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Bogdan Bojarski

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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Bogdan Bojarski

Helsinki University of Technology
Faculty of Information and Natural Sciences
Department of Mathematics and Systems Analysis

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Correspondence

Institute of Mathematics
Polish Academy of Sciences
00-956 Warsaw
Poland

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Helsinki University of Technology
Faculty of Information and Natural Sciences
Department of Mathematics and Systems Analysis
P.O. Box 1100, FI-02015 TKK, Finland
email: math@tkk.fi http://math.tkk.fi/

1 Introduction

The paper continues a series of our papers devoted to the pointwise theory of Sobolev spaces (see also e.g. [5], [6], [7] and some earlier, quoted in [4]). In [4] it was observed that the involved concepts in many respects are keen to and show many analogies, on the one hand, with the ideas of H. Whitney, exposed in his theory of smooth functions on arbitrary closed subsets of $\mathbb{R}^n$ and the famous extension theorems [46], [47], and, on the other hand, with the concepts elaborated by Marcinkiewicz–Zygmund in their deep works on Peano and Riemann differentiability of arbitrary measurable functions on the real line $\mathbb{R}$ [33], [34].

Intimately related to these papers is also the A. Denjoy paper [16] whose profound ideas on differentiability, including the concept of approximate derivative, go back to his fundamental works from the beginning of the last century [15]. However, the theory initiated by Denjoy–Marcinkiewicz–Zygmund in the above papers, referred in the sequel, for short, as DMZ theory, in view of the employed tools—Lagrange interpolation formulas, precise pointwise Taylor remainders estimates, Vandermonde determinants, and related estimates for solving algebraic equations (inequalities)—was rather one-dimensional, i.e. restricted to the functions of one real variable.

In [4] a programme for unifying the DMZ theory as well as the Sobolev theory of the function spaces $W^{m,p}(G)$, $p > 1$, $m \geq 1$ ($G \subset \mathbb{R}^n$—open or closed domain with regular boundary), with the Whitney's ideas [46] was proposed and some plan and basic concepts in this direction were suggested and developed. Important supplementary tools used were various forms of Markov inequalities for polynomials [1], [9], [13]; they appeared already in connection with some earlier work on function spaces, e.g. in [21], [25], [31] and [3].

The plan sketched in [4] was realized in a rather detailed manner in [5], [6], for the case of Sobolev spaces $W^{m,p}_{\mathrm{loc}}(G)$, continuing (in a natural, conceptual and technical sense) the paper [7] of the author and P. Hajłasz and [8]. Actually, after 1993, in numerous seminar lectures at several mathematical institutes in Europe, I advocated related concepts in various forms. Also, as a by-product of the development of the ideas of [7] along this way P. Hajłasz [22] proposed his theory of $W^{1,p}(X)$ spaces on general measure metric spaces $X = X(d, \mu)$ occupying, during the last decade, an important role in the development of Analysis on general measure metric spaces, [22], [23], [26].

Below we describe mainly

a) the extension of DMZ theory to arbitrary measurable subsets of $\mathbb{R}^n$,

b) a detailed proof of our Theorem 5 extending Whitney's Theorem 1 in [48], proved there for the 1-smooth Luzin approximation, to the general case of k-smooth Luzin approximation, $k > 1$. See Definition 3 below.

An extension of the type b) was first announced in the papers by F.-Ch. Liu [30] and Liu & Tai [31]. Our proof below should be considered as an alternative to [30], [31], directly and carefully elaborating on Whitney's original proof of Theorem 1 in [48] (for $k = 1$). It is apparently the first self-contained proof of the theorem for the case $k > 1$ in $\mathbb{R}^n$.

It is maybe worthwhile to say that our proof follows the idea of Whitney in [48], but the crucial estimates which we use below—Lemmata 4 and 5—were apparently not known to H. Whitney. Thus in his Theorem 4 in [48], which is weaker than Theorem 5, he had to assume that the higher order component f_{k-1} is totally differentiable on an open subset of $\mathbb{R}^n$, instead of the approximate total differentiability. It turns out that the problem of iterated approximate total differentiability on arbitrary measurable subsets of $\mathbb{R}^n$ is much more delicate than that of ordinary differentiability. See the example of Movshovich quoted below [35] and the conjecture (3.17) in [19]. For some other remarks in this respect see below (after the proof of Theorem 5).

Summing up, we can say that, together with our papers [5], [6], this work realizes rather in detail the main goals of the programme sketched in [4] for approximate differentiation of arbitrary functions. See also our final comments in Section 6 below.

The text presented here is a slight modification of two earlier versions, of March 19, 2008, and February 10, 2009, circulating as multiplied manuscripts. The 2008 version was discussed in detail in a series of my lectures in March and April 2008 at TKK in Helsinki at the seminar of Prof. Juha Kinnunen. After my colloquium talk at TKK in March 2009 it was proposed that the last version be published in the TKK report series[1].

I thank Prof. J. Kinnunen for invitation and the fruitful creative atmosphere during both my visits to TKK. I thank also Dr. M. Korobkov from the Sobolev Institute in Novosibirsk for careful reading and insightful comments to the earlier versions, improving the presentation.

We adopt the standard notation for multidimensional differential or polynomial algebra and, in the context of k-jets, we follow Malgrange [32] or Abraham and Robbin [1]. However, some comments are due: the letter k can mean the multiindex $k = (k_1, \ldots, k_n)$, $k_i \geq 0$, then $|k| = \sum_i k_i$, as well as the natural number k as in the expression k-jet $F = \{f_0, \ldots, f_k\}$, where each f_k is a shorthand for the "vector" $f_k(y) = \{f_\alpha(y)\}$, $|\alpha| = k$, for all n-multiindices α. As usual, $D^l_x = D^l_x(f)$ is the notation for the partial derivative of order l, whereas $D_l F$ is the formal jet differentiation: $F = \{f_0, \ldots, f_k\}$, $D_l F = \{f_l, f_{l+1}, \ldots, f_k\}$, which makes sense for k-jets F with arbitrary components f_α; we always have $D_{k+1}F \equiv 0$. Though our formulas symbolically look like formulas of one-dimensional Taylor polynomial algebra, as a matter of fact, they are formulas of multidimensional Taylor algebra. Also, for a multiindex k, $k! = k_1! \cdots k_n!$.

A l-jet $F \subset J^l(G)$ is continuous, k-smooth, $k \geq 0$, measurable, if all its components $(f_0, \ldots, f_l)$ are continuous, ..., etc. However, the expression: *the l-jet $F \subset J^l(G)$ is Whitney k-smooth on a closed subset $\Sigma \subset G$* has a

[1] A systematic, more comprehensive presentation of this and related topics, taking into account the most recent related publications, available in print or electronic form, is under preparation.

very special meaning (see the Basic Definition below).

In Section 2, after the Basic Definition is presented, we state the precise formulation of the fundamental Whitney Extension Theorem (WhET); however we do not take any effort to prove it here. As a matter of fact we never refer to the WhET in the course of the proofs of any one of our theorems in DMZ theory or Whitney–Luzin structure theory. However the WhETs play a fundamental role in the formulation of final statements and global "corollaries". As a primary example compare the structure theorem 5 and Corollary 3.

Loosely speaking: the Whitney theory in the proper sense is alike to work with "segmented" objects ("traces" of k-Whitney-continuous jets defined on "disconnected" closed subsets of $\mathbb{R}^n$) to produce new "segmented" objects, but these constructions are "subordinated" to some rules and "requirements"—Whitney conditions or "Whitney integrability conditions", and this allows us to "integrate" them and produce a "classical" object, e.g. a function $f \in C^k(\mathbb{R}^n)$, that is, an object well appealing to the standard understanding.

We can also say that the Whitney theory creates a fascinating world of "segmented", scattered pieces of graphs of well understood objects, examines and describes their properties, and "organizes" them to the form that they may be "integrated" or "glued together" to a function in $C^k(\mathbb{R}^n)$.

It is feasible that in some cases it is useful to apply a WhET in the research process, and, apparently, this is the case in F.-Ch. Liu's proof of the Whitney–Luzin structure theorem in [30] or [31].

We will return to these questions in our final remarks.

