in space there holds:

$$\int_{\partial S} H \cdot \tau \, ds = \int_S j_{tot} \cdot n \, ds$$

Both magnetic fields are related by a material law, i.e., $B = B(H)$. We assume a linear relation

$$B = \mu H,$$

where the scalar μ is called *permeability*. In general, the relation is non-linear (ferro magnetic materials), and depends also on the history (hysteresis).

Assuming properly smooth fields, the integral relations can be reformulated in differential form. Gauss´ theorem gives

$$\int_{\partial V} B \cdot n \, ds = \int_V \operatorname{div} B \, dx = 0 \qquad \forall \, V,$$

which implies

$$\operatorname{div} \; B = 0.$$

Similar, applying Stokes´ theorem leads to

$$\int_{\partial S} H \cdot \tau \, ds = \int_S \operatorname{curl} \; H \cdot n \, ds = \int_S j_{tot} \cdot n \, ds,$$

or

$$\operatorname{curl} \; H = j_{tot}.$$

Since div curl $= 0$, this identity can only hold true if $\operatorname{div} j_{tot} = 0$ was assumed !

Summing up, we have

$$\operatorname{div} B = 0 \qquad \operatorname{curl} H = j_{tot} \qquad B = \mu H. \tag{1}$$

The integral forms can also be used to derive interface conditions between different materials. In this case, we may expect piecewise smooth fields. Let S be a surface in the material interface, i.e.,

$$S \subset \overline{\Omega}_+ \cap \overline{\Omega}_-$$

and set $V_\varepsilon = \{x + t n_x : x \in S, t \in (-\varepsilon, +\varepsilon)\}$. Let $S_+ = \{x + \varepsilon n_x\}$, $S_- = \{x - \varepsilon n_x\}$, $M = \partial V_\varepsilon \setminus S_{+\varepsilon} \setminus S_{-\varepsilon}$.

From

$$0 = \int_{S_+} B \cdot n \, ds + \int_{S_-} B \cdot n \, ds + \int_M B \cdot n \, ds,$$

and $\int_{S_{+/-}} B \cdot n \, ds \to \int_S B_{+/-} \cdot n \, ds$, and $|M| \to 0$ as $\varepsilon \to 0$, there follows

$$\int_S B_+ \cdot n \, ds = \int_S B_- \cdot n \, ds \qquad \forall \, S \subset \overline{\Omega}_+ \cap \overline{\Omega}_-.$$

Since this is true for all surfaces S in the interface, there holds

$$B_+ \cdot n = B_- \cdot n.$$

The B-field has continuous normal components. If $\mu_+ \neq \mu_-$, the normal components of the H-field are not the same. Similar (exercise!), one proves that the tangential component of the H-field is continuous:

$$H_+ \times n = H_- \times n.$$

Instead of dealing with the first order system (1), one usually introduces a vector potential to deal with one second order equation. Since $\operatorname{div} B = 0$ (on the simply connected domain $\mathbb{R}^3$), there exists a vector potential A such that

$$\operatorname{curl} A = B.$$

Plugging together the equations of (1), we obtain the second order system

$$\operatorname{curl} \mu^{-1} \operatorname{curl} A = j_{tot}. \tag{2}$$

The vector potential A is not unique. Adding a gradient field $\nabla \Phi$ does not change the equation. One may choose a divergence free A field (constructed by $\tilde{A} = A + \nabla \Phi$, where Φ solves the Poisson problem $-\Delta \Phi = \operatorname{div} A$). Choosing a unique vector potential is called Gauging. In particular, $\operatorname{div} A = 0$ is called Coulomb gauging. Gauging is not necessary, one can also work with (compatible) singular systems.

1.2 The equations of the electric fields

The involved field quantities are

E	$\frac{V}{m}$	electric field intensity (germ: elektrische Feldstärke)
D	$\frac{AS}{m^2}$	displacement current density (germ: Verschiebungsstromdichte)
j	$\frac{AS}{m^2}$	electric current density (germ: elektrische Stromdichte)
ρ	$\frac{AS}{m^3}$	Charge density (germ: Ladungsdichte)

Faraday's induction law: Let a wire form a closed loop ∂S. The induced voltage in the wire is proportional to the change of the magnetic flux through the surface encluded by the wire:

$$\int_{\partial S} E \cdot \tau \, ds = -\int_S \frac{\partial B \cdot n}{\partial t} \, ds.$$

The differential form is

$$\operatorname{curl} E = -\frac{\partial B}{\partial t}$$

Ohm's law states a current density proportional to the electric field:

$$j = \sigma E,$$

where σ is the electric conductivity. This current is a permanent flow of charge particles.

The electric displacement current models (beside others) the displacement of atomar particles in the electric field:

$$D = \varepsilon E.$$

The material parameter ε is called permittivity. It is not a permanent flow of current, only the change in time leads to a flow. Thus, we define the total current as

$$j_{tot} = \frac{\partial D}{\partial t} + j.$$

There are no sources of the total current, i.e.,

$$\operatorname{div} j_{tot} = 0.$$

The charge density is

$$\rho = \operatorname{div} D.$$

Thus, the charge density is the cummulation of current-sources:

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} j.$$

Current sources result in the accumulation of charges. Only in the stationary limit, Ohm's current is divergence-free.

1.3 The Maxwell equations

Maxwell equations are the combination of magnetic and electric equations

$$\begin{aligned} \operatorname{curl} E &= -\frac{\partial B}{\partial t}, && (3) \\ \operatorname{curl} H &= \frac{\partial D}{\partial t} + j, && (4) \\ \operatorname{div} D &= \rho, && (5) \\ \operatorname{div} B &= 0, && (6) \end{aligned}$$

together with the (linear) material laws

$$B = \mu H, \qquad j = \sigma E, \qquad D = \varepsilon E.$$

Proper boundary conditions will be discussed later.

Remark: Equation (3) implies $\operatorname{div} \frac{\partial B}{\partial t} = 0$, or $\operatorname{div} B(x,t) = \operatorname{div} B(x,t_0)$. Equation (6) is needed for the initial condition only ! The same holds for the charge density ρ: The initial charge density $\rho(x,0)$ must be prescribed. The evolution in time follows (must be compatible!) with $\operatorname{div} j$.

Using the material laws to eliminate the fluxes leads to

$$\begin{aligned} \operatorname{curl} E &= -\mu \frac{\partial H}{\partial t}, && (7) \\ \operatorname{curl} H &= \varepsilon \frac{\partial E}{\partial t} + \sigma E, && (8) \end{aligned}$$

plus initial conditions onto E and H. Now, applying $\operatorname{curl} \mu^{-1}$ to the first equation, and differentiating the second one in time leads to second order equation in time

$$\varepsilon \frac{\partial^2 E}{\partial t^2} + \sigma \frac{\partial E}{\partial t} + \operatorname{curl} \mu^{-1} \operatorname{curl} E = 0. \tag{9}$$

As initial conditions, E and $\frac{\partial}{\partial t} E$ must be prescribed.

Till now, there is no right hand side of the equation. Maxwell equations describe the time evolution of a known, initial state.

Many application involve windings consisting of thin wires. Maxwell equations describe the current distribution in the wire. Often (usually) one assumes that the current density is equally distributed over the cross section of the wire, the flow is in tangential direction, and the total current is known. In this case, the (unknown) current density σE is replaced by the known *impressed* current density j_I. In the winding, the conductivity is set to 0. This substitution may be done locally. In some other domains, the current distribution might not be known a priori, and the unknown current σE must be kept in the equation.

We plug in this current sources into (9). Additionally, we do some cosmetics and define the vector potential A such that $E = -\frac{\partial}{\partial t} A$ to obtain

$$\varepsilon \frac{\partial^2 A}{\partial t^2} + \sigma \frac{\partial A}{\partial t} + \operatorname{curl} \mu^{-1} \operatorname{curl} A = j_I. \tag{10}$$

Now, a possible setting is to start with $A = 0$ and $\frac{\partial}{\partial t} A = 0$, and to switch on the current j_I after finite time. The differential operator in space is the same as in the case of magnetostatics. But now, the additional time derivatives lead to a unique solution.

Equation (10) can be solved by a time stepping method (exercise!). Often, one deals with time harmonic problems (i.e., the right hand side and the solution are assumed to be of the form $j_I(x,t) = real(j_I(x) e^{i\omega t})$ and $A(x,t) = real(A(x) e^{i\omega t})$, respectively).

The evaluation of time derivatives lead to multiplication with $i\omega$. The time harmonic equation is

$$\operatorname{curl} \mu^{-1} \operatorname{curl} A + (i\omega\sigma - \omega^2 \varepsilon) A = j_I. \tag{11}$$

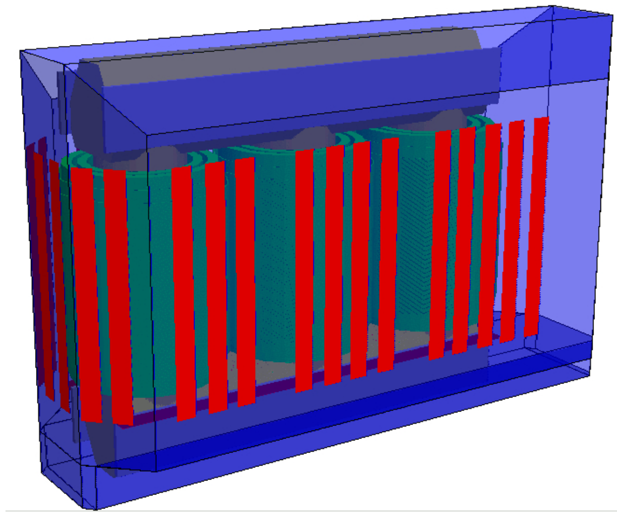

Figure 1: Three phase transformer

1.4 Technical Applications

Maxwell equations are applied in a wide range (limited by quantum effects in the small scale and by relativistic effects in the large scale). For different applications, different terms are dominating. In particular, if

$$L\omega \ll c = \frac{1}{\sqrt{\varepsilon\mu}},$$

where L is the length scale, and c is the speed of light, wave effects and thus the second order time derivative can be neglected. This case is called low frequency approximation.

1.4.1 Low frequency applications

This is the case of most electric machines, where the frequency is $50Hz$. A transformer changes the voltage and current of alternating current. Figure 1 shows a three phase transformer. It has an iron core with high permeability μ. Around the legs of the core are the windings (a primary and a secondary on each leg). The current in the windings is known. It generates a magnetic field mainly conducted by the core. A small amount of the field goes into the air and into the casing. The casing is made of steel and thus highly conducting, which leads to currents and losses in the casing. Thus, one places highly permeable shields in front of the casing to collect the magnetic flux. The shields are made of layered materials to prevent currents in the shields.

This problem is a real three dimensional problem, which can only be solved by numerical methods. The induced current density and loss density in the steel casing and interior conducting domains computed by the finite element method is plotted in Figure 2 and Figure 3.

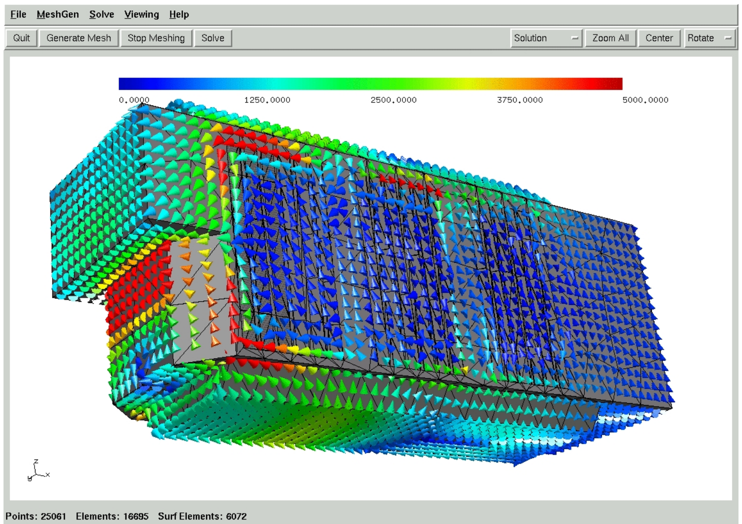

Figure 2: Induced currents

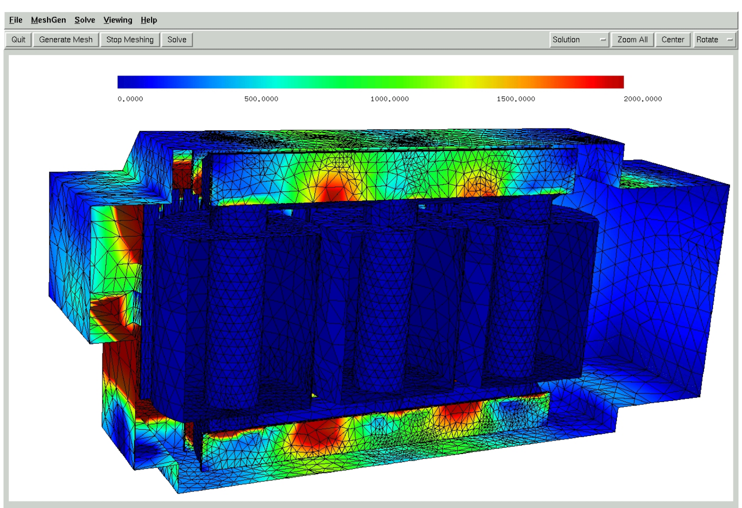

Figure 3: Loss density

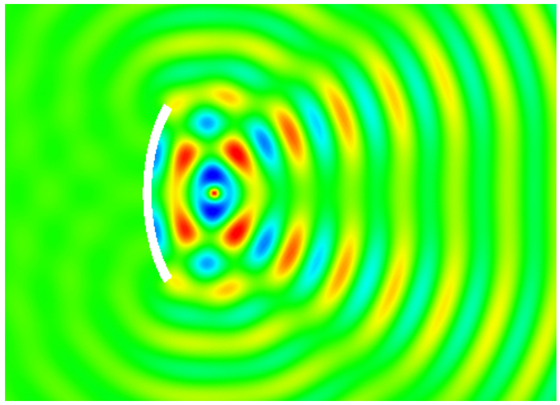

Figure 4: Parabola antenna

Other low frequency applications are electric motors and dynamos. Here, the mechanical force (Lorentz force) arising from electric current in the magnetic field is used to transform electromagnetic energy into motion, and vice versa. This requires the coupling of Maxwell equations (on moving domains!) with solid mechanics.

1.4.2 High frequency applications

Here, the wave phenomena play the dominating role. Conducting materials ($\sigma > 0$) lead to Ohm's losses. The conductivity term enters with imaginary coefficient into the time harmonic equations

Transmitting **Antennas** are driven by an electric current, and radiate electromagnetic waves (ideally) into the whole space. Receiving antennas behave vice versa. By combining several bars, and by adding reflectors, a certain directional characteristics (depending on the frequency) can be obtained. The radiation of an antenna with a parabolic reflector is drawn in Figure 4. The behavior of waves as $x \to \infty$ requires the formulation and numerical treatment of a radiation condition.

In a **Laser resonator** a standing electromagnetic wave is generated. At a certain, material dependent frequency, the wave is amplified by changing the atomar energy state. The geometry of the resonator chamber must be adjusted such that the laser frequency corresponds to a Maxwell eigenvalue. The case of imperfect mirrors at the boundary of the resonator leads to challenging mathematical problems.

Optical fibers transmit electromagnetic signals (light) over many kilometers. A pulse at the input should be obtained as a pulse at the output. The bandwidth of the fiber is limited by the shortest pulse which can be transmitted. Ideally, the (spatial) wave length λ of the signal is indirect proportional to the frequency. Due to the finite thickness of the fiber, this is not true, and the dependency of $1/\lambda$ on the frequency ω can computed and plotted as a dispersion diagram. This diagrams reflect the transmission behaviour of the fiber.

2 The Variational Framework

Several versions of Maxwell equations lead to the equation

$$\operatorname{curl} \mu^{-1} \operatorname{curl} A + \kappa A = j \tag{12}$$

for the vector potential A. Here, j is the given current density, and μ is the permeability. The coefficient κ depends on the setting:

- The case of magnetostatic is described by $\kappa = 0$.
- The time harmonic Maxwell equations are included by setting

$$\kappa = i\omega\sigma - \omega^2 \varepsilon.$$

- Applying implicit time stepping methods for the time dependent problem (10) leads to the equation above for each timestep. Here, depending on the time integration method, $\kappa \in \mathbb{R}^+$ takes the form

$$\kappa \approx \frac{\sigma}{\tau} + \frac{\varepsilon}{\tau^2}.$$

It is the main emphasis of the lecture to study equation (12) for the different choices of $\kappa \in \mathbb{C}$.

2.1 Maxwell equations in weak formulation

In the following, Ω denotes a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. The outer normal vector is denoted by n.

Lemma 1. *For smooth functions u and v there holds the integration by parts formula*

$$\int_\Omega \operatorname{curl} u \cdot v \, dx = \int_\Omega u \cdot \operatorname{curl} v \, dx - \int_{\partial\Omega} (u \times n) \cdot v \, ds.$$

Proof. Follows from component-wise application of the scalar integration by parts formula

$$\int_\Omega \frac{\partial u}{\partial x_i} v \, dx = - \int_\Omega u \frac{\partial v}{\partial x_i} \, dx + \int_{\partial\Omega} n_i u v \, ds.$$

□

We multiply the vector potential equation (12) with all proper test functions v, and integrate over the domain:

$$\int_\Omega \operatorname{curl} \mu^{-1} \operatorname{curl} A \cdot v + \kappa A \cdot v \, dx = \int_\Omega j \cdot v \, dx \qquad \forall\, v$$

We apply integration by parts for the curl − curl term to obtain

$$\int_\Omega \mu^{-1} \operatorname{curl} A \cdot \operatorname{curl} v + \kappa A \cdot v \, dx - \int_{\partial\Omega} (\mu^{-1} \operatorname{curl} A \times n) \cdot v \, ds = \int j \cdot v \, dx \qquad \forall\, v.$$

Now, we observe useful boundary conditions:

- Natural boundary conditions on Γ_N: Assume that $j_S := \mu^{-1} \operatorname{curl} A \times n$ is known at the boundary. This is a 90 deg rotation of the tangential component of the magnetic field H.
- Essential boundary conditions on Γ_D: Set $A \times n$ as well $v \times n$ to zero. Since $E = -\frac{\partial A}{\partial t}$, this corresponds to the tangential component of the electric field. It also implies $B \cdot n = \operatorname{curl} A \cdot n = 0$.

A third type of boundary condition which linearly relates $E \times n$ and $H \times n$ is also useful and called surface impedance boundary condition. We will skip it for the moment. Inserting the boundary conditions leads to: Find A such that $A \times n = 0$ on Γ_D such that

$$\int_\Omega \mu^{-1} \operatorname{curl} A \cdot \operatorname{curl} v + \kappa A \cdot v \, dx = \int_\Omega j \cdot v \, dx + \int_{\Gamma_N} H_\tau \cdot v_\tau \, ds \qquad \forall \, v. \tag{13}$$

Note the $j_S \perp n$, thus the boundary functional depends only on $v_\tau := (v \times n) \times n$.