2 Auxiliary formulas and pointwise estimates

Given a Whitney k-jet $F = \{f_0(y), \ldots, f_k(y)\}$, $f_0 \equiv f$, defined on a closed subset $\Sigma \subset \mathbb{R}^n$, the k-order Taylor polynomial of F, $T^k_y F(x) = T^k F(y, x)$ centred at the point $y \in \Sigma$ and its Taylor remainder $R^k F(y, x)$ are defined by the formulas

$$T^k F(y,x) = f_0(y) + f_1(y)(x-y) + \ldots + f_k(y)\,\frac{(x-y)^k}{k!} \qquad (1)$$

$$f(x) \equiv R^k F(y,x) + T^k F(y,x) \qquad (2)$$

For y fixed in Σ, $T^k F(y,x)$ is a polynomial of order k in x, the Whitney-Taylor polynomial or k-th order Whitney field on Σ with coefficients $f_i(y)$, indexed by the multiindex i, $|i| \le k$. The Whitney fields can be differentiated with respect to x and satisfy the conditions

$$D^l_x T^k F(y,\cdot)|_{x=y} = f_l(y), \quad |l| \le k \qquad (3)$$

$$D^l_x T^k F(y,x) = f_l(y) + f_{l+1}(y)(x-y) + \ldots + f_k(y)\,\frac{(x-y)^{k-l}}{(k-l)!} \qquad (4)$$

This can also be written in terms of the formal jet derivatives $D_l : J^k \to J^{k-|l|}$, $D_l F = \{f_l, f_{l+1}, \ldots, f_k\}$,

$$D_x^l T^k F(y,x) = T^{k-|l|}(D_l F)(y,x) \tag{5}$$

and the Taylor remainders $R^{k-|l|}(D_l F)(y,x) \doteq R_l^k F(y,x)$

$$R^{k-|l|}(D_l F)(y,x) + T^{k-|l|}(D_l F)(y,x) \equiv f_l(x) \tag{6}$$

for the l-th component of the k-jet F, or the first (i.e. zero) component of $D_l F$.

We recall and describe in a form convenient for our purposes some polynomial identities and derived inequalities which allow us to control various estimates for the Taylor remainders $R^k F(x,y)$, $R_l^k F(x,y)$. Crucial for the discussion of function spaces is the control of the behaviour of the remainders near the diagonal $\Delta = \{x = y\}$ of $\mathbb{R}^n \times \mathbb{R}^n$ described in Lemmata 4 and 5 below. They are analogous and closely related to estimates which in explicit or implicit form appeared and have been used by many authors starting from A. Denjoy [16], Marcinkiewicz and Zygmund [33], [34], Whitney [46], [47], [48], Glaeser [21], Malgrange [32], where the coefficients of the polynomials (their derivatives—thus Markov inequalities!) are controlled by the values of the polynomials at appropriate points of their domain or directly by Markov inequalities (Jonsson–Wallin [25] or F. Ch. Liu [30]). The presentation below allows us to give a rather unified approach to several questions discussed in some of the papers above.

For a triple of points x, y, z, or a point of the Cartesian product $\Sigma \times \Sigma \times \Sigma$, the difference

$$P^k F(x,y,z) = R^k F(x,z) - R^k F(y,z) \equiv T^k F(y,z) - T^k F(x,z) \tag{7}$$

is a polynomial in z of degree k

$$P^k F(x,y,z) = \sum_{|l| \le k} P_l^k F(x,y) \, \frac{(z-y)^l}{l!} \tag{8}$$

with coefficients $P_l^k F(x,y) = D_z^l P^k F(x,y,z)|_{z=y}$ equal to the lower degree remainders centered at x for the formal derivatives $D_l F$. Actually

$$P_l^k F(x,y) = R_l^k F(x,y) = R^{k-|l|} D_l F(x,y). \tag{9}$$

In particular

$$\begin{aligned} &P_0^k F(x,y) = R^k F(x,y) \\ &P_1^k F(x,y) = f_1(y) - f_1(x) - f_2(x)(y-x) - \ldots - f_k(x) \, \frac{(y-x)^{k-1}}{(k-1)!} \\ &\qquad \ldots\ldots\ldots\ldots \\ &P_k^k F(x,y) = f_k(y) - f_k(x) \end{aligned} \tag{10}$$

These and analogous formulas appear in various forms in many texts on classical analysis, difference calculus, approximation or interpolation theory.

It will be convenient for us to write (7), (8) in the form

$$R^k F(x,z) - R^k F(x,y) - R^k F(y,z) = \sum_{1\le |l| \le k} R^k_l F(x,y)\, \frac{(z-y)^l}{l!} \tag{11}$$

which allows us to control the Taylor remainder of degree k in the neighbourhood of the point x evaluated at y through Taylor remainders of degree k at x and y evaluated at some intermediate point z and lower degree remainder $R^k_l F$, $|l| \ge 1$, estimates for derivatives $D_l F$ at x, or vice versa: the lower degree estimates of errors for formal derivatives $D_l F$ allow us to control the highest degree remainders.

This observation gives important services in many delicate situations. In DMZ one-dimensional theory basic tool in this connection are Lagrange interpolation formulas, Vandermonde determinants, finite differences, and various representation formulas for polynomials. For the multivariate case G. Glaeser [21] proposed a special, regular step multigrid interpolation, procedure. We refer to Markov inequalities.

Lemma 1. *For any ball $B(x_0, r)$ in $\mathbb{R}^n$ and any polynomial $p(y)$ of degree k, $y \in \mathbb{R}^n$, the inequality*

$$|D^\alpha p(x_0)| \le \frac{C}{r^{|\alpha|}}\, |p|_{C(B)} \tag{12}$$

holds with the constant C depending on n and k only.

Here $|p|_{C(B)}$ is the supremum norm in the space of continuous functions on $B = B(x_0, r)$.

A slightly more refined is

Lemma 2. *Let E be a measurable subset of the ball $B(x_0, r)$ and define $\frac{|E|}{r^n} = \sigma > 0$. Then there exists a constant $C = c(n, k, \sigma)$ such that for all polynomials $p(y)$ of degree at most k, the inequality*

$$|D^\alpha p(x_0)| \le \frac{C}{r^{n+|\alpha|}} \int_E |p(y)|\, dy \tag{13}$$

holds.

This lemma is attributed by S. Campanato to de Giorgi [13].

Much more subtle is the following lemma due to Brudnyĭ–Ganzburg [9].

Lemma 3. *In the conditions of Lemma 2 the following estimate holds for $1 \le q \le \infty$*

$$|D^\alpha p(x_0)| \le \frac{\delta}{r^{|\alpha|}}\, \sigma^{-k} \mathcal{N}_q(p, E) \tag{14}$$

where $\delta = \delta(n, k, |\alpha|, q)$ and $\mathcal{N}_q(p, E) = \left(⨍_E |p|^q\, dy\right)^{1/q}$. Important is the case $q = \infty$ where $\mathcal{N}_q(p, E) = \max_E |p|$ and $\delta = \delta(n, k, |\alpha|) = \left(\frac{\sqrt{n}}{2} k\right)^{|\alpha|} \delta(n, k)$,

$$|D^\alpha p(x_0)| \le \frac{\delta(n, k, |\alpha|)}{r^{|\alpha|}}\, \sigma^{-k} \max_{z \in E} |p(z)|. \tag{15}$$

Lemma 3 is the deepest of the three in the sense that it specifies explicitly the dependence on the parameters σ, k and n, and its proof is most sophisticated [9]. In the sequel depending on the context it is convenient to refer to all three, and especially to Lemma 3 in the form of the inequality (15). However, roughly speaking, Lemmata 1 and 3 are consequences of Lemma 2.

If the k-jet $F \subset J^k(\mathcal{U})$ is a k-jet of a function $g(x) \in C^k(\mathcal{U})$, $f_\alpha = \frac{\partial^\alpha g}{\partial x^\alpha}$, $\mathcal{U}$—open set of $\mathbb{R}^n$, e.g., an open neighbourhood of a closed compact set Σ, then the k-jet $G = \left\{\frac{\partial^\alpha g}{\partial x^\alpha}\right\}$, $|\alpha| \le k$, satisfies the Taylor remainder conditions

$$|R_l^k G(y,x)| = |R_l^k F(y,x)| = o(r)|x-y|^{k-|l|}, \tag{16}$$

$$\text{for } r = |x-y| \to 0,\ x, y \in \Sigma,\ |l| \le k, \tag{17}$$

uniformly on compact subsets of $\mathcal{U}$, for all values of $|l| \le k$ and all $x, y \in \Sigma$. It is convenient to write (16) for a k-jet F defined on Σ in the form of the inequalities[2]

$$R_l^k F(x,y) \le \alpha_{F,\Sigma}(|x-y|)|x-y|^{k-|l|}, \quad x, y \in \Sigma,\ |l| \le k, \tag{18}$$

with $\alpha_F(t)$—a modulus of continuity of the k-jet F on the compact Σ. We recall that $\alpha_F(t)$ can be taken as a concave, i.e., nondecreasing, subadditive continuous function for $t \in \mathbb{R}^+$, $\alpha(0) = 0$. These properties hold in view of the general principle of minimum maximorum. A priori the continuity modulus α depends on the indices l, but, in fact, the moduli, for various l, may be taken equivalent in the sense: $\alpha_1 \sim \alpha_2$ if $\frac{1}{C}\alpha_1(t) \le \alpha_2(t) \le C\alpha_1(t)$ for some constant C depending on k and n only.

For smooth functions in the class $C^k(\mathcal{U})$, $\mathcal{U}$—open, all these facts follow directly from classical formulas for the Taylor remainders. See also formula (31) in Section 4 below. A deep idea of H. Whitney [46], [47] was to consider (18) or (16) as conditions defining the class of smooth functions, or rather smooth k-jets, denoted $C^k(\Sigma)$, on the compact set Σ.

Definition 1 (Whitney)**.** The subspace $C^k(\Sigma)$ of all continuous k-jets $J^k(\Sigma)$ satisfying the uniform estimates (18) or (16) is termed the *space of k-smooth functions on Σ in the sense of H. Whitney*, *k-smooth Whitney jets*, or *Whitney k-smooth functions* for short.

This concept is justified by the fundamental Whitney Extension theorem [46], [47].

Theorem. *For any k-jet $F \subset C^k(\Sigma)$ there exists a classical smooth function $g \in C^k(\mathcal{U})$ such that*

$$G = J^k g|_\Sigma = F$$

($G|_\Sigma$ means the restriction of $G \in J^k(\mathcal{U})$ to Σ). Here $\mathcal{U}$ is any open neighbourhood of Σ in $\mathbb{R}^n$, or just $\mathcal{U} \equiv \mathbb{R}^n$. Moreover, the extended k-jet G may be defined as a bounded linear operator $G = W_\Sigma^k(F)$ with the full control of the continuity moduli $\alpha_G \sim \alpha_F$.

[2] In the sequel we usually omit the sign of absolute value: $R_l^k = |R_l^k|$.

The Banach space structure, norms etc. are carefully described in [32] and [43].

More generally for a measurable subset $P \subset \mathbb{R}^n$ of positive Lebesgue measure $|P| > 0$, we shall consider measurable k-jets on P, using the same notation $J^k(P)$, or, if need will be to stress measurability, we write $\operatorname{mes} J^k(P)$. Thus $\operatorname{mes} J^k(P)$ is just the vector space of (real valued) measurable vector functions on P, while $C^0(P)$ is the space of continuous real valued functions on P.

Many operations on measurable functions, like restriction to measurable subsets $Q \subset P$, pointwise convergence, convergence in measure, uniform convergence on compact sets etc., are directly transferred to the general k-jets F in $J^k(P)$.

Since the Whitney smoothness conditions (16), (18) have pointwise formulations, it may very well happen that while the jet $F \subset J^k(P)$ does not satisfy (16), (18) its restriction $G = F|_Q$ satisfies the Whitney k-condition or $W_{k'}$-condition on a subset $Q \subset P$ for some k', $0 \le k' < k$.

Definition 2. A closed compact subset Σ of the domain P of the jet $F \subset J^k(P)$ such that $F|_\Sigma$ satisfies the Whitney conditions (16) or (18) will be called a *Whitney regularity set of the k-jet F*, for short WR-set of F, denoted by WR(F). The associated continuity modulus α_F, depending also on Σ, $\alpha = \alpha_{(F,\Sigma)}$ will be called the *modulus of the Whitney jet $F|_\Sigma$* on Σ. (Usually we drop the lower indices.)

Definition 3. A measurable function $f(x)$, or k-jet F, defined on the measurable set P is *k-quasismooth*[3] on P iff for each $\varepsilon > 0$ there exists a closed (perfect) subset $Q_\varepsilon \subset P$ such that $|P \setminus Q_\varepsilon| < \varepsilon$ and the restriction $F|_{Q_\varepsilon}$ is Whitney k-smooth on Q_ε.

The Whitney k-smooth jet $F|_{Q_e}$ or its k-smooth extension over some open neighbourhood of Q_ε is also called *the k-smooth Luzin approximation of F*. Alternatively k-quasismooth functions (k-jets) on P will be also said to have the Luzin property of order k on P.

After Luzin's fundamental structure theorem for measurable real valued functions and Egorov's theorem on pointwise convergence of sequences we shall be searching for closed subset $Q_\varepsilon = Q \subset P$, $|P \setminus Q| < \varepsilon$ (ε—small, given), such that $F|_Q$ is a k-Whitney jet.

Lemma 4. *Assume that we are given a k-jet $F \subset \operatorname{mes} J^k(P)$, and a closed compact subset Σ satisfying the following condition* (WRC-*condition, Whitney regularity condition*):

> *For any pair $x, y \in \Sigma$ the intersection set $\omega_r = S_r(x,y) \cap \Sigma$ where $S_r(x,y)$ is the spherical segment*
>
> $$S_r(x,y) = B(x,r) \cap B(y,r), \quad r = |x-y|, \tag{19}$$

[3] The term *quasicontinuous functions*—0-*quasismooth functions*—is being commonly used in recent advanced monographs on Real Analysis, see [17].

i.e. the set of all $z \in \Sigma$ such that $|z - x| \leq |x - y|$, $|z - y| \leq |x - y|$, satisfies the condition

$$\sigma = \frac{|\omega_r|}{|B(x,r)|}, \quad \sigma = \sigma(r) \geq \sigma_0, \tag{20}$$

with a positive constant σ_0 independent of r, for sufficiently small r ($r < \delta$, δ—fixed positive).

Moreover, assume that the k-jet F satisfies

$$R^k F(x,y) \leq \alpha^k(|x-y|)|x-y|^k, \quad x, y \in \Sigma. \tag{21}$$

Then the remainders $R_l^k F(x,y)$ admit uniform estimates ($x, y \in \Sigma$)

$$R_l^k F(x,y) \leq \alpha_l^k(|x-y|)|x-y|^{k-l}, \quad 0 < |l| \leq k, \tag{22}$$

and the continuity moduli $\alpha_l^k \sim \alpha_{F,\Sigma}^k$.

Proof. By Lemma 3, formulas (8), (9) and (11) applied to the triples x, y, z, $z \in \omega_r(x,y)$, and our assumption (21) imply

$$|R_l^k F(x,y)| \leq \frac{C_l}{|x-y|^l} \left(|R^k F(x,z)| + |R^k F(y,z)|\right)$$
$$\leq \frac{C_l}{|x-y|^l} \left(\alpha(|x-z|)|x-z|^k + \alpha(|y-z|)|y-z|^k\right) \leq 2C_l \alpha(|x-y|)|x-y|^{k-l}$$

by concavity of the modulus $\alpha_{F,\Sigma}$. □

Our lemma states that the general Whitney smoothness conditions (18), a priori required to hold for all l, $0 \leq |l| \leq k$, may be drastically reduced to the highest order, i.e., estimates (21) of order k, or $|l| = 0$. This was understood, probably for the first time, by G. Glaeser [21] in his result on the Converse Taylor Theorem on open subsets (cubes) of $\mathbb{R}^n$. Malgrange notes ([32], Remark 3.4) that for arbitrary closed subsets Σ this may not hold and in his presentation of the Whitney theory [32], he requires smoothness conditions (18) for all l, $0 \leq |l| \leq k$. Our lemma gives sufficient geometric conditions (20) for the set Σ admitting the reduction. It will be crucial for us that (20) holds at sufficiently close ($|x-y| \to 0$) density 1 points of Σ.

The idea to apply Markov or Markov type inequalities in the proof of the results analogous to our Lemma 4 and employ them in the study of subtle properties of k-smooth functions, $C^k(\mathcal{U})$ or Sobolev functions $W^{k,p}_{\text{loc}}(\mathbb{R}^n)$ occurred previously on several occasions. G. Glaeser in [21] invented for this purpose some kind of Lagrange interpolation lemma which we recall for completeness. We follow [1].

Interpolation Lemma. *There exist points $a_j \in \mathbb{R}^n$ and real valued polynomials $Q_j(y,a)$ in $y \in \mathbb{R}^n$, $j = 1, \ldots, (k+1)^n$ such that for every polynomial $P(y)$ of degree $\leq k$ the formula holds*

$$DP(y) = \sum_j \frac{1}{t} P(y + ta_j) DQ_j(0) \tag{23}$$

for all $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $t \neq 0$.

Formula (23) follows from the universal reproducing formula

$$P(y) = \sum_j P(a_j) Q_j(y) \quad \text{for all} \quad y \tag{24}$$

and its consequence

$$D^l P(y) = \sum_j P(a_j) D^l Q_j(y), \quad l > 0. \tag{25}$$

The discrete reproducing formula (24) requires that the universal polynomials Q_j satisfy the conditions $Q_j(a_i) = \delta^i_j$, δ^i_j—Kronecker's symbol, precisely in the same way as in the classical one-dimensional Lagrange interpolation. The interpolation lemma and the formulas (24)–(25) obviously can be considered as some kind of Markov reproducing formula.

Let us remark also that the classical finite difference calculus formulas for the Riemann derivatives of a function $f(x)$ in $C^k(\mathbb{R})$,

$$\lim_{k \to 0} \frac{\Delta_k(x,h;f)}{h^k} = D_k f(x)$$

with

$$\Delta_k(x,h;f) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f\big(x + (j - \tfrac{k}{2})h\big),$$

can be also considered as a substitute for Markov pointwise inequalities. Exactly in this role they have been used by Marcinkiewicz–Zygmund in their one-dimensional theory [33], [34] of differentiation of measurable functions.