2.2 Existence and Uniqueness Theorems

In this section, we give the framework to prove existence, uniqueness and stability estimates for the vector potential equation in weak form (13).

The proper norm is

$$\|v\|_{H(\operatorname{curl},\Omega)} := \left\{ \|u\|^2_{L_2(\Omega)} + \|\operatorname{curl} u\|^2_{L_2(\Omega)} \right\}^{1/2}.$$

The according inner product is $(u,v)_{H(\operatorname{curl})} = (u,v)_{L_2} + (\operatorname{curl} u, \operatorname{curl} v)_{L_2}$. Denote by $\mathcal{D}(\overline{\Omega})$ all indefinitely differentiable functions on $\overline{\Omega}$, and define

$$H(\operatorname{curl},\Omega) := \overline{\mathcal{D}(\overline{\Omega})}^{\|\cdot\|_{H(\operatorname{curl},\Omega)}} \tag{14}$$

This space is a Hilbert space (inner product and complete).

Theorem 2 (Riesz$'$ representation theorem)**.** *Let V be a Hilbert space, and $f(.) : V \to \mathbb{R}$ be a continuous linear form (i.e., $f(v) \leq \|f\|_{V^*} \|v\|_V$). Then there exists an $u \in V$ such that*

$$(u,v)_V = f(v) \qquad \forall \, v \in V.$$

Furthermore, $\|u\|_V = \|f\|_{V^*}$.

We call the operator $J_V : f \to u$ the Riesz isomorphism.

Theorem 3 (Lax-Milgram)**.** *Let $B(.,.) : V \times V \to \mathbb{R}$ be a bilinear-form. Assume that $B(.,.)$ is coercive, i.e.,*

$$B(u,u) \geq c_1 \|u\|_V^2 \qquad \forall \, u \in V,$$

and continuous, i.e.,

$$B(u,v) \leq c_2 \|u\|_V \|v\|_V \qquad \forall \, u, v \in V.$$

Let $f(.)$ be a continuous linear form. Then there exists a unique $u \in V$ such that

$$B(u, v) = f(v) \qquad \forall\, v \in V.$$

There holds the stability estimate

$$\|u\|_V \leq \frac{1}{c_1}\|f\|_{V^*}.$$

The Lax-Milgram Lemma can be applied in the case of $\kappa \in \mathbb{R}^+$. The Hilbert space is

$$V := \{v \in H(\text{curl}) : v \times n = 0 \text{ on } \Gamma_D\}$$

We will show later that the tangential trace $v \times n$ is a continuous operator on $H(\text{curl})$. The linear functional is

$$f(v) = \int_\Omega j \cdot v\, dx + \int_{\Gamma_N} j_S \cdot v_\tau\, ds.$$

For now, assume that $j_S = 0$. Then

$$f(v) \leq \|j\|_{L_2}\|v\|_{L_2} \leq \|j\|_{L_2}\|v\|_V.$$

The boundary term requires the trace estimate proved later.

The bilinear-form is

$$B(u, v) = \int \mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} v + \kappa u \cdot v\, dx.$$

It is coercive with constant

$$c_1 = \min\{\inf_{x\in\Omega} \mu^{-1}(x), \inf_{x\in\Omega} \kappa(x)\},$$

and continuous with constant

$$c_2 = \max\{\sup_{x\in\Omega} \mu^{-1}(x), \sup_{x\in\Omega} \kappa(x)\}.$$

Lax-Milgram proves a unique solution in V which depends continuously on the right hand side, i.e.,

$$\|A\|_V = \{\|A\|_{L_2}^2 + \|\operatorname{curl} A\|_{L_2}^2\}^{1/2} \leq \frac{1}{c_1}\|j\|_{L_2}$$

If $\kappa \to 0$, the stability estimate degenerates.

Theorem 4 (Babuška-Aziz)**.** *Let U and V be two Hilbert spaces, and let $B(.,.) : U \times V \to \mathbb{R}$ be a continuous bilinear-form. Assume that*

$$\sup_{u\in U} \frac{B(u, v)}{\|u\|_U} \geq c_1\|v\|_V \qquad \forall\, v \in V \tag{15}$$

and

$$\sup_{v \in V} \frac{B(u,v)}{\|v\|_V} \geq c_1 \|u\|_U \qquad \forall\, u \in U. \tag{16}$$

Let $f(.)$ be a linear form on V. Then there exists a unique $u \in U$ such that

$$B(u,v) = f(v) \qquad \forall\, v \in V.$$

There holds the stability estimate

$$\|u\|_U \leq \frac{1}{c_1} \|f\|_{V^*}.$$

This theorem can be used in the complex case. Assume $\kappa = \kappa_r + i\kappa_i$ with $\kappa_i \neq 0$. Here, the real part may be negative. We write down the complex equation as a real system:

$$\begin{aligned} \operatorname{curl} \mu^{-1} \operatorname{curl} u_r + \kappa_r u_r - \kappa_i u_i &= j_r, \\ \operatorname{curl} \mu^{-1} \operatorname{curl} u_i + \kappa_i u_r + \kappa_r u_i &= j_i. \end{aligned}$$

The first equation is multiplied with v_r, the second one with v_i, we integrate by parts, and add up both equations to obtain the weak problem: Find $u = (u_r, u_i) \in V := H(\operatorname{curl})^2$ such that

$$B(u,v) = \int j_r v_r + j_i v_i \, dx \qquad \forall\, v = (v_r, v_i) \in V,$$

with the bilinear form

$$\begin{aligned} B(u,v) &= \int \mu^{-1} \{\operatorname{curl} u_r \operatorname{curl} v_r + \operatorname{curl} u_i \operatorname{curl} v_i\} + \\ &\quad + \kappa_r \{u_r v_r + u_i v_i\} + \kappa_i \{u_r v_i - u_i v_r\}. \end{aligned}$$

Continuity of $B(.,.)$ is clear. We prove (15), condition (16) is equivalent. For given $v = (v_r, v_i) \in V$, we have to come up with an explicit $u = (u_r, u_i)$ such that $\|u\|_V \leq c\|v\|_V$ and $B(u,v) \geq c\|v\|_V^2$. We choose

$$u = (v_r, v_i) + \alpha(v_i, -v_r),$$

with some α to be specified below. Evaluation gives

$$B(u,v) = \mu^{-1}\{\|\operatorname{curl} v_r\|^2 + \|\operatorname{curl} v_i\|^2\} + (\kappa_r + \alpha\kappa_i)\{\|v_r\|^2 + \|v_i\|^2\}.$$

Set $\alpha = \frac{1-\kappa_r}{\kappa_i}$, to obtain

$$B(u,v) = \mu^{-1} \|\operatorname{curl} v\|_{L_2}^2 + 1\, \|v\|_{L_2}^2.$$

As long as $\kappa_i \neq 0$, the weak form has a unique solution. The continuity depends on $\frac{1}{\kappa_i}$. The imaginary coefficient stabilizes the problem !

Theorem 5 (Brezzi). *Let V and Q be Hilbert spaces. Let $a(.,.) : V \times V \to \mathbb{R}$ and $b(.,.) : V \times Q \to \mathbb{R}$ be continuous bilinear forms, and $f(.) : V \to \mathbb{R}$ and $g(.) : Q \to \mathbb{R}$ be linear forms. Denote the kernel of $b(.,.)$ by*

$$V_0 := \{v \in V : b(v,q) = 0 \ \forall q \in Q\}.$$

Assume that $a(.,.)$ is coercive on the kernel, i.e.,

$$a(v,v) \geq \alpha_1 \|v\|^2 \qquad \forall\, v \in V_0, \tag{17}$$

and assume that $b(.,.)$ satisfies the LBB (Ladyshenskaya-Babuška-Brezzi) condition

$$\sup_{v \in V} \frac{b(v,q)}{\|v\|_V} \geq \beta_1 \|q\|_Q \qquad \forall\, q \in Q. \tag{18}$$

Then there exists a unique $u \in V$ and $p \in Q$ such that

$$\begin{array}{lcllll} a(u,v) & + & b(v,p) & = & f(v) & \forall\, v \in V, \\ b(u,q) & & & = & g(q) & \forall\, q \in Q. \end{array} \tag{19}$$

There holds $\|u\|_V + \|p\|_Q \leq c(\|f\|_{V^*} + \|g\|_{Q^*})$, where c depends on α_1, β_1, $\|a\|$, and $\|b\|$.

This variational problem is called a mixed problem, or a saddle point problem. Brezzi's theorem will be applied to the case $\kappa = 0$. The original weak form is

$$\int_\Omega \mu^{-1} \operatorname{curl} A \cdot \operatorname{curl} v \, dx = \int_\Omega jv \, dx \qquad \forall\, v \in H(\operatorname{curl}).$$

The bilinear-form is not coercive: Take $u = \nabla\varphi$. Then $B(u,u) = 0$, but $\|u\|^2_{H(\operatorname{curl})} = \|\nabla\varphi\|^2_{L_2}$. To satisfy the equation for all test functions v, the source term j must be compatible. If $v = \nabla\psi$, the left hand side vanishes, thus also the right hand side must vanish, too. Integration by parts gives

$$0 = \int_\Omega j \cdot \nabla\psi = -\int_\Omega \operatorname{div} j \psi + \int_{\partial\Omega} j \cdot n \psi \, ds \qquad \forall\, \psi \in H^1(\Omega).$$

Thus, $\operatorname{div} j = 0$ as well as $j \cdot n = 0$ must be satisfied.

We reformulate the problem now as a saddle point problem. The vector potential A is defined only up to gradient fields. Thus, we add the constraint $A \perp \nabla H^1$:

$$\int A \nabla\psi \, dx = 0 \qquad \forall\, \psi \in H^1(\Omega).$$

We cast the problem now in the saddle point framework: Search $A \in H(\operatorname{curl})$ and $\varphi \in H^1/\mathbb{R}$ such that

$$\begin{array}{rclcll} \int \mu^{-1} \operatorname{curl} A \cdot \operatorname{curl} v & + & \int v \cdot \nabla\varphi & = & \int j \cdot v & \forall\, v \in H(\operatorname{curl}), \\ \int A \cdot \nabla\psi & & & = & 0 & \forall\, \psi \in H^1/\mathbb{R}. \end{array} \tag{20}$$

If the right hand side j is compatible, the newly introduced variable $\varphi \in H^1$ will be 0. To see this, take $v = \nabla\varphi$.

One condition of Brezzi's theorem is the LBB condition, i.e.,

$$\sup_{v \in H(\operatorname{curl})} \frac{\int v \nabla\varphi}{\|v\|_{H(\operatorname{curl})}} \geq \beta_1 \|\nabla\varphi\|.$$

This one holds trivially true by choosing $v = \nabla\varphi$. The kernel ellipticity reads as

$$\mu^{-1} \|\operatorname{curl} v\|^2 \geq \|v\|^2 + \|\operatorname{curl} v\|^2 \qquad \forall\, v \in V_0 = \{v : (v, \nabla\varphi) = 0 \,\forall\, \varphi\}.$$

The second one is non-trivial, and will be proven later. Brezzi's theorem proves the existence of a unique $A \in H(\operatorname{curl})$ depending continuously on j for general $j \in L_2$.

Up to now we have considered all cases κ except $\kappa \in \mathbb{R}^-$. Here, a unique solution is not guaranteed for all values κ. But, the operator is singular only for a discrete set of eigenvalues. This most general case can be handled with the Fredholm theorem:

Theorem 6 (Fredholm)**.** *Assume that K is a compact operator. Then $(I - \lambda K)$ is invertible up to a discrete set of singular values λ.*

If A solves the variational problem, then

$$\int \mu^{-1} \operatorname{curl} A \operatorname{curl} v \, dx = \int (j - \kappa A) \cdot v \, dx$$

Assuming $\operatorname{div} f = 0$, and testing with $v = \nabla\varphi$, we observe $\int \kappa A \nabla\varphi = 0$. We add this constraint, and add also a dummy - Lagrange parameter to obtain the mixed problem

$$\begin{array}{lclcll} \int \mu^{-1} \operatorname{curl} A \cdot \operatorname{curl} v & + & \int v \cdot \nabla\varphi & = & \int (j - \kappa A) \cdot v & \forall\, v \in H(\operatorname{curl}), \\ \int A \cdot \nabla\psi & & & = & 0 & \forall\, \psi \in H^1/\mathbb{R}. \end{array} \tag{21}$$

Brezzi's theory ensures a unique solution for given right hand side $(j - \kappa A) \in L_2$. We denote the solution operator by T, i.e., we have

$$A = T(j - \kappa A),$$

or

$$A + T\kappa A = j$$

We will prove that T is a compact operator on L_2. Thus, the Maxwell equation is solveable up to a discrete set of eigenvalues $\kappa \in \mathbb{R}^-$.

2.3 The function spaces $H(\operatorname{curl})$ and $H(\operatorname{div})$

We will define weak derivatives. First, consider a smooth function $u \in C^1(-1, 1)$, and let $g = u'$. This can be defined in weak sense, i.e.

$$\int (g - u') v \, dx = 0 \qquad \forall\, v \in C_0^\infty(-1, 1)$$

Now, integrate by parts to obtain

$$\int_{-1}^{1} gv\,dx = -\int_{-1}^{1} uv'\,dx \qquad \forall\, v \in C_0^\infty(-1,1).$$

Boundary terms do not appear, since v has 0-boundary values. This definition can be extended to distributions. Here, we are interested in weak derivatives, where the derivatives are still regular functions in L_2.

Definition 7 (Weak differential operators)**.** *Let $w \in L_2(\Omega)$, $u \in [L_2(\Omega)]^3$, and $q \in [L_2(\Omega)]^3$. We call $g = \nabla\, w \in [L_2(\Omega)]^3$ the weak gradient, $c = \operatorname{curl}\, u \in [L_2(\Omega)]^3$ the weak curl, and $d = \operatorname{div}\, q \in L_2(\Omega)$ the weak divergence if they satisfy*

$$\begin{aligned}
\int g\cdot v\,dx &= -\int w \operatorname{div}\, v\,dx \qquad \forall\, v \in [C_0^\infty(\Omega)]^3 \\
\int c\cdot v\,dx &= +\int u \operatorname{curl}\, v\,dx \qquad \forall\, v \in [C_0^\infty(\Omega)]^3 \\
\int d\cdot v\,dx &= -\int q\nabla\, v\,dx \qquad \forall\, v \in C_0^\infty(\Omega)
\end{aligned}$$

Definition 8 (Function spaces)**.** *We define the spaces*

$$\begin{aligned}
H(\mathrm{grad}) = H^1 &= \{w \in L_2 : \nabla w \in [L_2]^3\} \\
H(\mathrm{curl}) &= \{u \in [L_2]^3 : \operatorname{curl} u \in [L_2]^3\} \\
H(\mathrm{div}) &= \{q \in [L_2]^3 : \operatorname{div} q \in L_2\}
\end{aligned}$$

and the corresponding semi-norms and norms

$$\begin{aligned}
|w|_{H(\mathrm{grad})} &= \|\nabla w\|_{L_2}, & \|w\|_{H(\mathrm{grad})} &= \left(\|w\|_{L_2}^2 + |w|_{H(\mathrm{grad})}^2\right)^{1/2} \\
|u|_{H(\mathrm{curl})} &= \|\operatorname{curl} u\|_{L_2}, & \|u\|_{H(\mathrm{curl})} &= \left(\|u\|_{L_2}^2 + |u|_{H(\mathrm{curl})}^2\right)^{1/2} \\
|q|_{H(\mathrm{div})} &= \|\operatorname{div} q\|_{L_2}, & \|q\|_{H(\mathrm{div})} &= \left(\|q\|_{L_2}^2 + |q|_{H(\mathrm{div})}^2\right)^{1/2}
\end{aligned}$$

These spaces are related by the following sequence:

$$H(\mathrm{grad}) \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2$$

The (weak) gradients of $H(\mathrm{grad})$ are in L_2, and $\operatorname{curl}\operatorname{grad} = 0 \in L_2$. It is easy to check this also in weak sense. Further on, the (weak) curls of functions in $H(\mathrm{curl})$ are in L_2, and $\operatorname{div}\operatorname{curl} = 0$, thus $\operatorname{curl}[H(\mathrm{curl})] \subset H(\mathrm{div})$. Finally, $\operatorname{div}[H(\mathrm{div})] \subset L_2$. We will prove later the de Rham theorem, which tells us that (on simply connected domains) the range of an operator of this sequence is exactly the kernel of the next operator.

The de Rham theorem is elementary for smooth functions. To prove it for the Hilbert spaces, we need a density result.

Definition 9. *The boundary of the domain Ω is called Lipschitz, if there exist a finite number of domains ω_i, local coordinate systems (ξ_i, η_i, ζ_i), and Lipschitz-continuous functions $b(\xi_i, \eta_i)$ such that*

- $\partial\Omega \subset \cup\omega_i$
- $\Omega \cap \omega_i = \{(\xi_i, \eta_i, \zeta_i) \in \omega_i : \zeta_i > b(\xi_i, \eta_i)\}$

We are going to prove density results of smooth functions in H(curl and in H(div). For this, we will study mollification (smoothing) operators. In the following, we assume that Ω has a Lipschitz continuous boundary.

Theorem 10. *$C^m(\overline{\Omega})$ is dense in L_2.*

Proof: Analysis 3

We will smooth by local averaging. The influence domain of the smoothing operator must be contained in the domain. For this, we first shrink the domain by the order of ε.

Lemma 11. *There exists a family of smooth transformations $\phi^\varepsilon : \Omega \to \Omega$ such that*

$$\phi^\varepsilon \stackrel{\varepsilon \to 0}{\longrightarrow} id \qquad in\ C^m,$$

and

$$dist\{\phi^\varepsilon(\Omega), \partial\Omega\} \geq \varepsilon.$$

Proof. Let ψ_i be a smooth partitioning of unity on the boundary, i.e. $\psi_i \in [0,1]$, support $\psi_i \subset \omega_i$, and $\sum \psi_i(x) = 1$ for $x \in \partial\Omega$. Let $e_{\zeta,i}$ be the inner unit vector in the local coordinate system. Then

$$\phi^\varepsilon(x) = x + c\varepsilon \sum_i \psi_i(x) e_{\zeta,i}$$

is a transformation with this property. Here, c is a constant $O(1)$ depending on the Lipschitz-norm of the boundary. □

We define

$$F^\varepsilon(x) = (\phi^\varepsilon)' \qquad \text{and} \qquad J^\varepsilon(x) = \det F^\varepsilon(x).$$

There holds $F^\varepsilon \to I$, and thus F^ε is invertible for sufficiently small ε.

Let $B(x,r)$ be the ball with center x and radius r, and let ψ be a fixed function in $C_0^m(B(0,1))$ such that $\int_{B(0,1)} \psi(y)\,dy = 1$. When needed, ψ is extended by 0 to $\mathbb{R}^3$.

The family of smoothing operators is defined by

$$(S_g^\varepsilon w)(x) := \int_{B(0,1)} \psi(y) w(\phi^\varepsilon(x) + \varepsilon y)\,dy$$

Since $\phi^\varepsilon(\Omega)$ is separated from the boundary $\partial\Omega$, only values of w in Ω are envolved.

Lemma 12. *The smoothing operators S_g^ε map $L_2(\Omega)$ into $C^m(\overline{\Omega})$*

Proof. First, we prove that $S_g^\varepsilon w$ is a continuous function. By substituting $\xi = \phi^\varepsilon(x) + \varepsilon y$, we rewrite

$$S_g^\varepsilon w(x) = \int_{B(0,1)} \psi(y) w(\phi^\varepsilon(x) + \varepsilon y)\, dy = \int_\Omega \psi\Big(\frac{\xi - \phi^\varepsilon(x)}{\varepsilon}\Big)\, w(\xi)\, d\xi/\varepsilon^3$$

Next, bound

$$\begin{aligned}
(S_g^\varepsilon w)(x_1) - (S_g^\varepsilon w)(x_2) &= \varepsilon^{-3} \int_\Omega \Big\{\psi\Big(\frac{\xi - \phi(x_1)}{\varepsilon}\Big) - \psi\Big(\frac{\xi - \phi(x_2)}{\varepsilon}\Big)\Big\} w(\xi)\, d\xi \\
&\leq \varepsilon^{-3} \Big\| \psi\Big(\frac{\xi - \phi(x_1)}{\varepsilon}\Big) - \psi\Big(\frac{\xi - \phi(x_2)}{\varepsilon}\Big)\Big\|_{L_2(\Omega)} \|w\|_{L_2(\Omega)}
\end{aligned}$$

The first factor contains a continuous function in x_1 (resp. x_2), and thus it converges to 0 as $x_1 - x_2 \to 0$.

Next, we prove convergence for the derivatives:

$$\begin{aligned}
\frac{\partial}{\partial x_i} S_g w(x) &= \int_B \psi(y) \frac{\partial}{\partial x_i} w(\phi^\varepsilon(x) + y)\, dy \\
&= \int_B \psi(y) (\nabla w)(\phi^\varepsilon + \varepsilon y) \frac{\partial \phi^\varepsilon}{\partial x_i}\, dy \\
&= \frac{\partial \phi^\varepsilon}{\partial x_i} \cdot \int_B \psi(y) \nabla_y [w(\phi^\varepsilon + \varepsilon y)]/\varepsilon\, dy \\
&= -\frac{1}{\varepsilon} \frac{\partial \phi^\varepsilon}{\partial x_i} \cdot \int_B \nabla \psi(y) w(\phi^\varepsilon + \varepsilon y)\, dy.
\end{aligned}$$

The derivative of the smoothed function is expressed by smoothing with the new mollifier function $\psi_i(y,x) := \frac{-1}{\varepsilon} \nabla_y \psi(y) \cdot \frac{\partial \phi^\varepsilon}{\partial x_i}$. Thus, convergence of derivatives is reduced by induction to convergence in C^0. We have used classical calculus. This is allowed, since C^1 is dense in L_2, and the right hand side is well defined for $w \in L_2$. □

Lemma 13. *There holds $S_g^\varepsilon w \to w$ in L_2 for $\varepsilon \to 0$.*

Proof. First, we prove L_2-continuity of S_g^ε uniform in ε:

$$\begin{aligned}
\|S_g^\varepsilon w\|_{L_2}^2 &= \int_\Omega \Big(\int_B \psi(y) w(\phi^\varepsilon(x) + \varepsilon y) dy)^2\, dx \\
&\leq \int_\Omega \|\psi\|_{L_2(B)}^2 \int_B w^2(\phi^\varepsilon(x) + \varepsilon y) dy dx \\
&= \|\psi\|_{L_2}^2 \int_B \int_\Omega w^2(\phi^\varepsilon(x) + \varepsilon y) dx dy \\
&= \|\psi\|_{L_2}^2 \int_B \int_{\phi^\varepsilon(\Omega)} w^2(\hat{x} + \varepsilon y) J^{-1} d\hat{x}\, dy \\
&\leq \|\psi\|_{L_2} \max\{J^{-1}\} \int_B \int_\Omega w^2(\hat{x}) d\hat{x} dy \\
&= \|\psi\|_{L_2} |B| \max\{J^{-1}\} \|w\|_{L_2}
\end{aligned}$$

Next, assume that w_1 is Lipschitz-continuous with Lipschitz constant L. Then

$$S_g^\varepsilon w_1(x) - w_1(x) = \int_B \psi(y) \{ \underbrace{w_1(\phi^\varepsilon(x) + \varepsilon y) - w_1(x)}_{\leq L|\phi^\varepsilon(x)+\varepsilon y - x| \leq cL\varepsilon} \} \, dy \leq c\varepsilon L$$

Now, use density of C^1 (and thus of Lipschitz-functions) in L_2. Choose $w_1 \in C^{0,1}$ such that $\|w - w_1\|_{L_2} \leq \delta$, and $\|w_1\|_{C^{0,1}} \leq L$. Then

$$\begin{aligned} \|w - S_g^\varepsilon w\|_{L_2} &\leq \|w - w_1\|_{L_2} + \|w_1 - S_g^\varepsilon w_1\|_{L_2} + \|S_g^\varepsilon (w - w_1)\| \\ &\leq (1 + \|S_g^\varepsilon\|)\delta + cL\varepsilon. \end{aligned}$$

The bound on the right hand side can be made arbitrarily small: First choose a small δ, which leads to a (possible large) L. Then choose ε such that $L\varepsilon$ is small. □

Lemma 14 (Transformation of differential operators)**.** *Let $\phi : \Omega \to \hat{\Omega}$ be a smooth, one-to-one transfromation. Let $F = \phi'$, and $J = \det F$. Then there holds*

$$\nabla[w(\phi(x))] = F^T(x)(\nabla w)(\phi(x)) \tag{22}$$

$$\operatorname{curl}[F^T(x)u(\phi(x))] = JF^{-1}(\operatorname{curl} u)(\phi(x)) \tag{23}$$

$$\operatorname{div}[J(x)F^{-1}(x)q(\phi(x))] = J(x)(\operatorname{div} q)(\phi(x)) \tag{24}$$

The vector-transformation $F^T u$ is called covariant, and the transformation $JF^{-1}q$ is called the Piola transformation

Proof. For smooth functions, the first identity is the chain rule. We start to prove the last one in weak sense. Choose a test function $v \in C_0^\infty$, use the definition of the weak divergence, and the transformation rule for gradients to evaluate

$$\begin{aligned} \int_\Omega \operatorname{div}[JF^{-1}q(\phi(x))]v(x)dx &= -\int_\Omega [JF^{-1}q(\phi(x)] \cdot \nabla v(x) \, dx \\ &= -\int_\Omega q(\phi(x)) \cdot F^{-T}\nabla v(x) J dx \\ &= -\int_{\phi(\Omega)} q(\hat{x}) F^{-T} (\nabla v)(\phi^{-1}(\hat{x})) d\hat{x} \\ &= -\int_{\phi(\Omega)} q(\hat{x}) \nabla[v(\phi^{-1}(\hat{x}))] d\hat{x} \\ &= \int_{\phi(\Omega)} \operatorname{div} q(\hat{x}) v(\phi^{-1}(\hat{x})) d\hat{x} \\ &= \int_\Omega J(\operatorname{div} q)(\phi(x)) v(x) \, dx. \end{aligned}$$

Since this is true for all smooth testfunctions, and they are dense in L_2, equation (24) is proven. Similarly, one proves the first identity for the weak gradient. The identity (23) for the curl is proven by the chain rule in the classical form. The notation

$$\Gamma_{ijk} = \begin{cases} +1 & (i,j,k) \in (1,2,3),(2,3,1),(3,1,2) \\ -1 & (i,j,k) \in (1,3,2),(2,1,3),(3,2,1) \\ 0 & \text{else} \end{cases}$$

allows to write

$$(\operatorname{curl} u)_i = \sum_{jk} \Gamma_{ijk} \frac{\partial u_k}{\partial x_j}.$$

Now, expand matrix products and use the chain-rule to obtain

$$\begin{aligned} \Big\{ F \operatorname{curl}[F^T u(\phi(x))] \Big\}_i &= \sum_j F_{ij} \big\{ \operatorname{curl}[F^T u(\phi(x))] \big\}_j \\ &= \sum_{j,k,l} F_{ij} \Gamma_{jkl} \frac{\partial [F^T u(\phi(x))]_l}{\partial x_k} \\ &= \sum_{j,k.l,m} F_{ij} \Gamma_{jkl} \frac{\partial [F_{ml} u_m(\phi(x))]}{\partial x_k} \\ &= \sum_{j,k.l,m} F_{ij} \Gamma_{jkl} \frac{\partial F_{ml}}{\partial x_k} u_m(\phi(x)) + \sum_{j,k.l,m,n} F_{ij} \Gamma_{jkl} F_{ml} \frac{\partial u_m}{\partial x_n} F_{nk} \end{aligned}$$

Since $\frac{\partial F_{ml}}{\partial x_k} = \frac{\partial^2 \phi_m}{\partial x_l \partial x_k} = \frac{\partial F_{mk}}{\partial x_l}$, and $\Gamma_{jkl} = -\Gamma_{jlk}$, the first summand disappears. One verifies that

$$\sum_{j,k,l} F_{ij} F_{ml} F_{nk} \Gamma_{jkl} = \det F \, \Gamma_{inm},$$

and completes the proof with

$$\big\{ F \operatorname{curl}[F^T u(\phi(x))] \big\}_i = \sum_{mn} \det F \, \Gamma_{inm} \frac{\partial u_m}{\partial x_n} = J \left\{ (\operatorname{curl} u)(\phi(x)) \right\}_i.$$

□

Definition 15. *Define additional smoothing operators*

covariant transformation:

$$S_c^\varepsilon u : [L_2]^3 \to [C^m(\overline{\Omega})]^3 : \qquad (S_c^\varepsilon u)(x) := \int_{B(0,1)} \psi(y) F^T u(\phi^\varepsilon(x) + \varepsilon y) \, dy$$

Piola transformation:

$$S_d^\varepsilon q : [L_2]^3 \to [C^m(\overline{\Omega})]^3 : \qquad (S_d^\varepsilon q)(x) := \int_{B(0,1)} \psi(y) J F^{-1} q(\phi^\varepsilon(x) + \varepsilon y) \, dy$$

$$S_i^\varepsilon s : L_2 \to C^m(\overline{\Omega}) : \qquad (S_c^\varepsilon s)(x) := \int_{B(0,1)} \psi(y) J s(\phi^\varepsilon(x) + \varepsilon y) \, dy$$

These additinal smoothing operators converge point-wise as the original S_g^ε. The proofs need the additional argument that $F^\varepsilon \to I$ and $J^\varepsilon \to 1$ for $\varepsilon \to 0$.

Theorem 16. *The smoothing operators commute in the following sense:*

1. *Let $w \in H(\mathrm{grad})$. Then there holds*

$$\nabla S_g^\varepsilon w = S_c^\varepsilon \nabla w \tag{25}$$

2. *Let $u \in H(\mathrm{curl})$. Then there holds*

$$\operatorname{curl} S_c^\varepsilon u = S_d^\varepsilon \operatorname{curl} u \tag{26}$$

3. *Let $q \in H(\mathrm{div})$. Then there holds*

$$\operatorname{div} S_d^\varepsilon q = S_i^\varepsilon \operatorname{div} q \tag{27}$$

Proof. Follows (with classical calculus) directly from the transformation rules. Exercise: Prove the identities using weak derivatives. □

Corollary 17. *The space $[C^m(\overline{\Omega})]^3$ is dense in $H(\mathrm{curl})$ and in $H(\mathrm{div})$.*

Proof. Let $u \in H(\mathrm{curl})$. For $\varepsilon \to 0$, $S_c^\varepsilon u$ defines a sequence of smooth functions. There holds

$$S_c^\varepsilon u \xrightarrow{\varepsilon \to 0} u \quad (\text{in } L_2) \qquad \text{and} \qquad \operatorname{curl} S_c^\varepsilon u = S_d^\varepsilon \operatorname{curl} u \xrightarrow{\varepsilon \to 0} \operatorname{curl} u \quad (\text{in } L_2).$$

Thus

$$S_c^\varepsilon u \xrightarrow{\varepsilon \to 0} u \quad (\text{in } H(\mathrm{curl}))$$

The same arguments apply for $H(\mathrm{div})$ and $H(\mathrm{grad})$. □

Thanks to density, many classical theorems can be easily extended to the Hilbert-space context.

Theorem 18 (de Rham)**.** *Assume that Ω is simply connected. Then,*

$$\{u \in H(\mathrm{curl}) : \operatorname{curl} u = 0\} = \nabla H^1$$

Proof. The one inclusion $\nabla H^1 \subset H(\mathrm{curl})$ and $\operatorname{curl} \nabla H^1 = \{0\}$ is simple. Now, assume that $u \in H(\mathrm{curl})$ such that $\operatorname{curl} u = 0$. Define the sequence of smooth functions $u^\varepsilon = S_c^\varepsilon u$. They satisfy $\operatorname{curl} u^\varepsilon = S_c^\varepsilon \operatorname{curl} u = 0$. Smooth, curl-free functions are gradients, which follows from the path-independence of the integral. Thus, there exist smooth ϕ^ε such that $\nabla \phi^\varepsilon = u^\varepsilon$ and are normalized such that $\int_\Omega \phi^\varepsilon = 0$. The sequence u^ε is Cauchy in L_2, thus ϕ^ε is Cauchy in H^1, and converges to a $\phi \in H^1$ satisfying $\nabla \phi = u$. □

2.3.1 Trace operators

Functions in the Sobolve space $H^1(\Omega)$ have generalized boundary values (a trace) in the space $H^{1/2}(\Gamma)$. We recall properties of the trace operator in H^1, and investigate corresponding trace operators for the spaces $H(\text{curl})$ and $H(\text{div})$.

The trace operator $\operatorname{tr}|_\Gamma$ for functions in H^1 is constructed as follows:

1. Define the trace operator for smooth functions $u \in C(\overline{\Omega}) \cap H^1(\Omega)$ in the pointwise sense
$$(\operatorname{tr}_\Gamma u)(x) := u(x) \qquad \forall\, x \in \Gamma.$$

2. Prove continuity (the **trace theorem**)
$$\|\operatorname{tr}_\Gamma u\|_{H^{1/2}(\Gamma)} \leq c\|u\|_{H^1} \qquad \forall\, u \in C(\overline{\Omega}) \cap H^1$$
The $H^{1/2}$-norm is defined as
$$\|w\|^2_{H^{1/2}(\Gamma)} = \|w\|^2_{L_2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|w(x) - w(y)|^2}{|x-y|^2}\, dxdy,$$
the Hilbert space $H^{1/2}$ is the closure of smooth functions (e.g. C^∞) with respect to this norm.

3. Extend the definition of the trace operator to the whole $H^1(\Omega)$. Choose an arbitrary sequence (u_n) with $u_n \in C(\overline{\Omega}) \cap H^1(\Omega)$ such that $u_n \to u$ in H^1. Thanks to density of $C(\overline{\Omega}) \cap H^1$ in H^1 this is possible. Then define
$$\operatorname{tr}|_\Gamma u := \lim_{n\to\infty} \operatorname{tr}|_\Gamma u_n.$$
Since u_n is Cauchy in H^1, and $\operatorname{tr}|_\Gamma$ is a continuous operator, the sequence $\operatorname{tr}|_\Gamma u_n$ is Cauchy in $H^{1/2}(\Gamma)$. Since $H^{1/2}$ is a Hilbert space, the Cauchy sequence has a limit which we call $\operatorname{tr}|_\Gamma u$. Finally, check that the limit is independent of the chosen sequence (u_n).

Theorem 19 (inverse trace theorem)**.** *For a given $w \in H^{1/2}(\Gamma)$, there exists an $u \in H^1(\Omega)$ such that*
$$\operatorname{tr}|_\Gamma u = w.$$
and
$$\|u\|_{H^1(\Omega)} \leq c\|w\|_{H^{1/2}(\Gamma)}$$

The $H^{1/2}$ can be restricted to parts of the boundary. There are a few details which we do not discuss here. The trace theorem and inverse trace theorem are necessary to define boundary conditions. Dirichlet values are incorporated into the space
$$V_g = \{u \in H^1 : \operatorname{tr}_\Gamma u = u_D\}.$$

Thanks to the inverse trace theorem, boundary values $u_D \in H^{1/2}(\Gamma_D)$ are allowed. Neumann boundary values $\frac{\partial u}{\partial n} = g$ are included in the linear form:

$$f(v) = \int_{\Gamma_N} g \, \operatorname{tr}_\Gamma v \, ds \qquad \forall \, v \in H^1(\Omega)$$

The integral is understood as a duality product in $H^{1/2}$ and its dual $H^{-1/2}$. Then

$$|f(v)| = \langle g, \operatorname{tr}_\Gamma v \rangle_{H^{-1/2} \times H^{1/2}} \leq \|g\|_{H^{-1/2}} \| \operatorname{tr}_\Gamma v\|_{H^{1/2}} \leq c \, \|g\|_{H^{-1/2}} \|v\|_{H^1(\Omega)}$$

The linear-form is continuous on $H^1(\Omega)$ as long as $g \in H^{-1/2}(\Gamma)$.