In fact the Riemann formula is an approximate discrete interpolation formula for k-th order derivative.

Besides the discrete interpolation formulas also integral reproducing formulas have been employed for analogous goals. Let us recall here the Calderon–Zygmund reproducing formulas for polynomials of degree $\leq k$, introduced in [12], or the integral formulas discussed in [3] as a tool leading to integral Markov inequalities of the type described in our Lemma 1 or 2.

As an obvious remark we should also say that Lemmas 1–3 above for polynomials of degree $\leq k$, k finite, are nothing more than somewhat more precise formulations of the well understood general fact that all norms on a finite dimensional Banach space are equivalent.

Lemma 4 and its more refined form Lemma 5, which we formulate for sets P satisfying the very general topological and metric conditions

$$\begin{gathered} P \text{ is a perfect set (i.e., closed and without isolated points)} \\ \text{of positive measure } |P| > 0, \end{gathered} \tag{26}$$

have very far reaching consequences.

Actually we show below that most of Denjoy–Marcinkiewicz–Zygmund approximate differentiability results, proved by Denjoy–Marcinkiewicz–Zygmund for functions on subsets of $\mathbb{R} = \mathbb{R}^1$, hold for general measurable subsets of $\mathbb{R}^n$.

Lemma 5. *Assume that a measurable k-jet F defined on a compact perfect set $P \subset \mathbb{R}^n$ satisfies for $l = 0$ inequality* (16) *or* (21)*. Then for each $\varepsilon > 0$ there exist a closed subset $P_\varepsilon \subset P$, $|P \setminus P_\varepsilon| < \varepsilon$, and a positive number ρ_ε such that* (18), (22) *hold for $x, y \in P_\varepsilon$ and $|x - y| \le \rho_\varepsilon$ for all l, $|l| \le k$.*

Proof. In the notation from Lemma 4 consider the functions measurable in (x, y),

$$\widehat{\omega}_r(x, y; P) = \frac{|S_r(x, y) \cap P|}{|B(x, r)|}, \qquad r = |x - y|.$$

and the intermediate function

$$\widetilde{\omega}_r(x) = \inf_{y \in P, \rho \le r} \frac{|S_\rho(x, y) \cap P|}{|B(x, \rho)|}, \qquad \rho = |x - y|,$$
$$\widehat{\omega}_r(x, y; P) \ge \widetilde{\omega}_r(x) \quad \text{for } |y - x| = r.$$

At each density 1 point x_0 of P we have

$$\lim_{\substack{r \to 0 \\ r = |x_0 - x|}} \widetilde{\omega}_r(x) = \sigma > 0, \tag{27}$$

where σ is a positive constant, playing the role of the constant σ_0 from Lemma 4, depending on n only. The family of measurable functions $\widetilde{\omega}_r(x)$ satisfying (27) is monotone in r. Monotonicity condition makes it possible to apply Egorov's theorem to the family $\widetilde{\omega}_r(x)$, $r \to 0$, and to the family $\widehat{\omega}_r(x, y; P)$ and for each $\varepsilon > 0$ to construct the required subset P_ε, $|P \setminus P_\varepsilon| < \varepsilon$ on which the convergence (27) is uniform for $r \to 0$, and select the constant ρ_ε. We obtain thus a subset P_ε such that the estimate (20) in Lemma 4 holds with some positive $\sigma_0 > 0$. Now, application of Lemma 4 finishes the proof of Lemma 5. $\square$

3 Whitney jets on open subsets of $\mathbb{R}^n$

Here we give some direct consequences of Lemmas 1, 3, 4 for Whitney k-jets on open subsets of $\mathbb{R}^n$.

Theorem 1. *Let a measurable k-jet $F \in J^k(\mathcal{U})$ satisfy the Whitney condition* (21) *for $x, y \in \mathcal{U}$. Then F is a classical k-smooth jet on $\mathcal{U}$.*

In other words, the component $f_0(x) = f(x)$ is a k-smooth function, $f \in C^k(\mathcal{U})$, and for all $|\alpha| \le k$,

$$D^\alpha f(x) = f_\alpha(x). \tag{28}$$

Proof. Since $\mathcal{U}$ is open, then for sufficiently small r, $\omega_r = S_r(x, y)$ and thus (20) holds with some fixed σ_0. By Lemma 4 and formulas (10) we conclude that $f_\alpha(x)$, for $|\alpha| = k$, are uniformly continuous on compact subsets of $\mathcal{U}$. Iterating this we conclude that all components f_α of the k-jet F, for $|\alpha| < k$, have continuous classical derivatives and (28) holds for all $|\alpha| \le k$. $\square$

In particular, we have the classical formulas

$$D_x^{\alpha - \beta} f_\beta = f_\alpha \quad \text{for all } \beta,\ \beta < \alpha. \tag{29}$$

Corollary 1. *If a measurable k-jet* $F \in J^k(\mathcal{U})$ *satisfies the condition*

$$R^k F(x, y) \equiv 0 \quad \text{for all } x, y \in \mathcal{U}$$

then the initial term f of the k-jet F is a polynomial of degree $\leq k$.

As already remarked by Marcinkiewicz–Zygmund [33], [34], if the measurability assumption is dropped then the corollary and the whole DMZ theory fails (Hamel basis on $\mathbb{R}$!).

4 Denjoy–Marcinkiewicz–Zygmund theory

It is convenient and useful to represent the local Taylor fields $T_y^k F(x)$ and Taylor remainders $R^k F(y, x)$ in terms of the variables $(y, y + h)$, $x = y + h$, h "small". This is also in better agreement with the tradition [33], [34], [29], and makes easier the references to most of the existing literature (mainly for functions of one variable [18]). We recall the classical Taylor formula for functions $f(x) \in C^k(\mathcal{U})$, $x \in \mathcal{U}$, $\mathcal{U}$—open in $\mathbb{R}^n$, $f^r(x) \equiv D^r f(x)$, $|r| \leq k$, $f^0(x) = f(x)$.

$$f(x + h) = \sum_{r=0}^{k} \frac{f^r(x)h^r}{r!} + \widehat{R}^k f(x, h)h^k \tag{30}$$

$$\widehat{R}^k f(x, h) = \int_0^1 \frac{(1 - t)^{|k|-1}}{(k - 1)!} \left[f^k(x + th) - f^k(x)\right] dt. \tag{31}$$

It is also useful to employ the normalized Taylor remainders $e^k f(x, h) = e^k(x, h)$, if the function f is fixed, defined by

$$e^k f(x, h)|h|^k \doteq \widehat{R}^k f(x, h)h^k. \tag{32}$$

Since in the classical case of k-smooth functions on open subsets of $\mathbb{R}^n$ the derivatives f^r are $(k - |r|)$-smooth,

$$f \in C^k(\mathcal{U}) \Rightarrow f^r(x) = D^r f(x) \in C^{k-|r|}(\mathcal{U}),$$

it is obvious that formulae (30) are intimately connected with the formulae

$$\begin{gathered} f^l(x + h) = \sum_{j=0}^{k-|l|} \frac{f^{l+j}(x)h^j}{j!} + \widehat{R}_l^k f(x, h)h^{k-l}, \quad |l| \leq k, \\ e_l^k f(x, h)|h|^{k-l} \doteq \widehat{R}_l^k f(x, h)h^{k-l}. \end{gathered} \tag{33}$$

The remainders $\widehat{R}^k f(x, h)$, $\widehat{R}_l^k f(x, h)$ satisfy the conditions

1. They are continuous in (x, h) for $|h| \leq \delta$, δ small
2. $\widehat{R}_l^k f(x, 0) = 0, \quad \widehat{R}_l^k f(x, h) = o(1), \ |h| \to 0$

(34)

This holds for the normalized remainders $e_l^k(x,h)$ if, by definition, we set $e_l^k(x,0)=0$. In (32) and (33) $e_l^k(x,h)$ is defined for $|h|>0$ only.

On compact subsets $\Sigma\subset\mathcal{U}$, more precise formulation can be given

$$|\widehat{R}_l^k f(x,h)|\le\alpha_\Sigma^l(|h|),\quad x\in\Sigma,\ |l|=0,\dots,k, \tag{35}$$

for some concave functions $\alpha_\Sigma^l(t)$, $t>0$, $\alpha_\Sigma^l(0)=0$, depending on f in general.

All these facts are direct consequences of the explicit formulae for the Taylor remainders of type (31) or (33).

In the literature various generalized concepts of differentiation have been proposed based on formula (30) and the Taylor remainder estimates (34) and (35) for $h\to 0$. If for a (measurable) function $f(x)$ defined in a neighbourhood $\mathcal{U}_x$ of x the "values" $f_r(x)$ in the multilinear matrix sense, $|r|\le k$, can be assigned such that

$$f(x+h)=\sum_{r=0}^{k}\frac{f_r(x)h^r}{r!}+R^k f(x,h),\quad f_0(x)\equiv f(x), \tag{36}$$

for $x+h\in\mathcal{U}$, i.e., $|h|$ small, then the polynomial in h in (36) is the total k-Peano derivative (also called de la Vallée–Poussin derivative) of the function f at x (F. Ch. Liu [30], [31] uses the term k-order Taylor derivative, in the sequel we alternatively use both terms). We also write $f_r(x)=\mathrm{ap}D^r\mathrm{P}f(x)$, $|r|\le k$.