Lemma 20 (Integration by parts)**.** *There holds the integration by parts formula*

$$\int_\Omega \nabla u \cdot \varphi \, dx + \int_\Omega u \operatorname{div} \varphi \, dx = \int_{\partial\Omega} \operatorname{tr}_\Gamma u \, \varphi \cdot n \, ds \qquad \forall \, \varphi \in C^\infty(\overline{\Omega})$$

Lemma 21 ()**.** *Let $\Omega_1, \ldots \Omega_m$ be a domain decomposition of Ω, i.e., $\Omega_i \cap \Omega_j = \emptyset$ and $\overline{\Omega} = \cup \overline{\Omega}_i$, let $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. Let $u_i \in H^1(\Omega_i)$ such that $\operatorname{tr}_{\Gamma_{ij}} u_i = \operatorname{tr}_{\Gamma_{ij}} u_j$.*

Then

$$u \in H^1(\Omega) \qquad \textit{and} \qquad (\nabla u)|_{\Omega_i} = \nabla u_i$$

Proof. Let $g_i = \nabla u_i$ be the local weak gradients, and set $g = g_i$ on Ω_i. We use the integration by parts formula on Ω_i to obtain (for all $\varphi \in C_0^\infty(\Omega)$)

$$\begin{aligned}
-\int_\Omega u \operatorname{div} \varphi \, dx &= -\sum_{i=1}^m \int_{\Omega_i} u_i \operatorname{div} \varphi \, dx \\
&= \sum \int_{\Omega_i} \nabla u_i \cdot \varphi \, dx - \int_{\partial\Omega_i} \operatorname{tr}_{\partial\Omega_i} \varphi \cdot n_i \, dx \\
&= \sum \int_{\Omega_i} g_i \cdot \varphi \, dx - \int_{\partial\Omega_i} \operatorname{tr}_{\partial\Omega_i} \varphi \cdot n_i \, dx \\
&= \sum \int_{\Omega_i} g_i \cdot \varphi \, dx - \sum_{\Gamma_{ij}} \int_{\Gamma_{ij}} (\operatorname{tr}_{\Gamma_{ij}} u_i - \operatorname{tr}_{\Gamma_{ij}} u_j) \, \varphi \cdot n_i \, dx \\
&= \sum \int_{\Omega_i} g_i \cdot \varphi = \int_\Omega g \cdot \varphi \, ds
\end{aligned}$$

Thus, g is the weak gradient of u on Ω. □

Theorem 22 (trace theorems)**.**

- *There exists a unique continuous operator $\operatorname{tr}_n : H(\operatorname{div}) \to H^{-1/2}(\partial\Omega)$ which satisfies*

$$\operatorname{tr}_n u(x) = u(x) \cdot n(x) \qquad \forall \, x \in \partial\Omega \; (a.e.)$$

for functions $u \in [C(\overline{\Omega})]^3 \cap H(\operatorname{div})$.

- *There exists a unique continuous operator* $\operatorname{tr}_\tau : H(\operatorname{curl}) \to [H^{-1/2}(\partial\Omega)]^3$ *which satisfies*
$$\operatorname{tr}_\tau u(x) = u(x) \times n(x) \qquad \forall\, x \in \partial\Omega \text{ (a.e.)}$$
for functions $u \in [C(\overline{\Omega})]^3 \cap H(\operatorname{curl})$.

Proof. The construction follows the lines of the H^1-case. We have to prove continuity on a smooth, dense sub-space. Let $q \in H(\operatorname{div}) \cap C(\overline{\Omega})^3$, use the definition of the dual norm $H^{-1/2}$, and the inverse trace theorem on H^1:

$$\begin{aligned} \|\operatorname{tr}_n q\|_{H^{-1/2}} &= \sup_{w \in H^{1/2}} \frac{\int_{\partial\Omega} q \cdot n\, w ds}{\|w\|_{H^{1/2}}} \leq c \sup_{v \in H^1(\Omega)} \frac{\int_{\partial\Omega} q \cdot n \operatorname{tr} v\, ds}{\|v\|_{H^1}} \\ &= \sup_{v \in H^1(\Omega)} \frac{\int_\Omega q \cdot \nabla v + \operatorname{div} q\, v\, dx}{\|v\|_{H^1}} \leq \|q\|_{H(\operatorname{div})} \end{aligned}$$

The proof for $H(\operatorname{curl})$ is left as exercise. $\square$

To prove the trace theorem for $H(\operatorname{div})$, we needed the inverse trace theorem in H^1. The converse is also true:

Lemma 23 (inverse trace theorem)**.** *Let* $q_n \in H^{-1/2}(\Gamma)$. *Then there exists an* $q \in H(\operatorname{div})$ *such that*
$$\operatorname{tr}_n q = q_n \qquad \text{and} \qquad \|q\|_{H(\operatorname{div})} \leq c\|q_n\|_{H^{-1/2}}$$
If q_n *satisfies* $\langle q, 1\rangle = 0$, *then there exists an extension* $q \in H(\operatorname{div})$ *such that* $\operatorname{div} q = 0$.

Proof. We solve the weak form of the scalar equation $-\Delta u + u = 0$ with boundary conditions $\frac{\partial u}{\partial n} = q_n$. Since $q_n \in H^{-1/2}$, there exists a uniquie solution in H^1 such that
$$\|\nabla u\|^2 + \|u\|^2 \leq c\|q_n\|^2_{H^{-1/2}}$$
Now, set $q = \nabla u$. Observe that $\operatorname{div} q = u \in L_2$, and thus
$$\|q\|^2_{L_2} + \|\operatorname{div} q\|^2_{L_2} \leq c\|q_n\|^2_{H^{-1/2}}.$$
If q_n satisfies $\langle q, 1\rangle$, then we solve the Neumann problem of the Poisson equation $-\Delta u = 0$. It is possible, since the right hand side is orthogonal to the constant functions. Again, take $q = \nabla u$. $\square$

The inverse trace theorem shows also that the trace inequality is sharp. The stated trace theorem for $H(\operatorname{curl})$ is not sharp, and thus there is no inverse trace theorem. The right norm is $\|\operatorname{tr}_\tau u\|_{H^{-1/2}} + \|\operatorname{div}_\tau \operatorname{tr}_\tau u\|_{H^{-1/2}}$, which leads to an inverse trace theorem.

Lemma 24. *Let* $\Omega_1, \ldots \Omega_m$ *be a domain decomposition of* Ω, *i.e.,* $\Omega_i \cap \Omega_j = \emptyset$ *and* $\overline{\Omega} = \cup \overline{\Omega}_i$. *Let* $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. *Let* $q_i \in H(\operatorname{div}, \Omega_i)$ *such that* $\operatorname{tr}_{n_i, \Gamma_{ij}} q_i = \operatorname{tr}_{n_i, \Gamma_{ij}} q_j$. *Then*
$$q \in H(\operatorname{div} \Omega) \qquad \text{and} \qquad (\operatorname{div} q)|_{\Omega_i} = \operatorname{div} q_i$$

Lemma 25. *Let* $\Omega_1, \ldots \Omega_m$ *be a domain decomposition of* Ω, *i.e.,* $\Omega_i \cap \Omega_j = \emptyset$ *and* $\overline{\Omega} = \cup \overline{\Omega}_i$. *Let* $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. *Let* $u_i \in H(\operatorname{curl} \Omega_i)$ *such that* $\operatorname{tr}_{\tau_i, \Gamma_{ij}} u_i = \operatorname{tr}_{\tau_i, \Gamma_{ij}} u_j$. *Then*
$$u \in H(\operatorname{curl} \Omega) \qquad \text{and} \qquad (\operatorname{curl} u)|_{\Omega_i} = \operatorname{curl} u_i$$

2.3.2 Helmholtz decompositions

The Helmholtz decomposition splits a vector-function u into a gradient and into a curl function, i.e.,

$$u = \nabla \phi + \operatorname{curl} \psi.$$

Here, ϕ is called the scalar potential, and ψ is the vector potential. One can choose different boundary conditions for the two fields. Additionally, ψ is not uniquely defined, and one may select a particular one, e.g. by the constraint $\operatorname{div} \psi = 0$.

Lemma 26. *Assume that $q \in H(\mathrm{div})$ such that $\mathrm{div} = 0$ and $\operatorname{tr}_n q = 0$. Then there exists ψ such that*

$$q = \operatorname{curl} \psi.$$

The function ψ can be chosen such that

(i) $\psi \in [H^1]^3$ and $\operatorname{div} \psi = 0$ and $|\psi|_{H^1} \leq \|q\|_{L_2}$,

(ii) or $\psi \in [H^1_0]^3$ and $\|\psi\|_{H^1} \leq c\|q\|_{L_2}$,

(iii) or $\psi \in H_0(\mathrm{curl})$ and $\operatorname{div} \psi = 0$ and $\|\psi\|_{H(\mathrm{curl})} \leq \|q\|_{L_2}$.

Proof. The function q can be extended by zero to the whole $\mathbb{R}^3$. This q belongs to $H(\mathrm{div}, \Omega)$ and to $H(\mathrm{div}, \mathbb{R}^3 \setminus \Omega)$, and it has continuous normal trace. Due to Lemma 24, q belongs to $H(\mathrm{div}, \mathbb{R}^3)$, and the global weak divergence is zero.

The Fourier tansform $\mathcal{F} : L_2(\mathbb{R}^3) \to L_2(\mathbb{R}^3)$ is defined by

$$(\mathcal{F}v)(\xi) := \int_{\mathbb{R}^3} e^{-2\pi i x \cdot \xi} v(x)\, dx,$$

the inverse transformation is given by

$$(\mathcal{F}^{-1}\tilde{v})(\xi) := \int_{\mathbb{R}^3} e^{2\pi i x \cdot \xi} \tilde{v}(\xi)\, d\xi.$$

It is an isomorphism, i.e., $\|v\|_{L_2} = \|\mathcal{F}v\|_{L_2}$. Differentiation is reduced to multiplication, i.e.

$$\begin{aligned}
\mathcal{F}(\nabla w) &= 2\pi i\, \xi\, \mathcal{F}w \\
\mathcal{F}(\operatorname{curl} u) &= 2\pi i\, \xi \times \mathcal{F}u \\
\mathcal{F}(\operatorname{div} q) &= 2\pi i\, \xi \cdot \mathcal{F}q.
\end{aligned}$$

Let $\tilde{q} = \mathcal{F}q$. It satisfies $\|\tilde{q}\|_{L_2} = \|q\|_{L_2}$, and $\operatorname{div} q = 0$ implies $\xi \cdot \tilde{q} = 0$. We define

$$\tilde{\psi} = \frac{\xi \times \tilde{q}}{2\pi i |\xi|^2}$$

One easily verifies the relation

$$-\xi \times (\xi \times \tilde{q}) + \xi(\xi \cdot \tilde{q}) = |\xi|^2 \tilde{q}.$$

Thus

$$2\pi i\xi \times \tilde{\psi} = \frac{\xi \times (\xi \times q)}{|\xi|^2} = \tilde{q},$$

and the inverse Fourier transform $\psi := \mathcal{F}^{-1}\tilde{\psi}$ satisfies

$$\operatorname{curl}\psi = q.$$

Since $\mathcal{F}(\operatorname{div}\psi) = 2\pi i\xi \cdot \tilde{\psi} = 0$, the vector potential satisfies $\operatorname{div}\psi = 0$. The H^1-semi-norm is

$$\|\nabla\psi\|_{L_2(\mathbb{R}^3)} \;=\; \|2\pi i\xi\tilde{\psi}\|_{L_2(\mathbb{R}^3)} = \left\|\frac{\xi\;\xi \times \tilde{q}}{|\xi|^2}\right\| = \|\tilde{q}\|_{L_2(\mathbb{R}^3)} = \|q\|_{L_2(\Omega)},$$

which proves (i). Now, we modify ψ to satisfy zero boundary values. We have $\operatorname{curl}\psi = q$ in $\mathbb{R}^3$, and $q = 0$ in $\mathbb{R}^3 \setminus \Omega$. Thus, there exists a scalar potential $w \in H^1(\mathbb{R}^3 \setminus \Omega)$ such that $\psi = \nabla w$ outside Ω. Furthermore, $|w|_{H^2} = |\psi|_{H^1}$, and thus $\psi \in H^2_{loc}$. On Lipschitz domains, functions from H^k can be continuously extended. Extend w from $\mathbb{R} \setminus \Omega$ to $Ew \in H^2(\mathbb{R}^3)$. Now take

$$\psi_2 = \psi - \nabla Ew.$$

This ψ_2 is in $H^1(\mathbb{R}^3)$ and vanishes outside Ω. Since H^1 implies continuous traces, ψ_2 satisfies zero boundary conditions, as claimed in (ii). But, the div-free constraint is lost. To recover div-free functions, we perform an H^1_0-projection to obtain ψ_3:

$$\psi_3 = \psi_2 - \nabla\phi \qquad \text{with} \qquad \phi \in H^1_0 : (\nabla\phi, \nabla v) = (\psi_2, \nabla v)\; \forall\, v \in H^1_0.$$

This ψ_3 satisfies $\operatorname{div}\psi_3 = 0$. It still satisfies zero tangential boundary conditions, i.e. (iii). But now, ψ_3 is not in $[H^1]^3$ anymore, but still in $H(\operatorname{curl})$. $\square$

Now, we do not assume zero normal trace of the function q.

Lemma 27. *Assume that $q \in H(\operatorname{div})$ such that $\operatorname{div} q = 0$. Then there exist ψ such that*

$$q = \operatorname{curl}\psi.$$

The function ψ can be chosen such that

(i) $\psi \in [H^1]^3$ and $\operatorname{div}\psi = 0$ and $|\psi|_{H^1} \leq \|q\|_{L_2}$

(ii) or $\psi \in H(\operatorname{curl})$ and $\operatorname{div}\psi = 0$, $\operatorname{tr}_n \psi = 0$ and $\|\psi\|_{H(\operatorname{curl})} \leq \|q\|_{L_2}$.

Proof. We cannot directly extend q by zero onto $\mathbb{R} \setminus \Omega$. Now, let $\widetilde{\Omega}$ be a domain containing $\overline{\Omega}$. We construct a $\tilde{q} \in H_0(\operatorname{div}, \widetilde{\Omega})$, which coincides with q on Ω. For this let

$$\begin{aligned} \tilde{q} \cdot n &= q \cdot n \text{ on } \partial\Omega \\ \tilde{q} \cdot n &= 0 \text{ on } \partial\widetilde{\Omega}. \end{aligned}$$

Since $\operatorname{div} q = 0$ on Ω, there holds $\int_{\partial\Omega} q \cdot n \, ds = 0$. The boundary values for $\tilde{q}$ satisfy $\int_{\partial(\widetilde{\Omega}\setminus\Omega)} \tilde{q} \cdot n \, ds = 0$. Thus, according to Lemma 23, there exists a $\tilde{q} \in H(\operatorname{div})$ with $\operatorname{div} \tilde{q} = 0$ satisfying the prescribed boundary values. This $\tilde{q}$ has now zero boundary values at the outer boundary $\partial\widetilde{\Omega}$, and can be extended by zero to the whole $\mathbb{R}^3$. The proof of (i) follows now the previous lemma.

Now, we obtain the $H(\text{curl})$ function ψ_2 by performing the Poisson-projection with Neumann boundary conditions:

$$\psi_2 = \psi - \nabla\phi \qquad \text{with} \qquad \phi \in H^1(\Omega)/\mathbb{R} : \quad (\nabla\phi, \nabla v) = (\psi, \nabla v) \; \forall \, v \in H^1(\Omega)/\mathbb{R}$$

This ψ_2 satisfies $(\psi_2, \nabla v) = 0 \; \forall \, v \in H^1$, i.e., $\operatorname{div} \psi_2 = 0$ and $\psi_n = 0$. $\square$

Theorem 28 (Helmholtz decomposition)**.** *Let $q \in [L_2(\Omega)]^3$. Then there exists a decomposition*

$$q = \nabla\phi + \operatorname{curl} \psi$$

There are the following choices for the functions ϕ and ψ. The corresponding norms are bounded by $\|q\|_{L_2}$:

(i) $\phi \in H^1$ *and* $\psi \in [H^1]^3$ *such that* $\operatorname{div} \psi = 0$,

(ii) *or* $\phi \in H^1$ *and* $\psi \in [H_0^1]^3$,

(iii) *or* $\phi \in H^1$ *and* $\psi \in H_0(\text{curl})$ *and* $\operatorname{div} \psi = 0$,

(iv) *or* $\phi \in H_0^1$ *and* $\psi \in [H^1]^3$ *such that* $\operatorname{div} \psi = 0$,

(v) *or* $\phi \in H_0^1$ *and* $\psi \in H(\text{curl})$ *and* $\operatorname{div} \psi = 0$, $\operatorname{tr}_n \psi = 0$.

Proof. For the cases (i), (ii), (iii), we define $\phi \in H^1/\mathbb{R}$ by solving the Neumann problem

$$(\nabla\phi, \nabla v) = (q, \nabla v) \qquad \forall \, v \in H^1/\mathbb{R}.$$

The rest, $q - \nabla\phi$ is div-free, and satisfies zero normal boundary values. Lemma 26 proves the existence of the vector potential ψ. For the remaining cases, we solve a Dirichlet problem to obtain $\phi \in H_0^1$, and apply Lemma 27 for the construction of the vector potential ψ. $\square$

Theorem 29. *Let $u \in H(\text{curl})$. There exists a decomposition*

$$u = \nabla\phi + z$$

with $\phi \in H^1$ and $z \in [H^1]^3$ such that

$$\|\phi\|_{H^1} \le c\,\|u\|_{H(\text{curl})} \qquad \textit{and} \qquad \|z\|_{H^1} \le c\,\|\operatorname{curl} u\|_{L_2}$$

If $u \in H_0(\text{curl})$, then there exists a decomposition with $\phi \in H_0^1$ and $z \in [H_0^1]^3$.

Proof. Let $u \in H(\text{curl})$. Then $q := \operatorname{curl} u$ satisfies $\operatorname{div} q = 0$. Thus, there exists an $z \in [H^1]^3$ such that

$$\operatorname{curl} z = q = \operatorname{curl} u$$

and

$$\|z\|_{H^1} \preceq \|q\|_{L_2} = \|\operatorname{curl} u\|_{L_2}.$$

The difference $u - z$ is in the kernel of the curl, i.e. a gradient:

$$\nabla\phi = u - z.$$

We choose ϕ such that $\int \phi = 0$. The bound for the norm follows from

$$\|\phi\|_{H^1} \le \|u\|_{L_2} + \|z\|_{L_2} \le \|u\|_{L_2} + \|z\|_{H^1} \preceq \|u\|_{H(\text{curl})}$$

The proof follows the same lines for $u \in H_0(\text{curl})$. □

Theorem 30 (Friedrichs-type inequality). *(i) Assume that $u \in H(\text{curl})$ satisfies*

$$(u, \nabla\psi) = 0 \qquad \forall\, \psi \in H^1(\Omega).$$

Then there holds the Friedrichs'-type inequality

$$\|u\|_{L_2} \le c\,\|\operatorname{curl} u\|_{L_2}$$

(ii) Assume that $u \in H_0(\text{curl})$ satisfies

$$(u, \nabla\psi) = 0 \qquad \forall\, \psi \in H_0^1(\Omega).$$

Then there holds the Friedrichs'-type inequality

$$\|u\|_{L_2} \le c\,\|\operatorname{curl} u\|_{L_2}$$

Proof. To prove (i), let $u \in H(\text{curl})$, and choose $z \in [H^1]^3$ and $\phi \in H^1$ according to Theorem 29. Since

$$z = u - \nabla\phi \quad \text{and} \quad u \perp \nabla\phi,$$

there holds

$$\|z\|_{L_2}^2 = \|u\|_{L_2}^2 + \|\nabla\phi\|_{L_2}^2.$$

Thus we have

$$\|u\|_{L_2} \le \|z\|_{L_2} \le \|z\|_{H^1} \le \|\operatorname{curl} u\|_{L_2}.$$

□

3 Finite Element Methods for Maxwell Equations

We want to solve numerically the variational problem:

Find $u \in V := \{v \in H(\mathrm{curl}) : \mathrm{tr}_\tau v = 0 \text{ on } \Gamma_D\}$ such that

$$\int_\Omega \mu^{-1} \operatorname{curl} u \operatorname{curl} v \, dx + \int_\Omega \kappa u v \, dx = \int j v \, dx \qquad \forall v \in V.$$

For this purpose, we choose an N-dimensional subspace $V_N \subset V$, and define the Galerkin projection:

Find $u_N \in V_N$ such that

$$\int_\Omega \mu^{-1} \operatorname{curl} u_N \operatorname{curl} v_N \, dx + \int_\Omega \kappa u_N v_N \, dx = \int j v_N \, dx \qquad \forall v_N \in V_N.$$

We are interested in the behavior of the discretization error $\|u - u_N\|_{H(\mathrm{curl})}$ as $N \to \infty$. We choose finite element spaces as sub-spaces V_N.

3.1 Lowest order elements

We start with triangular elements in 2D and tetrahedral elements in 3D. Let the domain Ω be covered with a regular triangulation. This means, the intersection of two elements is empty, one edge, one face or the whole element. The diameter of the element T is denoted as h_T, it may vary over the domain. Otherwise, if $h_T \simeq h$, we call the triangulation quasi-uniform. Let ρ_T be the radius of the largest sphere contained in T. We assume shape regularity, i.e., h_T / ρ_T is bounded by a constant.

We call

$$\begin{aligned}
\text{the set of vertices} \qquad & \mathcal{V} = \{V_i\}, \\
\text{the set of edges} \qquad & \mathcal{E} = \{E_{ij}\}, \\
\text{the set of faces} \qquad & \mathcal{F} = \{F_{ijk}\}, \\
\text{the set of tetrahedra} \qquad & \mathcal{T} = \{T_{ijkl}\}.
\end{aligned}$$

In 2D, there is no set of faces, and $\mathcal{T}$ is the set of triangles. We define $N_\mathcal{V}$, $N_\mathcal{E}$, $N_\mathcal{F}$, and $N_\mathcal{T}$ as the number of vertices, edges, faces, and elements.

According to Ciarlet, a finite element consists of

- the geometric domain T
- a local element space V_T of dimension N_T
- a set of linearly independent functionals $\{\psi_{T,1}, \ldots, \psi_{T,N_T}\}$ on V_T. They are called degrees of freedom.

By identifying the local functionals with global functionals, one can control the continuity of the global space. The nodal basis $\{\varphi_\alpha\}$ is a basis for V_T biorthogonal to the functionals, i.e.,

$$\psi_\beta(\varphi_\alpha) = \delta_{\alpha,\beta} \qquad \alpha, \beta = 1, \dots N_T$$

Example: The continuous piecewise linear finite element space on triangles. The sets are triangles, the 3-dimensional element spaces are $V_T = P^1(T)$, the set of affine linear polynomials. The local dofs are the functionals $\psi_\alpha : v \mapsto v(V_\alpha)$, the vertex values. The nodal basis is

$$\varphi_\alpha = \lambda_\alpha,$$

the barycentric coordinates of the triangle. Two local functionals $\psi_{T,\alpha}$ and $\psi_{\widetilde{T},\beta}$ are identified, if they are associated with the same global vertex. We write $\psi_{T,\alpha} \equiv \psi_{\widetilde{T},\beta}$. The global finite element space is

$$V_h := \{v \in L_2 : v|_T \in V_T \text{ and } \psi_{T,\alpha} \equiv \psi_{\widetilde{T},\beta} \Rightarrow \psi_{T,\alpha}(v|_T) = \psi_{\widetilde{T},\beta}(v|_{\widetilde{T}})\}.$$

The global finite element functions are continuous at the vertices, and are linear along the edges. Thus, they are continuous functions. Polynomials on T belong to $H^1(T)$. Thus, according to Lemma 21, the finite element space is a sub-space of H^1. If the functionals would not be identified in the vertices, we would obtain a sub-space of L_2, only.

Now, we define the lowest order Nédélec elements to discretize $H(\text{curl})$.

Definition 31. *The triangular Nédélec finite element is*

- *a triangle T*
- *the local space*

$$\mathcal{N}_0 := \left\{ v = \begin{pmatrix} a_x \\ a_y \end{pmatrix} + b \begin{pmatrix} y \\ -x \end{pmatrix} \right\}.$$

- *the functionals*

$$\psi_{E_{\alpha\beta}} : v \mapsto \int_{E_{\alpha\beta}} v \cdot \tau \, ds$$

  associated with the three edges $E_{\alpha\beta}$ of the triangle.

It is called also the edge element.

We observe the following properties:

- There holds

$$[P^0]^2 \subset \mathcal{N}_0 \subset [P^1]^2,$$

-

$$\operatorname{curl} \mathcal{N}_0 = P^0$$

  with the vector-to-scalar curl operator $\operatorname{curl} v = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial v_y}$.

- The tangential component is constant on a line $\vec{p} + t\vec{w}$:

$$w \cdot \left[\begin{pmatrix} a_x \\ a_y \end{pmatrix} + b \begin{pmatrix} p_y + tw_y \\ -(p_x + tw_x) \end{pmatrix} \right] = w_x(a_x + bp_y) + w_y(a_y - bp_x)$$

Lemma 32. *The nodal basis function associated with the edge $E_{\alpha\beta}$ from vertex V_α to vertex V_β is*

$$\varphi_{\alpha\beta} := \lambda_\alpha \nabla \lambda_\beta - \lambda_\beta \nabla \lambda_\alpha$$

Proof. First, we check that $\varphi_{\alpha\beta}$ belongs to the space $\mathcal{N}_0$. The space $\mathcal{N}_0$ consists of all affine-linear functions $a + Bx$, where B is a skew-symmetric matrix. The basis function is affine-linear. Its gradient is

$$\nabla \varphi_{\alpha\beta} = \nabla \lambda_\beta (\nabla \lambda_\alpha)^T - \nabla \lambda_\alpha (\nabla \lambda_\beta)^T,$$

what is skew-symmetric. Next, we observe that $\varphi_{\alpha\beta} \cdot \tau = 0$ for edges other than $E_{\alpha\beta}$. Take the edge opposite to V_α: There is $\lambda_\alpha = 0$ and thus also $\tau \nabla \lambda_\alpha = 0$, and analogous for the edge opposite to V_β. Finally, consider the edge $E_{\alpha\beta}$. There holds $\lambda_\beta = 1 - \lambda_\alpha$, and $\tau \nabla \lambda_\beta = -\tau \cdot \nabla \lambda_\alpha$. Hence,

$$\tau \cdot \varphi_{\alpha\beta} = \lambda_\alpha (-\tau \nabla \lambda_\alpha) - (1 - \lambda_\alpha) \tau \nabla \lambda_\alpha = -\tau \cdot \nabla \lambda_\alpha,$$

and

$$\int_{E_{\alpha\beta}} \tau \cdot \varphi_{\alpha\beta} \, ds = -\int_{E_{\alpha\beta}} \tau \cdot \nabla \lambda_\alpha \, ds = \lambda_\alpha(V_\alpha) - \lambda_\beta(V_\beta) = 1.$$

$\square$

Lemma 33. *Let the local dofs associated with the same edge be identified. Then the global finite element space is a sub-space of $H(\text{curl})$*

Proof. On the element there holds $\mathcal{N}_0 \subset H(\text{curl}, T)$. We have to check continuity of the tangential trace: The tangential component on each edge is a constant function Thus, prescribing the same line integral ensures continuity of the tangential component. Now, according to Lemma 25, the global finite element space is a sub-space of $H(\text{curl}, \Omega)$.

$\square$

On simply connected domains in 2D, the spaces H^1, $H(\text{curl})$, and L_2 form a complete sequence:

$$H^1/\mathbb{R} \xrightarrow{\nabla} H(\text{curl}) \xrightarrow{\text{curl}} L^2$$

The operator ∇ has no kernel on $H^1/\mathbb{R}$. Its range is exactly the kernel of the curl, and the range of the curl is the whole L_2. By choosing the canonical finite element spaces, this property is inherited on the discrete level:

Theorem 34. *By choosing the finite element spaces*

$$\begin{aligned} W_h &= \{w \in H^1/\mathbb{R} : w|_T \in P^1\} \\ V_h &= \{v \in H(\text{curl}) : v|_T \in \mathcal{N}_0\} \\ S_h &= \{s \in L_2 : s|_T \in P^0\}, \end{aligned}$$

the discrete sequence is complete:

$$W_h \xrightarrow{\nabla} V_h \xrightarrow{\text{curl}} S_h$$

Proof. First, check that $\nabla W_h \subset V_h$. Since $\nabla W_h \subset \nabla H^1 \subset H(\text{curl})$, and ∇w_h is piecewise constant, $\nabla w_h \in V_h$. We have already observed that $\text{curl}\, V_h \subset S_h$.
Now, take a $v_h \in V_h$ such that $\text{curl}\, v_h = 0$. Thus, v_h is piecewise constant. There exists a $w \in H^1$ such that $\nabla w = v_h$. Since the gradient is piecewise constant, the function is piecewise linear, i.e., it belongs to W_h. Finally, we check that $\text{curl}\, V_h = S_h$ by counting dimensions:

$$\dim\{\text{curl}\, V_h\} = \dim\{V_h\} - \dim\{W_h\} = N_E - (N_V - 1).$$

On a simple connected domain there holds (proven by induction: remove vertex by vertex)

$$N_E = N_V + N_T - 1.$$

Thus, the dimension of $\text{curl}\, V_h$ is N_T, the dimension of S_h. $\square$

3.1.1 Transformation from the reference element

In both, analysis as well as implementation, it is useful to introduce one reference finite element and describe all elements in the mesh as transformations of the reference element. For this, let $T^R = [(0,0), (1,0), (0,1)]$ be the reference triangle, and define the affine linear mapping Φ_T such that

$$T = \Phi_T(T^R).$$

Define $F_T = \Phi_T'$. The element basis functions can be defined (implemented) for the reference element, and are mapped to the general element by the following transformation:

Lemma 35. *Let $E^R_{\alpha\beta}$ be an edge of the reference element, and $E_{\alpha\beta}$ the according edge of the general element. Then, the according edge basis functions $\varphi^R_{\alpha\beta}$ and $\varphi_{\alpha\beta}$ satisfy*

$$\begin{aligned} \varphi_{\alpha\beta} &= F^{-T} \varphi^R_{\alpha\beta}, \\ \text{curl}\, \varphi_{\alpha\beta} &= (\det F)^{-1}\, \text{curl}\, \varphi^R_{\alpha\beta}. \end{aligned}$$

This transformation is called covariant.

Proof. The barycentric coordinates (which are the vertex basis functions) satisfy

$$\lambda_\alpha(\Phi_T(x^R)) = \lambda_\alpha^R(x^R) \qquad \forall\, x^R \in T^R.$$

Take derivatives on both sides to obtain

$$F_T^T(\nabla\lambda_\alpha)(\Phi_T(x^R)) = \nabla\lambda_\alpha^R(x^R).$$

Now, the edge-shape function in $x \in T$ is

$$\begin{aligned} \varphi_{\alpha\beta}(x) &= \lambda_\alpha(x)\nabla\lambda_\beta(x) - \lambda_\beta(x)\nabla\lambda_\alpha(x) \\ &= \lambda_\alpha^R(x^R)F^{-T}\nabla\lambda_\beta^R(x^R) - \lambda_\beta^R(x^R)F^{-T}\nabla\lambda_\alpha^R(x^R) \\ &= F^{-T}\varphi_{\alpha\beta}^R(x^R) \end{aligned}$$

The proof of the transformation of the curls is based on the relation

$$\operatorname{curl}[F^T u(\Phi(x))] = (\det F)(\operatorname{curl} u)(\Phi(x))$$

for general smooth transformations Φ; see Lemma 14 for the 3D case, and exercises for 2D. Now, set $u = \varphi_{\alpha\beta}$ to obtain

$$(\operatorname{curl}\varphi_{\alpha\beta})(\Phi(x)) = (\det F)^{-1}\operatorname{curl}[F^T\varphi_{\alpha\beta}(\Phi(x))] = (\det F)^{-1}\operatorname{curl}\varphi_{\alpha\beta}^R(x).$$

□

3.1.2 Implementation aspects

One has to compute the global matrices

$$A_{ij} = \int_\Omega \mu^{-1}\operatorname{curl}\varphi_i\ \operatorname{curl}\varphi_j\,dx$$

and

$$M_{ij} = \int_\Omega \kappa\varphi_i\varphi_j\,dx,$$

where the indices i and j are associated with the edges of the mesh. The integrals are split over the elements. Thus, the global matrices are the sums of the local element matrices A^T and M^T

$$A = \sum_T C_T A^T C_T^T \qquad \text{and} \qquad M = \sum_T C_T M^T C_T^T.$$

The C_T are the connectivity matrices (of dimension $N_E \times 3$) connecting the numbering of the local basis function to the global basis functions. Here, also the orientation of the edges must be taken into account: If the local edge is opposite to the global one, the entry in C is -1.

For the computation of the local element matrices, one has to form integrals

$$M_{kl}^T = \int_T \kappa \varphi_k \cdot \varphi_l \, dx,$$

which are transformed to the reference element by

$$M_{kl}^T = \int_{T^R} \kappa \, (F^{-T} \varphi_k^R) \cdot (F^{-T} \varphi_l^R) \, \det F \, dx,$$

and similar for the curl − curl matrix:

$$A_{kl}^T = \int_{T^R} \mu^{-1} \, (\det F)^{-1} \operatorname{curl} \varphi_k^R (\det F)^{-T} \operatorname{curl} \varphi_l^R \, \det F \, dx,$$

The shape functions on the reference element are coded once and for all. The implementation is as simple as implementing scalar finite elements.

3.1.3 Interpolation operators and error estimates

The definition of functionals and biorthogonal nodal basis functions lead immediately to the interpolation operators

$$I_h u = \sum_{i=1}^{N} \psi_i(u) \varphi_i.$$

They are projectors, since

$$\begin{aligned} I_h I_h u &= \sum_{j=1}^{N} \psi_j \Big(\sum_{i=1}^{N} \psi_i(u) \varphi_i \Big) \varphi_j \\ &= \sum_{j=1}^{N} \sum_{i=1}^{N} \psi_i(u) \psi_j(\varphi_i) \varphi_j \\ &= \sum_{i=1}^{N} \psi_i(u) \varphi_i. \end{aligned}$$

We used the biorthogonality $\psi_j(\varphi_i) = \delta_{ij}$.

Interpolating a function on the element T should result in the same function as interpolation on the reference element T^R. This is trivial for the nodal elements:

$$I_h^R [u \circ \Phi] = (I_h u) \circ \Phi.$$

Remember that we are working with triangles. This is not true for curved elements, which leads to more technical difficulties.

Lemma 36. *When using the covariant transformation, the interpolation by a general Nédélec element is equivalent to interpolation by the reference element:*

$$I_h^R [F^T u \circ \Phi] = F^T (I_h u) \circ \Phi$$

Proof. The left hand side evaluates to

$$\sum_{E_{\alpha\beta} \subset T^R} \psi^R_{\alpha\beta}(F^T u \circ \Phi) \varphi^R_{\alpha\beta}.$$

The right hand side is

$$F^T \sum_{E_{\alpha\beta} \subset T} \psi_{\alpha\beta}(u) \, \varphi_{\alpha\beta} \circ \Phi = \sum_{E_{\alpha\beta}} \psi_{\alpha\beta}(u) \varphi^R_{\alpha\beta},$$

where we used the covariant transformation of the basis functions. It remains to show that the functionals give the same values, i.e.,

$$\psi^R_{\alpha\beta}(F^T u \circ \Phi) = \psi_{\alpha\beta}(u),$$

or

$$\int_{E^R} F^T u(\Phi(x)) \cdot \tau \, ds_x = \int_{\Phi(E^R)} u \cdot \tau \, ds.$$

This relation holds true for general curves. Assume that E^R is parametrized with $\gamma : [0, l] \to \mathbb{R}^2$. Then, the left hand side reads as

$$\int_0^l [F^T u(\Phi(\gamma(s)))] \cdot \gamma'(s) ds,$$

the right hand side is

$$\int_0^l u(\Phi(\gamma(s))) \cdot [\Phi(\gamma(s)]' ds = \int_0^l u(\Phi(\gamma(s))) \cdot F\gamma'(s) \, ds$$

$\square$

The analysis of the finite element error is based on the interpolation error. The transformation to the reference element allows to use the scaling technique, and the Bramble-Hilbert lemma.

Theorem 37. *The Nédélec interpolation operator satisfies the error estimate*

$$\|u - I_h u\|_{L_2(T)} \leq ch \, |u|_{H^1(T)}.$$

Proof. We transform to the reference element and define

$$u^R(x^R) = F^T u(\Phi_T(x^R))$$

The scaling gives $\|u^R\|_{L_2(T^R)} \simeq \|u\|_{L_2(T)}$, and $|u^R|_{H^1(T^R)} \simeq h \, |u|_{H^1(T)}$. Note that the factor $\det F$ from the transformation of integrals cancels out with two factors $|F| \simeq h$. Thus, the estimate is equivalent to prove

$$\|u^R - I^R_h u^R\|_{L_2(T^R)} \leq c \, |u^R|_{H^1(T^R)}$$

This follows from the Bramble-Hilbert lemma which needs that

- the operator $(id - I_h^R)$ vanishes for constant functions
- the operator $(id - I_h^R) : H^1 \to L_2$ is continuous

The first holds since the Nédélec space contains the constants, the second one follows from the trace inequality. Thus, the operator is continuous with respect to the H^1-semi-norm. $\square$

3.1.4 The commuting diagram

Let I_h^V be the vertex interpolation operator for H^1 (vertex) elements. Unfortunately, it is not defined on the whole H^1 in two or three dimensions, but only on smoother (e.g., continuous) subspaces. Let I_h^E be the edge interpolation operators to $H(\text{curl})$ (edge) elements. Also this one is not defined on the whole $H(\text{curl})$. Finally, let I_h^T be the element interpolation operator into piecewise constant elements. Here, the functionals $\psi(s) = \int_T s dx$ are well defined for L_2.

The interpolation operators can be drawn in the commuting diagram, called also the de Rham complex:

$$\begin{array}{ccccc} H^1 \cap C^1 & \stackrel{\nabla}{\longrightarrow} & H(\text{curl}) \cap C^0 & \stackrel{\text{curl}}{\longrightarrow} & L^2 \\ \big\downarrow I_h^V & & \big\downarrow I_h^E & & \big\downarrow I_h^T \\ W_h & \stackrel{\nabla}{\longrightarrow} & V_h & \stackrel{\text{curl}}{\longrightarrow} & S_h^0 \end{array} \tag{28}$$

It says that first interpolating, and then applying the differential operator results in the same function as going the other way.

Theorem 38. *There holds*

$$\nabla I_h^V w = I_h^E \nabla w \tag{29}$$

for all continuous H^1 functions w.