If in (36) the points $h\in\mathbb{R}^n$, $|h|\to 0$, are restricted to a subset $H(x)$ with density 1 at the point x, then we speak about *approximate* (approximative) *derivatives* (derivative along the set $H(x)$, varying with x, denoted ap.der., ap.lim. etc.), see e.g. [36]. The concepts of approximate derivatives were introduced by A. Denjoy [15] and A. Khintchine [27] independently at the beginning of the XX century (∼1915). A. Khintchine used the term *asymptotic derivative*.

It is rather obvious that, if meaningful, the total k-Peano derivative, i.e., all terms $f_r(x)$, $|r|\le k$, are uniquely determined.

If, in analogy with (30) and (33) we would like to extend formula (36) to the formulas

$$f_l(x+h)=\sum_{j=0}^{k-|l|}\frac{f_{l+j}(x)h^j}{j!}+R_l^k f(x,h), \tag{37}$$

$$|R_l^k f(x,h)|=o(|h|^{k-l}),\quad |h|\to 0, \tag{38}$$

$$e_l^k f(x,h)|h|^{k-l}\equiv|R_l^k f(x,h)|,\quad |h|>0,$$

$$e_l^k f(x,h)\le\alpha_{l,\Sigma}^k(|h|),\quad x\in\Sigma \tag{39}$$

for some concave functions $\alpha_{l,\Sigma}^k(t)$, $\alpha(0)=0$,

then we would be faced with the discussion of the existence problem of $(k-|l|)$-Peano derivatives of the terms f_l for various values of l, $|l|<k$, at

the point x etc. Though, up to my knowledge, there are practically no publications available up to most recent times on these topics in the area of real analysis in $\mathbb{R}^n$, there is an immense, and permanently growing in volume number of papers on differentiation of functions on the real line ($n = 1$). It would be very difficult to review these papers—cf. the papers [10], [11], [18], [20], [45] and the numerous references quoted therein! The amount of conceptual variations and counterexamples to various possibilities here is enormous.

It is our good luck that the concept of approximate Peano differentials of higher order, as proposed in $\mathbb{R}^n$ by Federer and Whitney, and essentially initiated by Marcinkiewicz–Zygmund in 1935 [33], [34], allows to control the situation precisely as is the case with the notion of approximate continuity a.e. in the Luzin–Egorov–Denjoy structure theorems for measurable functions. The general answer is sketched in our Theorem 4 below, Section 4.2.

In terms of k-jets F, $F \subset J^k(\Sigma)$, introduced above, defined on an arbitrary subset Σ of $\mathbb{R}^n$, $\Sigma \subset \mathbb{R}^n$, $F = (f_0(x), f_1(x), \ldots, f_k(x))$, expressions (36), (37) with the conditions (38) can be considered as Taylor formulas of order k for the k-smooth real valued functions on Σ (or a smooth mapping $f : \Sigma \to V$ of Σ into some finite dimensional vector space over the reals). They should be considered as expressing the necessary conditions to be satisfied by "k-smooth functions" (mappings) on Σ, whatever meaning the term "k-smooth on Σ" could have. The simplest and most natural should be: f is k-smooth on Σ if f is the restriction of a $g \in C^k(\mathcal{U})$ for some open set $\mathcal{U}$ containing Σ strictly in the interiour

$$f = g|_\Sigma \quad \text{or} \quad J^k(g)|_\Sigma = F, \quad \Sigma \Subset \mathcal{U},\ F \subset J^k(\Sigma). \tag{40}$$

4.1 Marcinkiewicz-Zygmund converse Taylor theorems

The most general "existence" problem of DMZ theory is the description of the "simplest conditions" which a k-jet F on Σ, $F \subset J^k(\Sigma)$ should satisfy so that it may be identified with some $J^k(g)$, $g \in C^\infty(\mathbb{R}^n)$. The possible answers are called converse Taylor theorems (or CTT for short). This term was probably first used by Abraham–Robbin in [1] and repeated in [29].

In the rather trivial classical case $\Sigma = \mathcal{U}$, $\mathcal{U}$—open subset of $\mathbb{R}^n$, $F \in \operatorname{mes} J^k(\mathcal{U})$, two simplest examples occur:

Theorem 2. *Let F be a measurable k-jet on $\mathcal{U}$ satisfying all conditions* (36), (37) *and* (38),

$$e_l^k F(x,h) \to 0 \quad \textit{for} \quad (x,h) \to (x_0, 0) \ \textit{and all}\ |l| \le k. \tag{41}$$

Then the component $f_0 \equiv f(x) \in C^k(\mathcal{U})$ and it generates the k-jet F, i.e.,

$$f_\alpha(x) = D^\alpha f(x) \quad \textit{for all} \quad |\alpha| \le k. \tag{42}$$

In particular,

$$f_\beta = D^{\beta-\alpha} f_\alpha \ \textit{in}\ \mathcal{U}\ \textit{for}\ \beta \ge \alpha,\ |\beta| \le k. \tag{43}$$

Proof. Direct checking: (36), (37) imply for $h \to 0$, continuity of all $f_l(x)$ and the differentiability $f_{l+1}(x) = Df_l$ for all l, $|l| \leq k$. Iterating we get (42) and (43). □

Theorem 3. *Let F be a measurable k-jet in $\mathcal{U}$ satisfying* (36) *with the condition*

$$e^k F(x,h) \to 0 \quad \textit{for} \quad (x,h) \to (x_0, 0). \tag{44}$$

Then $f \in C^k(\mathcal{U})$ and $f_l = D^l f$ for all $|l| \leq k$.

Proof. By our Lemma 4 condition (44) implies (41) and the proof reduces to the proof of Theorem 2. For the case of continuous k-jets Theorem 3 was proved by Glaeser in [21]. See also [1]. Instead of the Markov inequality argument employed in our proof of Lemma 4 he uses some kind of multivariate Lagrange interpolation lemma—recalled above—allowing to reproduce an arbitrary polynomial $P(y)$ in $\mathbb{R}^n$, and consequently the derivatives $DP(y)$ at an arbitrary point $y \in \mathbb{R}^n$ from its values at a sequence of suitably chosen points $a_j \in \mathbb{R}^n$, $j = 1, \ldots, (r+1)^n$. Glaeser's proof is presented in detail in [1]. □

4.2 DMZ theory on arbitrary measurable sets

As a main result of DMZ theory in n variables we have the following Theorem 4. It is the n-dimensional extension of the Denjoy–Marcinkiewicz–Zygmund main results on approximate differentiability of measurable functions on arbitrary measurable subsets of the real line [16], [33], [34].

Theorem 4. *Let $f(x)$ be a measurable function defined on a measurable set $P \subset R^n$, $|P| > 0$ (P-perfect), approximately k-th Peano differentiable* (*in the sense of formal definition* (36)–(37)) *with the k-th normalized remainder $e^k f(x,h)$ such that*

$$\operatorname{ap}\lim_{h \to 0} e^k f(x,h) = 0 \quad \textit{for a.e.} \quad x \in P.$$

Then the (*vector valued*) *l-th Peano derivatives $f_l(x)$ in* (35) *are approximately $(k - |l|)$-th Peano differentiable a.e. and the iterative formulas*

$$\operatorname{ap} D^j P f_l(x) = f_{l+j}(x), \quad |j| \leq k - |l| \tag{45}$$

hold for almost a.e. point of P.

Proof. The auxiliary results and concepts described in Section 2 above give us all tools sufficient to describe the complete proof of Theorem 4. Especially helpful in this respect are our Lemmas 4 and 5. However here, instead of following that route, we consider worthwhile to expose the fact that apparently the shortest way to Theorem 4 is to use the point III of our main Theorem 5 below. Here is the rather precise sketch of this proof: It is immediate that almost each point x of P is a density point of some Q_{i_0}, $i_0 = i_0(x)$, in the decomposition $\{Q_i\}$ of Theorem 5 III. The k-jet $F|_{Q_{i_0}}$ is then generated by the k-jet of the Whitney k-smooth function $g|_{Q_{i_0}}$ on Q_{i_0}, $g \in C^k(U)$, $x \in U$—open set. In particular the approximate Peano derivatives of f at the point x are equal to the classical derivatives of g at x, for which the iterative formulas (45) hold. This is precisely the way how J. Marcinkiewicz proved in 1936, [33], his main Theorem 3 and its Corollary II. 10, though he did not explicitly formulate the decomposition III, which for $k = 1$, was given as late as in 1951 by H. Whitney [48].

5 Whitney–Luzin theory

Whitney–Luzin theory for measurable k-th Peano differentiable functions is an alternate version of DMZ-theory.

It is the central topic of the present paper. It seems proper therefore to recall the precise definition. Let f be a measurable (real-valued) function on a measurable set $P \subset \mathbb{R}^n$. Let k be an integer, $k \geq 1$.

Definition 4. f is *k-Peano differentiable*—or *Peano differentiable of order k*—at the point $y \in P$ if there exists a k-jet $F = (f_0, \ldots, f_\alpha)$, $|\alpha| \leq k$, $f_0(y) = f(y)$, such that for all $h \in \mathbb{R}^n$ such that $y + h \in P$

$$\begin{aligned} f(y+h) &= \sum_{|\alpha|=0}^{k} \frac{f_\alpha(y)h^\alpha}{\alpha!} + R^kF(y, y+h) \\ &\equiv T^kF(y,h) + R^kF(y,y+h) \qquad (46) \\ |R^kF(y,y+h)| &= o(|h|^k) \\ \text{or } R^kF(y,y+h) &= e^kF(y,h)|h|^k \quad \text{with} \quad |e^kF(y,h)| = o(1),\ |h| \to 0. \end{aligned}$$

Equivalently,

$$\lim_{|h|\to 0} e^kF(y,h) = 0 \quad (y+h \in P). \qquad (47)$$

If for a given y, $y \in P$, there exists a set $H(y)$ of points of the form $y + h \in P$ such that $H(y)$ has density 1 at y, and

$$\lim_{|h|\to 0, y+h\in H(y)} e^kF(y,h) = 0, \qquad (48)$$

we say that the Taylor k-order polynomial defined by (46) or the k-jet $F = (f_0, \ldots, f_\alpha(y), \ldots, f_k)$ is the approximate total k-th Peano derivative of f at y, and define

$$F = \mathrm{apP}^kDf(y).$$

It is uniquely determined at the (positive density) points of P.

The class of measurable functions on P, admitting approximate k-th Peano derivatives at each point of the set P, will be denoted by $\mathrm{apP}^kD(P)$. In the sequel we freely identify the function f in $\mathrm{apP}^kD(P)$ with the corresponding k-jet F of its k-th Peano derivatives $f_\alpha(y)$, $y \in P$.

Theorem 5. *Let F be a measurable k-jet, $F \subset \operatorname{mes} J^k(P)$ on a perfect set P, $|P| > 0$. Then the following three conditions are equivalent*:

I. *F has an approximate total k-differential at a.e. point.*

II. *F is k-quasismooth on P.*

III. *There exists a sequence of disjoint closed subsets $Q_1, Q_2, \dots$*[4] *such that $P = Z \cup \bigcup Q_i$, $|Z| = 0$ and $F|_{Q_i} \in C^k(Q_i)$ for each i.*

Proof. Together with the remainders $R_l^{k-|l|}F(y,x)$ it is convenient to introduce the normalized remainders $e_l^k F(y,x)$,

$$|x-y|^{k-|l|} e_l^k F(y,x) \equiv R_l^{k-|l|}F(y,x), \quad e_0^k F \equiv e^k F.$$

We agree that $e_l^k F(x,x) \equiv 0$. The remainders $R_l^{k-|l|}F(y,x)$ and $e_l^k F(y,x)$ are measurable functions in the pair of variables x, y from the cartesian product $P \times P$.

I $\Rightarrow$ II. The assumption implies that for a.e. point $x \in P$ and each $\varepsilon > 0$ the set

$$H(x,\varepsilon) = \{t \in P : e^k F(x,t) < \varepsilon\} \tag{49}$$

has density 1 at x. The same holds at the points y, and for sufficiently small δ the intersection set[5]

$$\omega_r(x,y) = \big[H(y,\varepsilon) \cup H(x,\varepsilon)\big] \cap S_r(x,y), \quad r < \delta, \tag{50}$$

is a portion of $P \cap S_r(x,y)$ such that for $t \in \omega_r(x,y)$, $e^k F(x,t) \le \varepsilon$ and $e^k F(y,t) \le \varepsilon$,

$$|\omega_r| \ge \sigma |P \cap S_r(x,y)|$$

with σ independent of r, x, y. Hence for $x, y \in P$, $|x-y| < \delta$, by Lemma 3

$$\|R_l^{k-|l|}F(y,x)\| \le C_l \varepsilon |x-y|^{k-|l|}, \quad |l| \ge 1, \ |x-y| < \delta, \tag{51}$$

where C_l depends on k, n, l, σ only.

To prove II we have to construct, for any given $\varepsilon > 0$, a closed subset $Q = Q_\varepsilon \subset P$ such that $|P \setminus Q| < \varepsilon$ and for each $\varepsilon' > 0$ there is a $\delta > 0$ such that

$$|R^k F(y,x)| \le \varepsilon' |x-y|^k, \quad x, y \in Q, \tag{52}$$

and $F|_Q$ is continuous.

Let the positive number $a > 0$ be defined by

$$|S_r(x,y)| = 2a|B_r(x)|. \tag{53}$$

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[4] Decomposition III depends on the function f of the k-jet F. It realizes what can be called a Whitney smooth free interpolation of F by Whitney k-smooth functions.

[5] It was already Stepanoff [42] who, long before Whitney [48], employed the spherical segments $S_r(x,y)$ in his studies of approximate differentiability of approximately Lipschitz functions.

By similarity the number a does not depend on r. We take $r = |x - y|$. Consider a sequence $r_i \to 0$, e.g. $r_i = \frac{1}{i}$, $i = 1, 2, \ldots$. For $x \in P$ set $V_i = |B_{r_i}|$ and

$$\psi_i(x, \eta) = |B_{r_i}(x) \setminus H(x, \eta)|. \tag{54}$$

Then for each η

$$\frac{\psi_i(x, \eta)}{V_i} \to 0 \quad \text{as} \quad i \to \infty. \tag{55}$$

Define also a real valued function

$$\phi_i(x) = \underset{\eta}{\text{g.l.b.}}\{\psi_i(x, \eta) < aV_i\}. \tag{56}$$

$\psi_i(x, \eta)$ is measurable in x for fixed η, and for fixed x it is decreasing in η, continuous on the left. Thus

$$\phi_i(x) \leq \eta \quad \text{iff} \quad \psi_i(x, \eta) < aV_i \tag{57}$$

and $\phi_i(x)$ is measurable.

From (55) and (57) it follows that

$$\lim_{i \to \infty} \phi_i(x) = 0 \tag{58}$$

at each point of the set P_1 of approximate total differentiability of F.

By Luzin's and Egorov's theorems there is a closed set $Q \subset P_1$ satisfying $|P_1 \setminus Q| = |P \setminus Q| < \varepsilon$, and such that $F|_Q$ is continuous on Q and $\phi_i(x) \to 0$ uniformly in Q.

We have to prove that for each $\varepsilon' > 0$ there is a $\delta > 0$ such that

$$\|R^k F(x, y)\| \leq \varepsilon' |x - y|^k \quad \text{if} \quad x, y \in Q, \ |x - y| < \delta. \tag{59}$$

Setting $\varepsilon_1 = \frac{\varepsilon'}{C}$ (C—an absolute constant to be fixed later) we may choose δ so that

$$\begin{aligned} &|R_l^{k-|l|} F(x, y) v^l| \leq \varepsilon_1 |v|^l |x - y|^{k-|l|} \\ &|R_l^{k-|l|} F(y, x) v^l| \leq \varepsilon_1 |v|^l |y - x|^{k-|l|} \\ &|R_l^{k-|l|} F(y, x) v^l| \leq \varepsilon_1 |v|^l \quad \text{if} \quad x, y \in Q, \ |x - y| < 2\delta, \end{aligned} \tag{60}$$

and

$$\phi_i(x) < \varepsilon_1 \quad \text{for} \quad x \in Q, \ \tfrac{1}{i+1} < \delta. \tag{61}$$

Now take $x, y \in Q$ with $|x - y| < \delta$. Let j be the largest integer such that $\frac{1}{j} \geq |y - x|$ and consider the spherical segment $S_j = S_{r_j}$. Since $\frac{1}{j+1} < |y - x| < \delta$, (61) and (57) give

$$\psi_j(x, \varepsilon_1), \ \psi_j(y, \varepsilon_1) < aV_j.$$

Therefore there is a point z in $S_j \cap P$ belonging neither to $B_i(x) \setminus H(x, \varepsilon_1)$ nor to $B_i(y) \setminus H(y, \varepsilon_1)$. Thus for this z

$$|R^k F(x, z)| < \varepsilon_1 |x - z|^k \quad \text{and} \quad |R^k F(y, z)| < \varepsilon_1 |y - z|^k. \tag{62}$$

We are now in the position to finish the proof of (59) by the use of our basic identity (11). Indeed, using (11), (51), (60) for $v = z - y$ and (62) with the found intermediate point z we obtain

$$|R^kF(x,y)| \le |R^kF(x,z)| + |R^kF(y,z)| + \sum_{1\le|l|\le k} \left|\frac{(z-y)^l}{l!}\, R^k_l F(x,y)\right|$$

$$\le \varepsilon_1|x-z|^k + \varepsilon_1|y-z|^k + \sum_{1\le|l|\le k} \frac{|z-y|^l}{l!}\, |R^k_l F(x,y)|$$

$$\le 2\varepsilon_1|x-y|^k + \sum_{1\le|l|\le k} |x-y|^{|l|}|x-y|^{k-|l|}\,\frac{2\varepsilon_1 C_l}{l!} = 2\varepsilon_1\Big(1 + \sum_{1\le|l|\le k}\frac{C_l}{l!}\Big)|x-y|^k$$

which, with the choice $C = 2\big(1 + \sum_{|l|\ge 1}\frac{C_l}{l!}\big)$ gives (59).