Proof. Both operations end up in the space V_h. Thus, it is enough to compare all edge-functionals $\psi_{E_{ij}}$. We start with the left hand side of (29), and integrate the tangential derivatives along the edges

$$\psi_{E_{ij}}(\nabla I_h w) = \int_{E_ij} \tau \cdot \nabla I_h^V w \, ds \quad = \quad (I_h^V w)(V_j) - (I_h^V w)(V_i) = w(V_j) - w(V_i).$$

The functionals applied to the right hand side of (29) lead to

$$\psi_{E_{ij}}(I_h^E \nabla w) = \psi_{E_{ij}}\Big(\sum_{E_{kl}} \psi_{E_{kl}}(\nabla w) \varphi_{E_{kl}}\Big) = \psi_{E_{ij}}(\nabla w) = w(V_j) - w(V_i)$$

$\square$

Theorem 39. *There holds*

$$\operatorname{curl} I_h^E u = I_h^T \operatorname{curl} u$$

for all continuous $H(\operatorname{curl})$ functions u.

The proof is left as an exercise.

Corollary 40. *There holds the error estimate*

$$\|\operatorname{curl}(u - I_h^E u)\|_{L_2(T)} \leq ch\,|\operatorname{curl} u|_{H^1(T)}. \tag{30}$$

Proof. By scaling and a Bramble-Hilbert argument one proves that

$$\|s - I_h^T s\|_{L_2(T)} \leq ch|s|_{H^1(T)}.$$

Now, apply commutativity to bound

$$\|\operatorname{curl}(u - I_h^E u)\|_{L_2(T)} = \|(id - I_h^T)\operatorname{curl} u\|_{L_2(T)} \leq ch\,|\operatorname{curl} u|_{H^1(T)}.$$

□

3.2 Higher order triangular elements

The lowest order Nédélec element introduced above is between order 0 and order 1. There are Nédélec elements of the second type which are complete polynomials.

Definition 41. *The lowest order Nédélec-II element $\mathcal{N}_1^{II}$ is given by*

- *a triangle T*
- *the local space*
$$\mathcal{N}_1^{II} := [P_1]^2$$
- *the functionals*
$$\psi_{E_{\alpha\beta},k} : v \mapsto \int_{E_{\alpha\beta}} q_k v \cdot \tau \, ds \qquad k = 0, 1$$
associated with the three edges $E_{\alpha\beta}$ of the triangle. The q_k are a hierarchical polynomial basis on the edge.

The dimension of the element space is $2 \times 3 = 6$, and there are $3 \times 2 = 6$ functionals. A possible choice for the q_k is

$$q_0(s) = 1 \qquad q_1(s) = -\frac{3}{4}s,$$

assuming that the edge is parameterized with $s \in (-1, 1)$.

Computing the nodal basis leads to the two functions associated with each edge

$$\begin{aligned}\varphi_{E_{\alpha\beta},0} &= \lambda_\alpha \nabla \lambda_\beta - \lambda_\beta \nabla \lambda_\alpha, \\ \varphi_{E_{\alpha\beta},1} &= \nabla(\lambda_\alpha \lambda_\beta) = \lambda_\alpha \nabla \lambda_\beta + \lambda_\beta \nabla \lambda_\alpha.\end{aligned}$$

The tangential components along the edges are linear. Thus, enforcing two conditions for tangential continuity are enough to obtain $H(\text{curl})$-continuity.

The element is still linear. Thus, the space for the curl did not increase:

$$\text{curl}\, \mathcal{N}_1^{II} = P^0$$

As we can see from the basis functions $\varphi_{E_{\alpha\beta},1}$, the element is enriched by gradients of second order H^1 basis functions $\lambda_\alpha \lambda_\beta$. There holds the complete sequence

$$\mathcal{L}_2/\mathbb{R} \xrightarrow{\nabla} \mathcal{N}_1^{II} \xrightarrow{\text{curl}} \mathcal{S}_0,$$

where $\mathcal{L}_k$ is the space of k^{th}-order continuous elements, and $\mathcal{S}_k$ is the space of k^{th}-order non-continuous elements.

There are different possibilities to define the functionals for $\mathcal{L}_2$. Different functionals lead to different nodal interpolation operators. All functionals have to include the vertex functionals

$$\psi_\alpha(v) = v(V_\alpha).$$

Then, one has to choose one functional for each edge, for example, the function value in the edge mid-point. An alternative is

$$\psi_{E_{\alpha\beta}}(v) = \int_{E_{\alpha\beta}} q_1 \frac{\partial v}{\partial \tau}\, ds$$

These functionals lead to interpolation operators commuting with the $\mathcal{N}_1^{II}$-interpolation operators (exercise).

Higher order $H(\text{curl})$ elements also have degrees of freedom involving domain integrals:

Definition 42. *The k^{th}-order Nédélec-II element $\mathcal{N}_k^{II}$ is given by*

- *a triangle T*
- *the local space*

$$\mathcal{N}_k^{II} := [P^k]^2$$

- *the functionals*

$$\psi_{E_{\alpha\beta},l} : v \mapsto \int_{E_{\alpha\beta}} q_l\, v \cdot \tau\, ds \qquad l = 0, \ldots, k$$

associated with the three edges $E_{\alpha\beta}$ of the triangle, and the functionals

$$\begin{aligned}\psi_{T,l}^c &: v \mapsto \int_T s_l \,\text{curl}\, v\, dx \qquad \text{with } s_l \text{ a basis for } P^{k-1}/\mathbb{R} \\ \psi_{T,l}^g &: v \mapsto \int_T \nabla p_l \cdot v\, dx \qquad \text{with } p_l \text{ a basis for } \lambda_1\lambda_2\lambda_3 P^{k-2}\end{aligned}$$

associated with the triangle T.

The element functionals act either on the curl, thus ψ^c, or are related to gradients, thus called ψ^g.

Lemma 43. *The degrees of freedom are linearly independent.*

Proof. The dimension of the space is

$$\dim \mathcal{N}_k^{II} = 2\frac{(k+1)(k+2)}{2} = k^2 + 3k + 2$$

The number of edge functionals is

$$3(k+1),$$

the number of curl-element functionals is

$$\dim[P^{k-1}/\mathbb{R}] = \frac{k(k+1)}{2} - 1 = \frac{k^2+k-2}{2},$$

the number of gradient-element functionals is

$$\dim P^{k-2} = \frac{(k-1)k}{2}.$$

The number of degrees of freedom is equal to the space dimension. We check that $\psi_i(v) = 0$ implies $v = 0$. The tangential trace is a polynomial of order k. Thus, the edge functionals imply $v_t = 0$. Since $\int_T \operatorname{curl} v\, dx = \int_{\partial T} v_\tau\, ds = 0$, the $\operatorname{curl} v \perp P^0$. Together with the curl functionals ψ^c, this implies $\operatorname{curl} v = 0$. Thus, v is a gradient, say $v = \nabla\phi$. Since $v \in [P^k]^2$, and $v_\tau = 0$, there is $\phi \in P^{k+1}$ and ϕ is constant on the boundary. W.l.o.g, we may set $\phi = 0$ on the boundary. Since $\phi \in \lambda_1\lambda_2\lambda_3 P^{k-2}$, and $v = \nabla\phi$ is orthogonal to $\nabla\lambda_1\lambda_2\lambda_3 P^{k-2}$, there holds $v = 0$. $\square$

The basis functions satisfy

- assume that $q_0 = 1$ and $\int_E q_l\, dx = 0$ for $l \geq 1$. Then $\varphi_{E_{\alpha\beta},0}$ is the lowest order edge basis function.
- Assume that $q_0 = 1$. Then the high order edge basis functions $\varphi_{E_{\alpha\beta},l}$ with $l \geq 1$ are gradient functions.
- The basis functions according to $\psi^g_{T,l}$ are gradient functions.

The elements satisfy the complete sequence

$$\mathcal{L}_{k+1}/\mathbb{R} \xrightarrow{\nabla} \mathcal{N}_k^{II} \xrightarrow{\operatorname{curl}} \mathcal{S}_{k-1}.$$

The first family of Nédélec elements is obtained by increasing the order of the curl by one. The complete sequence is

$$\mathcal{L}_{k+1}/\mathbb{R} \xrightarrow{\nabla} \mathcal{N}_k^{I} \xrightarrow{\operatorname{curl}} \mathcal{S}_k.$$

The element space is

$$\mathcal{N}_k^I = \left\{ a + b \begin{pmatrix} y \\ -x \end{pmatrix} : a \in [P^k]^2, b \in P^k \right\}.$$

The functionals are the same as for the second family, but the order for the curl-functionals is increased by one. As for the lowest order element, the space consists of incomplete polynomials

$$[P_k]^2 \subset \mathcal{N}_k^I \subset [P_{k+1}]^2$$

3.3 Tetrahedral elements

A difference between 2D and 3D is the length of the complete sequence. In 3D, it contains also the space $H(\mathrm{div})$:

$$H^1/\mathbb{R} \xrightarrow{\nabla} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2$$

Similar to 2D, we define the edge element as follows:

Definition 44. *The lowest order tetrahedral Nédélec element is given by*

- *a tetrahedron T*
- *the local space*
$$\mathcal{N}_0 := \left\{ v = a + b \times x : a, b \in \mathbb{R}^3 \right\}$$
- *the functionals*
$$\psi_{E_{\alpha\beta}} : v \mapsto \int_{E_{\alpha\beta}} v \cdot \tau \, ds$$
associated with the 6 edges $E_{\alpha\beta}$ of the tetrahedron.

As in 2D, the nodal basis function associated with the edge $E_{\alpha\beta}$ is

$$\varphi_{\alpha\beta} := \lambda_\alpha \nabla \lambda_\beta - \lambda_\beta \nabla \lambda_\alpha.$$

Its tangential trace onto a face is exactly the 2D Nédélec triangle. The curl of the element is piecewise constant. Furthermore, the normal component of the curl is continuous across faces. The curl is contained in the following finite element sub-space of $H(\mathrm{div})$:

Definition 45. *The lowest order tetrahedral Raviart-Thomas element is given by*

- *a tetrahedron T*
- *the local space*
$$\mathcal{R}T_0 := \left\{ v = a + bx : a \in \mathbb{R}^3, b \in \mathbb{R} \right\}$$

- *the functionals*

$$\psi_{F_{\alpha\beta\gamma}} : v \mapsto \int_{F_{\alpha\beta\gamma}} v \cdot n \, ds$$

  associated with the 4 faces $F_{\alpha\beta\gamma}$ of the tetrahedron.

Computing the nodal basis function for the face $F_{\alpha\beta\gamma}$ leads to

$$\varphi_{\alpha\beta\gamma} = \lambda_\alpha \nabla\lambda_\beta \times \nabla\lambda_\gamma + \lambda_\beta \nabla\lambda_\gamma \times \nabla\lambda_\alpha + \lambda_\gamma \nabla\lambda_\alpha \times \nabla\lambda_\beta.$$

The Raviart-Thomas element satisfies

- The element space is an incomplete polynomial space between $[P^0]^3$ and $[P^1]^3$.
- The divergence satisfies
$$\operatorname{div} \mathcal{R}T_0 = P^0.$$
- The normal components on the faces are constant.
- The functionals ensure continuity of the normal components across interfaces.

The global finite element spaces satisfy the complete sequence

$$\mathcal{L}_1/\mathbb{R} \xrightarrow{\nabla} \mathcal{N}_0 \xrightarrow{\text{curl}} \mathcal{R}T_0 \xrightarrow{\text{div}} \mathcal{S}_0.$$

In particular, the range of the curl applied to $\mathcal{N}_0$ is exactly the divergence-free sub-space of $\mathcal{R}T_0$.

3.4 Hierarchical high order elements

For the implementation of high order elements, one may take a short-cut and may define immediately the basis functions without considering the functionals. This is possible as long as there is no need for interpolating functions such as initial conditions or boundary conditions.

We start with the Legendre polynomials $P_i : [-1, 1] \to P^i$. Legendre polynomials are defined to be L_2-orthogonal, and normalized such that $P(1) = 1$. They can be computed by the 3-term recurrency

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_i(x) &= \frac{2i-1}{i} x P_{i-1}(x) - \frac{i-1}{i} P_{i-2}. \end{aligned}$$

We also define the so called integrated Legendre polynomials

$$L_i(x) := \int_{-1}^{x} P_{i-1}(s) \, ds \qquad i \geq 2$$

They can be computed with a 3-term recurrency as well (by choosing initial values for L_0 and L_1)

$$\begin{aligned} L_0(x) &= -1, \\ L_1(x) &= x, \\ L_i(x) &= \frac{2i-3}{i} x L_{i-1}(x) - \frac{i-3}{i} L_{i-2}. \end{aligned}$$

The integrated Legendre polynomials satisfy

$$L_i(-1) = L_i(1) = 0.$$

3.4.1 Hierarchical basis functions for H^1 elements

We start to define the 1D reference element of order p. The domain is $T = (-1,1)$. Basis functions are the two vertex basis functions, and the so called bubble functions vanishing at the boundary:

$$\begin{aligned} \varphi_{V_1}(x) &= \frac{x+1}{2} \\ \varphi_{V_2}(x) &= \frac{1-x}{2} \\ \varphi_{T,k}(x) &= L_k(x) \qquad k = 2, \ldots, p. \end{aligned}$$

On the quadrilateral $T = (-1,1)^2$, basis functions are defined by tensor products. There are 4 vertex basis functions. E.g., the basis function for the vertex $(1,1)$ is

$$\varphi_{V_1}(x) = \frac{x+1}{2}\frac{y+1}{2}.$$

Basis functions associated with an edge must span $P_0^p(E)$, and must vanish on all other edges. E.g., for the edge $E_1 = (-1,1) \times \{-1\}$, the basis functions are

$$\varphi_{E_1,k}(x) = L_k(x)\frac{1-y}{2} \qquad k = 2, \ldots, p.$$

Finally, there are $(p-1)^2$ basis functions vanishing on the whole boundary of the element:

$$\varphi_{T,kl}(x) = L_k(x)L_l(y) \qquad k, l = 2, \ldots, p.$$

To define high order basis functions for triangular elements, we define the scaled Legendre and scaled integrated Legendre polynomials as

$$P_i^S(x,t) = P_i(\tfrac{x}{t})t^i \qquad \text{and} \qquad L_i^S(x,t) = L_i(\tfrac{x}{t})t^i.$$

These are polynomials in x and t, and can be directly evaluated by recursion.

The basis functions for the triangle are the vertex basis functions of the linear triangular element, i.e., the barycentric coordinates

$$\varphi_{V_\alpha} = \lambda_\alpha.$$

Next, there are $p-1$ edge-based basis functions defined as

$$\varphi_{E_{\alpha\beta},k} = L_k^S(\lambda_\alpha - \lambda_\beta, \lambda_\alpha + \lambda_\beta) \qquad k = 2, \ldots, p$$

for each edge. On the edge $E_{\alpha\beta}$, there holds $\lambda_\alpha + \lambda_\beta = 1$, and thus the basis function is equal to L_k on the edge. On the other two edges, the basis function vanishes since $\frac{\lambda_\alpha - \lambda_\beta}{\lambda_\alpha + \lambda_\beta}$ is either -1 or $+1$.

Finally, there are internal basis functions defined as

$$\varphi_{T,kl} = \underbrace{L_k^S(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2)}_{u_k} \underbrace{P_l(2\lambda_3 - 1)\lambda_3}_{v_l} \qquad k \geq 2, l \geq 0, k + l \leq p - 1.$$

The factor u_k vanishes on the edges with $\lambda_1 = 0$ and $\lambda_2 = 0$, the factor v_l vanishes for the edge $\lambda_3 = 0$.

3.4.2 Hierarchical basis functions for triangular $H(\mathrm{curl})$ elements

A basis for high order triangular Nédélec elements can be defined as follows:

- Low order edge basis functions

$$\varphi_{E_{\alpha\beta},0} = \lambda_\alpha \nabla \lambda_\beta - \lambda_\beta \nabla \lambda_\alpha$$

- High order edge basis functions:

$$\varphi_{E_{\alpha\beta},k} = \nabla L_{k+1}^S(\lambda_\alpha - \lambda_\beta, \lambda_\alpha + \lambda_\beta), \qquad k = 1, \ldots, p$$

- Internal basis functions of gradient type:

$$\varphi_{T,kl}^g = \nabla(u_k v_l) = \nabla u_k v_l + u_k \nabla v_l \qquad k \geq 2, l \geq 0, k + l \leq p.$$

  Internal basis functions of curl type:

$$\begin{aligned} \varphi_{T,kl}^{c,1} &= (\nabla u_k) v_l - u_k \nabla v_l \qquad k \geq 2, l \geq 0, k + l \leq p, \\ \varphi_{T,l}^{c,2} &= \varphi_{E_{12},0} v_l \qquad 0 \leq l \leq p - 2 \end{aligned}$$

  with $u_k = L_k^S(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2)$ and $v_l = P_l(2\lambda_3 - 1)\lambda_3$.

3.5 Finite Element Convergence Theory

We consider the variational problem: find $u \in V := H_0(\text{curl})$ such that

$$a(u, v) = (j, v) \qquad \forall\, v \in H_0(\text{curl}) \tag{31}$$

with the bilinear-form

$$a(u, v) = (\operatorname{curl} u, \operatorname{curl} v)_{L_2} + \kappa(u, v)_{L_2}$$

We assume that

- $\kappa \in \mathbb{C}$
- The source j satisfies $\operatorname{div} j = 0$.

Let $u_h \in V_h$ denote the corresponding finite element solution in a Nédélec finite element sub-space.

3.5.1 Regularity Theory for Maxwell equations

The regularity theory for Maxwell equations follows from regularity results for the Poisson equation.

Definition 46 (s-regularity for the Poisson equation)**.** *The Poisson equation*

$$\begin{array}{rcll} -\Delta\Phi & = & f & \text{in } \Omega, \\ \Phi & = & 0 & \text{on } \partial\Omega \end{array}$$

is called s-regular, if $f \in L_2$ implies $\Phi \in H^{1+s}$ with the regularity estimate

$$\|\Phi\|_{H^{1+s}} \preceq \|f\|_{L_2}. \tag{32}$$

If Ω is either convex or smooth, then the Poisson problem is regular with $s = 1$. On Lipschitz domains, regularity holds with some $s \in (0, 1)$.