Thus we have found the closed set Q such that F is Whitney k-smooth on Q, $|P \setminus Q| < \varepsilon$ and II holds.

Proof II $\Rightarrow$ III goes along a rather standard way. We successively construct disjoint closed sets $Q_1, Q_2, \ldots$ so that F is Whitney k-smooth in each of them and $|P_i| \le \frac{|P|}{2^i}$ where

$$P_i = P \setminus \bigcup_{k=1}^{i} Q_k.$$

Set $Q_1 = Q_\varepsilon$, $\varepsilon = \frac{|P|}{2}$ and having found $Q_1, \ldots, Q_{i-1}$ choose a closed Q_i' so that F is k-smooth in Q_i' and $|P \setminus Q_i'| \le |P|/2^{i+1}$. Let $\mathcal{U}_\delta(\widehat{Q})$ be the open δ-neighbourhood of $\widehat{Q}$. Then for δ small enough we can set $Q_i = Q_i' \setminus \mathcal{U}_\delta\big(\bigcup_{k=1}^{i-1} Q_k\big)$. With this choice III obviously holds.

III $\Rightarrow$ I. Suppose III holds. Let Q_i^* be the set of density points of Q_i and set $Q^* = \bigcup_{i=1}^{\infty} Q_i^*$. Then $|P \setminus Q^*| = 0$ and it is immediate that F is approximately totally k-differentiable at any point $x \in Q^*$, or $x \in Q_{i_0}^*$ for some i_0, since being a density 1 point of $Q^* \subset P$ x is also a density 1 point of P and I holds. □

The proof presented above is deliberately organized in detailed analogy to the original proof of Whitney to his Theorem 1 in [48]; we even use the same notation for functions ϕ_i, ψ etc. It is worthwhile to stress that the proof above operates for the fixed k directly. The proof of F. Ch. Liu [30] proceeds by induction on k and, as already remarked in the introduction, this inductive process should be performed very carefully. In somehow related proof of his Theorem 4 in [48] H. Whitney additionally assumes that the considered $(k-1)$-jet F is defined on an open neighbourhood $\mathcal{U}$ of the set P and a.e. totally differentiable in $\mathcal{U}$. In the light of Movshovich's counterexample in [35] the inductive step should be performed with special care.

Also the discussed questions are somehow related with points 3.1.15, 3.1.16 and 3.1.17 in Chapter 3 of Federer's famous book [19].

For completeness we formulate also

Theorem 6. *A k-jet F satisfying the assumptions of Theorem 5 has approximate partial k-th order derivatives a.e. iff F has an approximate total k-differential a.e.*

This follows from a general theorem of S. Saks [38] to the effect that in the framework of approximate differentiability the mixed partial derivatives, though they may differ at individual points, coincide a.e.

We also have

Corollary 2. *A $(k-1)$-jet F approximately Lipschitz at a.e. point of a perfect set P of positive measure has an approximate total differential a.e. in P.*

This is Stepanoff's theorem on approximate total differentiability [42]. See [31].

Actually in the course of the above proof of Theorem 5, II, we proved II only in the sense of Whitney k-smoothness of F on a closed subset $Q_\varepsilon \subset P$ such that $|P \setminus Q_\varepsilon| < \varepsilon$. If we apply the full strength of WhET from Section 2 we can formulate the corollary.

Corollary 3. *In the conditions of Theorem 5 for each $\varepsilon > 0$ there exists a smooth k-jet G (i.e. G is generated by a smooth function $g \in C^k(\mathbb{R}^n)$) such that $F|_{Q_\varepsilon} \equiv G|_{Q_e}$.*

6 Conclusions and final remarks

Luzin's structure theorem ([17, p. 133]) plays a crucial role in our proof of Theorem 5 above. To see this theorem from a broader perspective we recall the main points of Luzin's theory.

Let P be a measurable subset of $\mathbb{R}^n$ and $\mathcal{M}(P)$ the class of real valued, finite a.e., measurable functions on P.

Theorem 7.[6] *$f \in \mathcal{M}(P)$ iff f is quasicontinuous on P (i.e. 0-quasismooth).*

Closely related with Theorem 7 is Egorov's theorem (∼1912).

Theorem 8. *Let f_n be a sequence of functions in $\mathcal{M}(P)$ converging pointwise a.e. on P, $f_n \to f$ a.e. Then for every $\varepsilon > 0$ there exists a closed set P_ε such that*

$$|P \setminus P_\varepsilon| < \varepsilon$$

and, for a subsequence f_{n_k}, the convergence of f_{n_k} is uniform on P_ε: $f_{n_k} \to g_\varepsilon$ and $g_\varepsilon \in C(P_\varepsilon)$.

Notice that by the Tietze extension theorem Egorov's Theorem 8 implies Theorem 7.

Another deep fact of Luzin's theory is

[6] N. N. Luzin, ∼1912.

Theorem 9 (Luzin, Denjoy)**.** *$f \in \mathcal{M}(P)$ iff f is approximately continuous at almost every point of P.*

Theorems 7 and 9 combined give

Theorem 10 (Luzin)**.** *A measurable function $f \in \mathcal{M}(P)$ is quasicontinuous on P iff f is approximately continuous at almost every point $x \in P$.*

Theorem 10 can be interpreted as an infinitesimal (pointwise, local) description of 0-quasismoothness (quasicontinuity).

A geometric description of continuity behaviour of a quasicontinuous function on a measurable set P may be also described by the decomposition of P into a countable family of disjoint closed subsets and a null-set.

Theorem 11. *Let $f : P \to \mathbb{R}$ be quasicontinuous, then P may be decomposed*

$$P = Z \cup \sum_{i=1}^{\infty} P_i \tag{63}$$

into an at most countable family of closed disjoint sets and a set Z such that $|P_i| > 0$ and $|Z| = 0$ and $f|_{P_i} \in C(P_i)$.

The function f is then approximately continuous at each density 1 point of some P_i.

The decomposition (63) may be called the *Luzin decomposition* of P for the given $f \in \mathcal{M}(P)$.

Actually each of the assertions of Theorems 7–11 can be viewed as a characterization of measurability.

The equivalence of statements 7–11 above may be then formulated in the following Luzin–Denjoy theorem.

Theorem 12. *For any measurable set $P \subset \mathbb{R}^n$ and any real-valued function $f : P \to \mathbb{R}$ the following are equivalent*

a) *$f \in \mathcal{M}(P)$,*

b) *f is quasicontinuous on P,*

c) *f induces a countable Luzin decomposition of P,*

d) *f is approximately continuous at almost every point of P.*

For the classical proofs of this fundamental theorem see e.g. Natanson [36] or Federer [19] or hundreds of other papers. They all study the behaviour of local oscillations of f measured by the 0-order remainder term $Rf(x; y) = f(x) - f(y)$. The proof, e.g. see [19], may be organized as a sequence of implications a) $\implies$ b) $\implies$ c) $\implies$ d) $\implies$ a)[7]. In hundreds of papers devoted to various aspects of Theorem 12 this fundamental fact of real analysis was brought to its natural borders of generality when $f : X \to Z$ is a

[7] even though the precise formulation of c) is, or may be, missing in [19].

mapping of measure metric space X into a separable metric space Z. In this context it is presented in [19] with the remainder estimated as $Rf(x;y) = \rho(f(x), f(y))$, where ρ is the notation for the metric in Z.

Now our Theorem 5 is the precise extension of the Luzin–Denjoy theorem to the subclass of k-quasismooth functions of $\mathcal{M}(P)$ characterizing it as the class ap$\mathrm{P}D^k(P)$ of approximately k-Peano differentiable functions at almost every point of the perfect set P. Both theorems have the same simplicity and, in some sense, very similar conceptual structure. Also in either case the notion of approximate continuity or total differentiability cannot be avoided.

k-quasismooth functions on P, by definition, are restrictions to closed subsets of P_ε. In particular, obviously, for any k, k-quasismooth function on the perfect set P can be extended to k-quasismooth function on the whole enhancing space.