Lemma 47. *Assume that the the Poisson problem is s-regular. Let either*

$$u \in H_0(\text{curl}) \cap H(\text{div})$$

or

$$u \in H(\text{curl}) \cap H_0(\text{div})$$

Then there holds

$$\|u\|_{H^s} \preceq \|\operatorname{curl} u\|_{L_2} + \|\operatorname{div} u\|_{L_2} \tag{33}$$

Proof. We prove the case $u_\tau = 0$, the other one is similar. According to Theorem 29, there exists a decomposition

$$u = \nabla\Phi + z$$

with $\Phi \in H^1_0$ and $z \in [H^1_0]^3$ such that $\|z\|_{H^1} \preceq \|\operatorname{curl} u\|$. The Φ satisfies

$$(\nabla\Phi, \nabla\Psi) = (u - z, \nabla\Psi) \qquad \forall\, \Psi \in H^1_0,$$

i.e., the Dirichlet problem

$$-\Delta\Phi = -\operatorname{div}(u - z).$$

The right hand side is per assumption on u, and the estimates for z in L_2, and thus $\Phi \in H^{1+s}$. Thus, the gradient $\nabla\Phi$ is in $[H^s]^3$. □

Note that one boundary condition is really necessary. Take some non-constant harmonic function Φ (i.e., $\Delta\Phi = 0$), and set $u = \nabla\Phi$. It satisfies $\operatorname{div} u = 0$ and $\operatorname{curl} u = 0$, but $\|u\|_{H^1} \neq 0$. Each one of the boundary conditions of Lemma 33 implies that Φ is constant.

Theorem 48. *Assume that equation (31) satisfies the stability estimate*

$$\|\operatorname{curl} u\|_{L_2} + \|u\|_{L_2} \preceq \|j\|_{L_2}.$$

Assume s-regularity. Then there also holds

$$\|u\|_{H^s(\text{curl})} \preceq \|j\|_{L_2}$$

with the norm

$$\|u\|_{H^s(\text{curl})} := \left\{\|\operatorname{curl} u\|^2_{H^s} + \|u\|^2_{H^s}\right\}^{1/2}$$

Proof. Testing equation (31) with $\nabla\psi$, $\psi \in H^1_0$ is

$$\kappa \int u\nabla\psi\, dx = \int j\nabla\psi\, dx,$$

i.e.

$$\operatorname{div} u = \operatorname{div} j = 0$$

Thus $u \in H_0(\text{curl})$ is also in $H(\text{div})$, and thus

$$\|u\|_{H^s} \preceq \|\operatorname{curl} u\| + \|\operatorname{div} u\| = \|\operatorname{curl} u\| \preceq \|j\|_{L_2}$$

Now, set $B = \operatorname{curl} u$. It satisfies $B \in H_0(\text{div})$ with $\operatorname{div} B = 0$. Furthermore, from

$$(B, \operatorname{curl} v) + \kappa(u, v) = (j, v)$$

there follows

$$\operatorname{curl} B = j - \kappa u \qquad \in L_2.$$

Again, from Lemma 33 there follows

$$\|B\|_{H^s} \preceq \|\operatorname{div} B\| + \|\operatorname{curl} B\| = \|j - \kappa u\| \preceq \|j\|_{L_2}.$$

□

3.5.2 Error estimates

In Section 2.2 we have discussed several techniques to prove stability of the continuous problem, i.e.,

$$\inf_{u \in V} \sup_{v \in V} \frac{a(u,v)}{\|u\|_V \|v\|_V} \geq \alpha.$$

For the cases $\kappa \notin \mathbb{R}_0^-$, the stability condition follows with the same techniques also for the discrete case:

$$\inf_{u_h \in V_h} \sup_{v_h \in V_h} \frac{a(u_h,v_h)}{\|u_h\|_V \|v_h\|_V} \geq \alpha.$$

Convergence is shown by standard techniques:

Theorem 49. *Assume that*

- *the problem is s-regular*
- *the discrete problem is inf-sup stable*

Then there holds the error estimate

$$\|u - u_h\|_{H(\mathrm{curl})} \leq c h^s \|j\|_{L_2}$$

Proof. Let I_h be an $H(\mathrm{curl})$ interpolation operator satisfying

$$\|u - I_h u\|_{H(\mathrm{curl})} \leq c h^s \|u\|_{H^s(\mathrm{curl})}.$$

Then

$$\begin{aligned}
\|u - u_h\|_V &\leq \|u - I_h u\|_V + \|I_h u - u_h\|_V \\
&\leq \|u - I_h u\|_V + \alpha^{-1} \sup_{v_h} \frac{a(I_h u - u_h, v_h)}{\|v_h\|_V} \\
&\leq \|u - I_h u\|_V + \alpha^{-1} \sup_{v_h} \frac{a(I_h u - u, v_h)}{\|v_h\|_V} \\
&\leq \|u - I_h u\|_V + \|a\| \alpha^{-1} \|u - I_h u\|_V \\
&\leq c h^s \|u\|_{H^s(\mathrm{curl})} \\
&\leq c h^s \|j\|_{L_2}
\end{aligned}$$

□

For H^1 problems, the Aubin-Nitsche theorem gives an improved convergence in the weaker L_2 norm. This cannot be completely obtained for $H(\mathrm{curl})$ problems, since on the gradient sub-space, the L_2 norm is of the same order as the $H(\mathrm{curl})$-norm. On the complement, the equation is of second order, and one obtains the improved convergence. Although the gradient functions do not converge better in L_2, the converge is better in the H^{-1}-norm.

Theorem 50. *Let u and u_h be the continuous solution and the finite element solution to (31). Assume s-regularity. Then, the Helmholtz decomposition of the error*

$$u - u_h = \nabla \Phi + z \qquad \text{with } \; \Phi \in H_0^1, \; z \perp \nabla H_0^1$$

satisfies

$$\|\Phi\|_{L_2} + \|z\|_{L_2} \leq c h^s \|u - u_h\|_{H(\mathrm{curl})}.$$

Proof. The part z is divergence free. We pose the dual problem

$$a(w, v) = (z, v)_{L_2} \qquad \forall \, v \in H_0(\mathrm{curl}),$$

and the dual finite element problem: find $w_h \in V_h$ such that

$$a(w_h, v_h) = (z, v_h) \qquad \forall \, v \in V_h.$$

By Theorem 49, the error is bounded by

$$\|w - w_h\|_{H(\mathrm{curl})} \leq c h^s \|z\|_{L_2}.$$

Since z is the L_2-projection of $u - u_h$ onto $[\nabla H_0^1]^\perp$, there holds $\|z\|_{L_2}^2 = (z, u - u_h)$. We conclude with

$$\begin{aligned} \|z\|_{L_2}^2 &= (z, u - u_h) = a(w, u) - a(w_h, u_h) \\ &= a(w - w_h, u - u_h) \preceq c h^s \|z\|_{L_2} \|u - u_h\|_{H(\mathrm{curl})}. \end{aligned}$$

The scalar Φ satisfies

$$\|\nabla \Phi\| \leq \|u - u_h\|_{L_2}$$

and

$$(\nabla \Phi, \nabla \eta_h) = (u - u_h - z, \nabla \eta_h) = 0 \qquad \forall \eta_h \in W_h \subset H_0^1.$$

The later is true since $(u - u_h, \nabla \eta_h) = a(u - u_h, \nabla \eta_h) = 0$, and $z \perp \nabla \eta_h$. Posing the dual problem

$$(\nabla \Psi, \nabla \eta) = (\Phi, \eta) \qquad \forall \, \eta \in H_0^1$$

leads to

$$\begin{aligned} \|\Phi\|_{L_2}^2 &= (\nabla \Psi, \nabla \Phi) = (\nabla(\Psi - I_h \Psi), \nabla \Phi) \\ &\leq c h^s \|\Psi\|_{H^{1+s}} \|\nabla \Phi\| \leq c h^s \|\Phi\|_{L_2} \|\nabla \Phi\|_{L_2}, \end{aligned}$$

and thus

$$\|\Phi\|_{L_2} \leq c h^s \|\nabla \Phi\|_{L_2} \preceq h^s \|u\|_{H(\mathrm{curl})}$$

which proves the theorem. ∎

3.5.3 Discrete divergence free functions

A function u_h is called discrete divergence free if there holds

$$(u_h, \nabla \varphi_h) \qquad \forall\, \varphi_h \in W_h \subset H^1.$$

We are interested in discrete divergence free Nédélec finite element functions. The goal is to construct close exact divergence free functions u with the same curl. We will build the functions by solving a mixed variational problem.

Lemma 51. *For all $q_h \in Q_h = \mathcal{R}T_0 \subset H(\mathrm{div})$ such that $\operatorname{div} q_h = 0$ there exists an $u_h \in V_h = \mathcal{N}_0 \subset H(\mathrm{curl})$ such that*

$$\operatorname{curl} u_h = q_h$$

and

$$\|u_h\|_{H(\mathrm{curl})} \leq c \|q_h\|_{L_2}$$

Proof. By Lemma 26 and Lemma 27 there exist an $u \in H(\mathrm{curl})$ such that $\operatorname{curl} u = q_h$ and $\|u\|_{H(\mathrm{curl})} \preceq \|q\|_{L_2}$.

There exist quasi-interpolation operators $\pi^V : H^1 \to W_h$, $\pi^E : H(\mathrm{curl}) \to Q_h$, $\pi^F : H(\mathrm{div}) \to Q_h$, and $\pi^T : L_2 \to S_h$ which are continuous on L_2, commute, and preserve finite element functions (see later).

Set $u_h = \pi^E u$. It satisfies

$$\operatorname{curl} u_h = \operatorname{curl} \pi^E u = \pi^F \operatorname{curl} u = \pi^F q_h = q_h,$$

and

$$\|u_h\|_{H(\mathrm{curl})} \preceq \|u\|_{H(\mathrm{curl})} \preceq \|q_h\|_{L_2}.$$

□

From Lemma 51 there follows the discrete LBB condition

$$\sup_{u_h \in V_h} \frac{(\operatorname{curl} u_h, q_h)}{\|u_h\|_{H(\mathrm{curl})}} \succeq \|q_h\|_{H(\mathrm{div})} \qquad \forall\, q_h \in Q_h : \operatorname{div} q_h = 0$$

Simply take the u_h according to the lemma.

Theorem 52. *Let u_h be a discrete divergence free Nédélec finite element function. Then there exists a unique $u \in H(\mathrm{curl})$ satisfying*

$$\operatorname{curl} u = \operatorname{curl} u_h, \qquad (u, \nabla\varphi) = 0 \;\; \forall \varphi \in H^1, \qquad \|u - u_h\|_{L_2} \preceq h^s \|\operatorname{curl} u_h\|_{L_2}. \tag{34}$$

Proof. We define the funciton u as solution of the mixed variational problem: find $u \in H(\mathrm{curl})$ and $p \in H^0(\mathrm{div}) := \{q \in H(\mathrm{div}) : \operatorname{div} q = 0\}$ such that

$$\begin{array}{lcllll} \int uv & + & \int \operatorname{curl} v\, p & = & 0 & \forall\, v \in H(\mathrm{curl}) \\ \int \operatorname{curl} u\, q & & & = & \int \operatorname{curl} u_h q & \forall\, q \in H^0(\mathrm{div}). \end{array} \tag{35}$$

The variational problem satisfies the conditions of Brezzi (Theorem 5): Continuous bilinear-forms and linear-forms, the LBB condition for $\int \operatorname{curl} uq$ (non-trivial), and the kernel ellipticity of $\int uv$ (trivial). By choosing test functions $v = \nabla\varphi$ in the frist line we obtain

$$(u, \nabla\varphi) = 0.$$

Choosing $q = \operatorname{curl}(u - u_h) \in H^0(\operatorname{div})$ in the second line, we obtain

$$\int |\operatorname{curl}(u - u_h)|^2 = 0, \qquad \text{i.e.,} \qquad \operatorname{curl} u = \operatorname{curl} u_h.$$

We are left to prove that u is close to u_h. Since

$$\operatorname{curl} u = \operatorname{curl} u_h \in L_2, \qquad \operatorname{div} u = 0 \in L_2, \qquad u \cdot n = 0,$$

Theorem 48 gives the regularity estimate

$$\|u\|_{H^s} \preceq \|\operatorname{curl} u_h\|_{L_2}.$$

Now, we pose the corresponding finite element problem: find $u_h^* \in V_h$, and $p_h \in Q_h \subset H^0(\operatorname{div})$ such that

$$\begin{array}{rcrcll} \int u_h^* v_h & + & \int \operatorname{curl} v_h \, p_h & = & 0 & \forall\, v \in V_h \\ \int \operatorname{curl} u_h^* \, q_h & & & = & \int \operatorname{curl} u_h q_h & \forall\, q_h \in Q_h. \end{array} \tag{36}$$

Again, the discret variational problem satisfies the conditions of Brezzi, and thus has a unique solution. Indeed, the solution u_h^* is equal to u_h. The second line proves that $\operatorname{curl}(u_h^* - u_h) = 0$. Thus, the difference must be a discrete gradient, say $\nabla\varphi_h$. Now, test the first line with $\nabla\varphi_h$ to obtain $\nabla\varphi_h = 0$.

We have constructed a variational problem such that u_h is the finite element approximation to u. Now, we bound the discretization error. Choose the test function $v = v_h := \pi^E u - u_h$, and subtract the finite element problem (36 from the continuous problem (35) to obtain

$$\int (u - u_h)(\pi^E u - u_h) + \int \operatorname{curl}(\pi^E u - u_h)\,(p - p_h) = 0. \tag{37}$$

There holds

$$\operatorname{curl}(\pi^E u - u_h) = \pi^F \operatorname{curl} u - \operatorname{curl} u_h = \pi^F \operatorname{curl} u_h - \operatorname{curl} u_h = 0,$$

and thus the second term of (37) vanishes. Inserting an u in the first term leads to

$$\int (u - u_h)(u - u_h) = \int (u - u_h)(u - \pi^E u) \leq \|u - u_h\|\, \|u - \pi^E u\|,$$

and thus

$$\|u - u_h\| \leq \|u - \pi^E u\| \preceq h^s \|u\|_{H^s} \preceq h^s \|\operatorname{curl} u_h\|_{L_2}.$$

$\square$

3.5.4 Error estimates for the general case

We now consider the bilinear-form

$$a(u, v) = (\operatorname{curl} u, \operatorname{curl} v) + \kappa\,(u, v)$$

with a general $\kappa \in \mathbb{C}$. We assume that the continuous problem is solveable, i.e., $-\kappa$ is not an eigenvalue of the Maxwell eigenvalue problem: find $u \in H(\text{curl})$ and $\lambda \in \mathbb{C}$ such that

$$(\operatorname{curl} u, \operatorname{curl} v) = \lambda\,(u, v) \qquad \forall\, v \in H(\text{curl}).$$

It is not guaranteed that the corresponding finite element problem is solveable. Even if $-\kappa$ is not an eigenvalue of the continuous eigenvalue problem, it can be an eigenvalue of the finite element eigenvalue problem, and thus the discrete problem is not solveable. We will prove that for sufficiently fine meshes, the discrete solution exists and converges to the true one.

We define the $H(\text{curl})$-projection $P_h : H(\text{curl}) \to V_h$ by

$$(P_h u, v_h)_{H(\text{curl})} = (u, v_h)_{H(\text{curl})} \qquad \forall\, v_h \in V_h.$$

This is the finite element solution of a problem with $\kappa = 1$.

Theorem 53. *There exists a constant $C > 0$ such that for $h^s \leq C^{-1}$ there holds*

$$\|u - u_h\|_{H(\text{curl})} \leq \frac{1}{1 - Ch^s} \|u - P_h u\|_{H(\text{curl})}.$$

Proof. Assume that the discrete problem is solveable. If not, replace κ by the small perturbation $\kappa + \varepsilon$. All estimates will depend continuously on ε, and thus we can send $\varepsilon \to 0$.

Let u_h be the finite element solution, i.e.,

$$a(u_h, v_h) = a(u, v_h) \qquad \forall\, v_h \in V_h.$$

There holds

$$\begin{aligned}
\|u - u_h\|^2_{H(\text{curl})} &= \|\operatorname{curl}(u - u_h)\|^2 + \|u - u_h\|^2 \\
&= a(u - u_h, u - u_h) + (1 - \kappa)\,\|u - u_h\|^2 \\
&= a(u - u_h, u - P_h u_h) + (1 - \kappa)\,\|u - u_h\|^2 \\
&= (u - u_h, u - P_h u_h)_{H(\text{curl})} + (\kappa - 1)(u - u_h, u - P_h u_h) + (1 - \kappa)\,\|u - u_h\|^2 \\
&= (u - u_h, u - P_h u_h)_{H(\text{curl})} + (\kappa - 1)(u - u_h, u_h - P_h u) \\
&\leq \|u - u_h\|_{H(\text{curl})} \|u - P_h u_h\|_{H(\text{curl})} + |\kappa - 1| \sup_{v_h} \frac{(u - u_h, v_h)}{\|v_h\|_{H(\text{curl})}} \|u_h - P_h u\|_{H(\text{curl})}
\end{aligned}$$

From the orthogonality $u - P_h u \perp P_h u - u_h$ there follows

$$\|u - u_h\|^2_{H(\text{curl})} = \|u - P_h u\|^2_{H(\text{curl})} + \|u_h - P_h u\|^2_{H(\text{curl})}.$$

We divide by $\|u-u_h\|_{H(\mathrm{curl})}$ in the estimates above to obtain

$$\|u-u_h\|_{H(\mathrm{curl})} \leq \|u-P_h u\|_{H(\mathrm{curl})} + |\kappa-1| \sup_{v_h} \frac{(u-u_h, v_h)_{L_2}}{\|v_h\|_{H(\mathrm{curl})}}. \tag{38}$$

We will show that the second term on the right hand side is of smaller order. For this, we apply carefully continuous and discrete Helmholtz decompositions. Consider the inner product $(u-u_h, v_h)_{L_2}$. Let

$$u-u_h = \nabla\varphi + z \qquad \text{withz} \perp \nabla H^1.$$

A version of the Aubin-Nitsche technique, Theorem 50, can be applied for general $\kappa \in \mathbb{C}$ to obtain

$$\|z\|_{L_2} \preceq h^s \|u-u_h\|_{H(\mathrm{curl})}$$

The involved constant depends only on the stability of the continuous problem. Now, let

$$v_h = \nabla\psi + r = \nabla\psi_h + r_h \qquad \text{with} \qquad r \perp \nabla H^1 \quad \text{andr}_h \perp \nabla W_h.$$

There holds $\operatorname{curl} r = \operatorname{curl} r_h = \operatorname{curl} v_h$, and r_h is discrete divergence free, and r is divergence free. From Theorem 52 there follows

$$\|r-r_h\|_{L_2} \preceq h^s \|\operatorname{curl} v_h\|_{L_2}.$$

Applying the Helmholtz decompositions, Galerkin orthogonality $a(u-u_h, \nabla\psi_h) = \kappa(u-u_h, \nabla\psi_h)_{L_2} = \kappa\,(\nabla\phi, \nabla\psi_h) = 0$, and the obtained error estimates we continue with

$$\begin{aligned}
(u-u_h, v_h)_{L_2} &= (\nabla\varphi, v_h) + (z, v_h) \\
&= (\nabla\varphi, \nabla\psi_h + r_h) + (z, v_h) \\
&= (\nabla\varphi, r_h) + (z, v_h) \\
&= (\nabla\varphi, r-r_h) + (z, v_h) \\
&\leq \|\nabla\varphi\|_{L_2} \|r-r_h\|_{L_2} + \|z\|_{L_2} \|v_h\|_{L_2} \\
&\preceq \|u-u_h\|_{L_2}\, h^s \,\|\operatorname{curl} v_h\| + h^s \|u-u_h\|_{H(\mathrm{curl})} \|v_h\|_{L_2} \\
&\leq h^s \,\|u-u_h\|_{H(\mathrm{curl})}\, \|v_h\|_{H(\mathrm{curl})}.
\end{aligned}$$

Plug this bound into (38) to obtain

$$\|u-u_h\|_{H(\mathrm{curl})} \leq \|u-P_h u\|_{H(\mathrm{curl})} + |\kappa-1|\, c h^s \|u-u_h\|_{H(\mathrm{curl})}$$

Move the last term to the left hand side, assume that the mesh size is sufficiently small to fulfill $|\kappa-1| c h^s < 1$, and divide by $1-|\kappa-1| c h^s$ to finish the proof. $\square$

4 Iterative Equation Solvers for Maxwell Equations

We want to solve the linear system of equations

$$Au = f$$

arising from finite element discretization of the Maxwell equation: find $u_h \in V_h \subset H(\mathrm{curl})$

$$\int \mathrm{curl}\, u_h \,\mathrm{curl}\, v_h + \kappa u_h v_h \, dx = \int j v_h \, dx \qquad \forall\, v_h \in V_h.$$

Now, we assume that $\kappa \in \mathbb{R}^+$, but are concerned with possibly very small κ. Such small κ occur, e.g., when the singular magnetostatic problem is regularized by adding a small L_2-term. A small (but complex) κ is also obtained from the time harmonic formulation for frequencies $\omega \to 0$.

For 3D problems, the linear system might become large, and iterative solvers must be applied for CPU-time and memory reasons. A simple iterative method is the preconditioned Richardson iteration

$$u^{k+1} := u^k + \tau C^{-1}(f - Au^k),$$

where C is a symmetric matrix called a preconditioner for A. A good preconditioner satisfies

- The matrix-vector product

  $$w = C^{-1} d$$

  can be computed fast,

- and it is a good approximation to A in the sense of quadratic forms:

  $$\gamma_1 \leq \frac{u^T A u}{u^T C u} \leq \gamma_2 \qquad \forall\, 0 \neq u \in \mathbb{R}^N.$$

  The relative spectral condition number of $C^{-1}A$ is the ratio

  $$\kappa := \frac{\gamma_2}{\gamma_1}.$$

  It should be small. There holds $\sigma\{C^{-1}A\} \subset [\gamma_1, \gamma_2]$, with the spectrum σ, i.e., the set of eigenvalues.

A faster convergent method is the preconditioned conjugate gradient iteration. In the case of general coefficients κ, other Krylov-space solvers such as GMRES, QMR, etc. can be applied with a real-valued preconditioner C.

The simplest preconditioner (except $C = I$) is the diagonal one

$$C = \mathrm{diag}\{A\}.$$

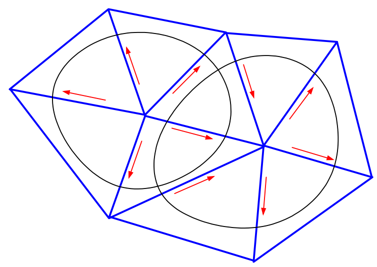

Figure 5: Arnold Falk Winther smoothing blocks

We will observe that it has a large condition number

$$\kappa = \frac{1}{\kappa h^2}.$$

The factor h^{-2} comes from the second order operator curl curl. It is similar to the Poisson case, is usually not too large, and can be overcome by multigrid methods. The other factor κ^{-1} comes from the singular curl-operator. On the rotational sub-space of a Helmholtz decomposition, the curl-operator with coefficient 1 dominates. On the gradient sub-space, the bilinear-form is of 0^{th} order with a small coefficient κ. As $\kappa \to 0$, some eigenvalues of A converge to 0. But, the limit of $C = \operatorname{diag} A$ is a regular matrix.

A robust preconditioner is the Arnold-Falk-Winther one. It is an overlapping block Jacobi preconditioner. Each block is connected with a vertex of the mesh. A block contains all unknowns on edges connected to the vertex, see Figure 5. To build the block-Jacobi method, one takes the sub-matrices according to the blocks, inverts them, and adds them together to obtain the block-Jacobi preconditioner C^{-1}. This one has the improved condition number

$$\kappa = \frac{1}{h^2}.$$

Again, by multigrid methods, the condition number can be improved to $O(1)$.

4.1 Additive Schwarz preconditioning

The additive Schwarz (AS) theory is a general framework containing block-preconditioning. For $i = 1, \ldots, M$ let $E_i \in \mathbb{R}^{N \times N_i}$ be rectangular matrices of rank N_i such that each $u \in \mathbb{R}^N$ can be (not necessarily uniquely) written as

$$u = \sum_{i=1}^{M} E_i u_i \qquad \text{with} \quad u_i \in \mathbb{R}^{N_i}.$$

The additive Schwarz preconditioning operation is defined as

$$C^{-1} d = \sum_{i=1}^{M} E_i A_i^{-1} E_i^T d \qquad \text{with} \qquad A_i = E_i^T A E_i.$$

In the AFW - preconditioner there is M = number of vertices. The block-size N_i corresponds to the number of connected edges. The columns of the matrix E_i are unit-vectors according to the dof-numbers of the connected edges.

The following lemma gives a useful representation of the quadratic form. It was proven in similar forms by many authors (Nepomnyaschikh, Lions, Dryja+Widlund, Zhang, Xu, Oswald, Griebel, ...) and is called also Lemma of many fathers, or Lions' Lemma:

Lemma 54 (Additive Schwarz lemma)**.** *There holds*

$$u^T C u = \inf_{\substack{u_i \in \mathbb{R}^{N_i} \\ u = \sum E_i u_i}} \sum_{i=1}^{M} u_i^T A_i u_i$$

Proof: The right hand side is a constrained minimization problem for a convex function. The feasible set is non-empty, the CMP has a unique solution. It can be solved by means of Lagrange multipliers. Define the Lagrange-function for $(u_i) \in \Pi \mathbb{R}^{N_i}$ and Lagrange multipliers $\lambda \in \mathbb{R}^N$:

$$L((u_i), \lambda) = \sum u_i^T A u_i + \lambda^T (u - \sum E_i u_i).$$

Its stationary point (a saddle point) is the solution of the CMP:

$$\begin{aligned} 0 = \nabla_{u_i} L((u_i), \lambda) &= 2 A_i u_i - E_i^T \lambda \\ 0 = \nabla_{\lambda} L((u_i), \lambda) &= u - \sum E_i u_i \end{aligned}$$

The first line gives

$$u_i = \frac{1}{2} A_i^{-1} E_i^T \lambda.$$

Use it in the second line to obtain

$$0 = u - \frac{1}{2} \sum E_i A_i^{-1} E_i \lambda = u - \frac{1}{2} C^{-1} \lambda,$$

i.e., $\lambda = 2Cu$, and

$$u_i = A_i^{-1} E_i^T C u.$$

The minimal value is

$$\begin{aligned} \sum u_i^T A_i u_i &= \sum u^T C E_i A_i^{-1} A_i A_i^{-1} E_i^T C u \\ &= \sum u^T C E_i A_i^{-1} E_i^T C u \\ &= u^T C C^{-1} C u = u^T C u \end{aligned}$$

$\square$

The linear algebra framework is needed for the implementation. For the analysis, it is more natural to work in the finite element space. For this, introduce the Galerkin isomorphism

$$G : \mathbb{R}^N \to V_h : u \mapsto \sum_{i=1}^N u_i \varphi_i$$

The range of the matrices E_i are linked to sub-spaces $V_i \subset V_h$

$$V_i := G \,\mathrm{range}\{E_i\} = \{\sum_{j=1}^N \sum_{k=1}^{N_i} \varphi_j E_{jk} \lambda_k : \lambda \in \mathbb{R}^{N_i}\}.$$

In the case of the AFW preconditioner, the subspace V_i is spanned by the edge-basis functions connected with the edges of the vertex.

The quadratic form of the preconditioner can be written as

$$u^T C u = \inf_{\substack{u_i \in V_i \\ Gu = \sum u_i}} \sum_{i=1}^M \|u_i\|_A^2$$

Now, the task is to analyzed the bounds in the norm estimates

$$\gamma_1 \inf_{\substack{u_i \in V_i \\ u = \sum u_i}} \sum_{i=1}^M \|u_i\|_A^2 \leq \|u\|_A^2 \leq \gamma_2 \inf_{\substack{u_i \in V_i \\ u = \sum u_i}} \sum_{i=1}^M \|u_i\|_A^2 \qquad \forall\, u \in V_h$$

Usually, the right inequality is the simpler one. If only a finite number of sub-spaces overlap, then $\gamma_2 = O(1)$.

4.2 Analysis of some $H(\mathrm{curl})$ preconditioners

We start with some scaling and inverse inequalities:

Lemma 55. *Let d be the space dimension, and let E be an edge of the element T. The according $\mathcal{N}_0$ edge basis function is φ^E. There holds*

1. $$\|\varphi^E\|_{L_2}^2 \simeq h^{d-2}$$

2. $$\|\operatorname{curl} \varphi^E\|_{L_2}^2 \simeq h^{d-4}$$

3. $$\int_E v_h \cdot \tau \, ds \preceq h^{(2-d)/2} \|v_h\|_{L_2(T)} \qquad \forall\, v_h$$

The lemma is proven by transformation to the reference element.

Theorem 56. *Diagonal preconditioning for the matrix arising from the L_2-bilinear-form*

$$M(u,v) = \int uv\,dx \qquad \forall\, u,v \in H(\text{curl})$$

leads to optimal condition numbers.

Proof. Let $u = \sum u_i$ be the decomposition of u into $u_i \in V_i := \text{span}\{\varphi^{E_i}\}$. This decomposition is unique. We have to show that

$$\sum \|u_i\|_M^2 \preceq \|u\|_M^2.$$

The function u_i is given by

$$u_i = \{\int_E u\cdot\tau\,ds\}\varphi^E,$$

and thus

$$\|u_i\|_{L_2}^2 = \{\int_E u\cdot\tau\,ds\}^2 \|\varphi^E\|_{L_2}^2 \preceq h^{2-d}\,\|u\|_{L_2(T)}^2 h^{d-2},$$

where T is an arbitrary element sharing the edge E. Since each element is used at most 6 times, summing up leads the desired estimate

$$\sum_{i=1}^{N_E} \|u_i\|_{L_2}^2 \preceq \sum \|u\|_{L_2(T_{E_i})}^2 \preceq \|u\|_{L_2(\Omega)}^2.$$

□

Theorem 57. *Diagonal preconditioning for the matrix arising from the bilinear-form*

$$A(u,v) = \int \text{curl}\,u\,\text{curl}\,v + \kappa uv\,dx$$

leads to condition numbers bounded by

$$\kappa \preceq \frac{1}{\kappa h^2}$$

Proof. Again, we decompose $u = \sum u_i$. Now there holds

$$\|u_i\|_A^2 \preceq \{\int_E u\cdot\tau\,ds\}^2 \|\varphi^E\|_A^2 \preceq h^{2-d}\,\|u\|_{L_2(T)}^2\,\{h^{d-4} + \kappa h^{d-2}\},$$

and thus

$$\sum \|u_i\|_A^2 \preceq \{h^{-2} + \kappa\}\|u\|_{L_2(\Omega)}^2 \preceq \{\frac{1}{h^2\kappa} + 1\}\|u\|_A^2$$

□

For scalar problems with small L_2-coefficient, we can use Friedrichs' inequality to bound the L_2-term by the H^1 term. This avoids the dependence of κ. For the $H(\text{curl})$ equation, we cannot apply the Friedrichs' on the whole space, but on the complement of the gradients. The gradient sub-space is analyzed separately:

Theorem 58. *The AFW block preconditioner leads to the condition number*

$$\kappa \preceq \frac{1}{h^2}.$$

Proof. Choose an $u \in V_h \subset H(\text{curl})$. The goal is to decompose u into local functions contained in the AFW blocks. We start with the discrete Helmholtz decomposition

$$u = \nabla w + z \qquad w \in W_h \subset H^1, \quad z \perp_{L_2} \nabla W_h$$

From Lemma 51 there follows the discrete Friedrichs' inequality

$$\|z\|_{L_2} \preceq \|\operatorname{curl} z\|_{L_2} = \|\operatorname{curl} u\|_{L_2}$$

We can now decompose $z = \sum z_i$ into basis functions satisfying

$$\sum \|z_i\|_A^2 \preceq \{h^{-2} + \kappa\} \|z\|_{L_2}^2 \preceq \{h^{-2} + 1\} \|u\|_A^2. \tag{39}$$

The bad factor κ^{-1} is avoided. A decomposition into basis functions implies also the coarser decomposition into the AFW blocks.

Now, we continue with the gradient functions. They satisfy

$$\kappa \|\nabla w\|_{L_2}^2 \leq \kappa \, \|u\|_{L_2}^2 \leq \|u\|_A^2$$

Decompose the scalar function w into vertex basis functions

$$w = \sum_{i=1}^{N_V} w_i = \sum_{i=1}^{N_V} w(V_i) \varphi^{V_i}$$

This decomposition satisfies

$$\sum \|\nabla w_i\|^2 \preceq h^{-2} \|w\|_{L_2(\Omega)}^2 \preceq h^{-2} \|\nabla w\|_{L_2(\Omega)}^2 \preceq h^{-2} \|u\|_{L_2(\Omega)}^2.$$

For gradient fields the curl-term vanishes:

$$\sum \|\nabla w_i\|_A^2 = \sum \kappa \|\nabla w_i\|_{L_2}^2 \preceq \kappa h^{-2} \|u\|_{L_2(\Omega)}^2 \leq h^{-2} \|u\|_A^2. \tag{40}$$

Finally observe that $\nabla w_i \subset V_i$: The gradient of a vertex basis function can be represented by the edge-basis functions connected with this vertex. Thus

$$\nabla w = \sum \nabla w_i$$

is a decomposition compatible with the AFW blocks. The final decomposition is $u_i = z_i + \nabla w_i$. Combining estimates (39) and (40) provides the stable decomposition of u. $\square$

4.3 Multigrid Methods

The condition number of local preconditioners get worse as the mesh size decreases. Multigrid methods involve several grids and (may) lead to condition numbers $O(1)$.

Assume we have a sequence of nested grids. On each level l, $0 \leq l \leq L$, we build a lowest order order Nédélec finite element space V_l of dimension N_l. These spaces are nested:

$$V_0 \subset V_1 \subset \ldots \subset V_L.$$

A function u_{l-1} in the coarser space is also in the finer space. It can be represented with respect to the coarse grid basis, or with respect to the fine grid basis:

$$u_{l-1} = \sum_{i=1}^{N_{l-1}} u_{l-1,i} \varphi_{l-1}^{E_i} = \sum_{i=1}^{N_l} u_{l,i} \varphi_l^{E_i}$$

Let $I_l \in \mathbb{R}^{N_l \times N_{l-1}}$ denote the prolongation matrix which transfers the coarse grid coefficients $u_{l-1,i}$ to the fine grid coefficients $u_{l,i}$.

On each level we define a cheap iterative method called smoother. It might be the block-Jacobi preconditioner by Arnold, Falk, and Winther. We call the local preconditioners D_l:

$$u_l^{k+1} = u_l^k + \tau D_l^{-1}(f_l - A_l u_l^k)$$

We define multigrid preconditioners on each level:

$$C_l^{-1} : \mathbb{R}^{N_l} \to \mathbb{R}^{N_l} : d_l \mapsto w_l$$

On the coarsest grid we use the inverse of the system matrix:

$$C_0^{-1} = A_0^{-1}$$

On the finer grids, the preconditioning actions $C_l^{-1} : d_l \mapsto w_l$ are defined recursively by the following algorithm:

Given $d_l \in \mathbb{R}^{N_l}$. Set $w_0 = 0$.

(1) Pre-smoothing:

$$w_1 = w_0 + \tau D_l^{-1}(d_l - A w_0)$$

(2) Coarse grid correction:

$$w_2 = w_1 + I_l C_{l-1}^{-1} I_l^T (d_l - A w_1)$$

(3) Post-smoothing:

$$w_3 = w_2 + \tau D_l^{-1}(d_l - A w_2)$$

Set $w_l = w_3$

This is a multigrid V-cycle with 1 pre-smoothing and 1 post-smoothing step. One can perform more pre- and post-smoothing iterations in step (1) and (3). One could also apply 2 coarse grid correction steps in step (2), which leads to the W-cycle.

4.3.1 Multigrid - Analysis

We sketch the application of the classical Braess-Hackbusch multigrid analysis. All we have to verify can be formulated in the estimate

$$\|u_l - I_l A_{l-1}^{-1} I_l^T A_l u_l\|_{D_l} \preceq \|u_l\|_{A_l} \tag{41}$$

This estimate is usually broken into two parts, the approximation property and the smoothing property. The approximation property states that the coarse grid approximation

$$u_{l-1} := A_{l-1}^{-1} I_l^T A_l u_l$$

is close to u_l in a weaker norm. For a scalar problem, the approximation property is

$$\|u_l - I_l u_{l-1}\|_{L_2} \preceq h \, \|u_l\|_{H^1}.$$

The smoothing property says that the matrix D_l of the smoother is related to the weaker norm. For a scalar problem, this is

$$\|u_l\|_{D_l} \preceq h^{-1} \|u_l\|_{L_2}.$$

Both together give estimate (41). If this estimate is established for all levels $1 \leq l \leq L$, the Braess-Hackbusch theorem proves that the condition number of the multigrid preconditioner is $O(1)$ uniformely in the number of refinement levels L.

The approximation property is proven similar to the Aubin-Nitsche technique. For the $H(\text{curl})$ case, the Aubin Nitsche theorem, Theorem 50 gives estimates for the Helmholtz decomposition of the error

$$u_l - I_l u_{l-1} = \nabla \varphi_l + z_l \qquad z_l \perp \nabla W_l,$$

namely

$$\|\varphi_l\|_{L_2} + \|z_l\|_{L_2} \preceq h_l \, \|u_l\|_{H(\text{curl})}.$$

In contrast to Theorem 50, we need the discrete Helmholtz decomposition. Its proof additionally needs the results of discrete divergence free functions of Section 3.5.3. By definition of the norm

$$\|v_l\|_{\tilde{0}} := \inf_{\varphi_l \in W_l} \left\{ \|\varphi_l\|_{L_2} + \|v_l - \nabla \varphi_l\|_{L_2} \right\},$$

the approximation property can be written as

$$\|u_l - I_l u_{l-1}\|_{\tilde{0}} \preceq h \, \|u_l\|_{H(\text{curl})}.$$

Similar to the proof of the AFW - preconditioner (Theorem 58), one verifies the smoothing property

$$\|u_l\|_{D_l} \preceq h_l^{-1} \|u_l\|_{\tilde{0}}.$$