Apparently the notion of k-quasismoothness ($k \geq 0$) gives more precise geometric information about the behaviour of the function than the classical k-th or even k-Peano derivative. It is well known that the classical k-th derivative and the k-th Peano derivative for $k \geq 2$ are not comparable, i.e., the existence of the one at a point does not imply the existence of the other at the same point or even a.e. It was also some surprise to real analysis specialists on differentiation [11], [18], [45][8] when Buczolich in 1988 published [10] his example of 2-Peano differentiable function on a measurable closed perfect subset H of the real line $\mathbb{R}$ which is not extendable to the 2-Peano differentiable function on the whole $\mathbb{R}$. This is in contrast with the V. Jarnik result of 1923 (we refer also to [45]) that 1-Peano differentiable≡1-classical differentiable functions on an arbitrary perfect set $H \subset \mathbb{R}$ are extendable with the same first derivative to the whole $\mathbb{R}$. Together with the Whitney result ([48], Theorem 1) at that time not available for $k > 1$, all these results were apparently confirming the view that the order of differentiation $k = 2$ creates a kind of threshold in the qualitative global understanding of the differentiation problem in dimension $n = 1$ and $n > 1$ as well. Apparently the longstanding query—conjecture of H. Federer in [19, 3.16–3.17], supported also by the Movshovich [35] example of a $C^1[0,1]$ function $f(x)$ with the second approximate derivative ap$f''(x) \equiv 0$ on a perfect set $A \subset [0,1]$, with meas$([0,1] \setminus A)$ arbitrarily small, contributed to considering this "analysis folklore idea" as an established "fact". As a consequence, the isolated publications and seminar lectures, [30], [31], [4], which expressed the contrary idea, were unnoticed or accepted with doubt, perhaps due to their insufficiently detailed presentations.

We stress here once more the opinion that the papers [30], [31], especially when properly complemented, maybe also with the natural application of Theorem 4 in Whitney's fundamental paper [48] apparently give a complete proof of our Theorem 5. The proof of Theorem 5 presented in detail here, and schematically discussed and proposed in [4], realizes for $k \geq 2$ the alternate idea initiated by H. Whitney in [48], using the methods elaborated also by

[8] and the vast literature recursively quoted in those papers

his followers, B. Malgrange [32] et al., including the work of F. Ch. Liu and W. Tai [31].

In our discussion above we "tacitly" assumed that all involved functions and maps are measurable: the main reason for that was to avoid undue technical complications at the initial stage of the study. As a matter of fact it is feasible to start with "arbitrary" functions and formulate later additional assumptions implying measurability, e.g., of the consecutive approximate Peano derivatives $f_k(y)$. Thus in [48] Whitney assumes that the Taylor remainders $R^k f(y;x)$ are measurable functions on the cartesian product $P \times P$ and deduces, via Fubini type theorems, measurability of approximate derivatives $f_k(y) = \mathrm{P}D^k f(y)$ (for $k = 1$). The measurability of higher order Peano derivatives ($k \geq 2$) on subsets of $\mathbb{R}^n$ is a more subtle problem, though it also follows from measurability of the normalized remainders $e^k f(y;x)$, and their vanishing, in the sense of Egorov theorem, at the diagonal $\Delta \subset P \times P$,

$$\lim_{y \to x} e^k f(y;x) = 0, \quad (y,x) \in P \times P. \tag{64}$$

This means, by definition, that for each $\varepsilon > 0$, the convergence (64) holds uniformly on some closed bounded subset $P_\varepsilon \subset P$, for which $|P \setminus P_\varepsilon| < \varepsilon$ (i.e., on $P_\varepsilon \times P_\varepsilon$).

The methods proposed in this paper, apparently, allow to obtain the following form of our main Theorem 5 for arbitrary functions, unifying the Whitney theory and the classical Luzin theory.

Theorem 13. *Let P be a measurable subset of $\mathbb{R}^n$. For any (real valued) function $f(x)$ on P and any integer $k \geq 0$ the following holds: $f(x)$ is k-quasismooth on P iff f is approximately k-Peano differentiable at almost each point of P.*

In particular for $k = 0$ a quasicontinuous (0-quasismooth) function is approximately continuous at a.e. point and, thus, measurable. This seems to be the appropriate form of the classical Luzin theorem for a priori not necessarily measurable functions. Also part III of Theorem 5 holds for quasicontinuous functions.

Measurability of approximate k-th Peano derivatives for functions on the real line follows from the results of Marcinkiewicz–Zygmund [33], [34], where the Peano derivatives $f_k(y)$ are identified with the Riemann derivatives or limits of $\Delta^k f$, expressed as finite difference quotients, which obviously preserve measurability. For higher dimensional differentiation processes apparently analogous formulas hold, though, probably, it would be difficult at this moment to give the precise reference, cf. [1]. The Glaeser interpolation formulas quoted above [1] may be useful in the precise proof of this fact.

Approximate total differentiability is a much weaker property than classical (Fréchet) total differentiability though, when both exist, they coincide. Therefore it is of interest to ask about additional conditions of global and infinitesimal character which imply total differentiability a.e. The classical general theorem is Rademacher–Stepanoff theorem on a.e. differentiability of locally Lipschitz functions (maps). Higher order Rademacher–Stepanoff theorems have been also published in [31]. However, a simple geometrical characterization of functions admitting a.e. a total Peano differential is apparently missing.

An interesting insight into "smoothness" and pointwise differentiability of functions (maps) is obtained when intersection sets of their graphs are considered. Typically, given two function spaces $\mathcal{F}_1 = \mathcal{M}(\Omega)$ and $\mathcal{F}_2 = C(\Omega)$ with $f \in \mathcal{F}_1$ measurable and $g \in \mathcal{F}_2$ continuous, the intersection set $E_{f,g}$ is defined as $E_{f,g} = \{x \in \Omega : f(x) = g(x)\}$.

With $A = E_{f,g}$ given and $\mathcal{F}_2 \subset \mathcal{F}_1$ we say that g (a "good", "smoother" function) interpolates a "bad" function on the set A or that f restricted to A, $f|_A$, extends (extrapolates) to g. In the classical interpolation problem the "irregular" function $f \in \mathcal{F}_1$ and the interpolation set A are given and, what is looked for, is the interpolating "regular" function $g \in \mathcal{F}_2$. We speak about "free" interpolation when the interpolating set A, possibly well "approximating" the domain of f, is also to be defined. In Luzin's theory A is required to be a closed compact subset of Ω and the answer is that for any $\varepsilon > 0$ the free interpolation is possible on some closed set A_ε approximating Ω in measure or such that $|\Omega \setminus A_\varepsilon| < \varepsilon$. Obviously the free interpolation problem admits a rich variety of natural (and interesting) modifications by varying the smoothness classes of $\mathcal{F}_1$ and $\mathcal{F}_2$ and the measure of "massiveness" of the coincidence set.

Besides Luzin's theory also the Ulam–Zahorski question (1951) on the free interpolation of a continuous function $f \in C[0,1]$ by a real analytic function g on some nonempty perfect set $A \subset [0,1]$ was classical. Zahorski showed that the answer is negative. However, as late as in 1984 Lachkovich showed that any continuous function in $C[0,1]$ can be free interpolated by a $C^\infty(H)$-function on some nonempty perfect set H. See [45] and the references quoted there. On the other hand, Olevski and Weil in 1995 constructed an everywhere differentiable real valued function on the interval $I = [0,1]$ which cannot be free interpolated by any twice Peano differentiable function on any nonempty perfect subset of I. Also the Movshovich paper [35], quoted above, constructs a monotone function $f \in C^{1,\alpha}(I)$ for some (any) $0 < \alpha < 1$, such that $E_{f,g}$ is an isolated subset of the open interval $(0,1)$ for any $g \in C^2(I)$, or even $g \in C^{1,1}(I)$. However, f is not approximately twice Peano differentiable a.e. on I!

It is important to notice that by varying the classes $\mathcal{F}_1$ and $\mathcal{F}_2$ interesting and, in general, nontrivial problems of free interpolation type arise. Thus, if instead of the class of k-smooth functions C^k we consider the less stringent class of Lipschitz–Zygmund function spaces $\Lambda(\alpha, \mathbb{R}^n)$, $0 \leq k < \alpha \leq k+1$, for some integer $k \geq 0$, we come to free interpolation of measurable (or continuous) functions by Lipschitz–Zygmund spaces (k-quasi Lipschitz–Zygmund classes etc.). In each case the natural problem arises of characterizing the "quasi class" by some pointwise "infinitesimal differentiation" process—in the case considered in this paper this is the approximate k-th Peano differentiation at almost every point of the domain. Obviously many problems in the area are open. For $\alpha = 1$ we obtain the classical Lipschitz spaces, which make sense for arbitrary measure metric spaces, leading to the Lipschitz Analysis of J. Heinonen [23], [24]. For arbitrary real α, or $\alpha = k$, $k > 1$, we come to Lipschitz Analysis of higher order in the spirit of [24] on arbitrary closed subsets of $\mathbb{R}^n$.

The general scheme to consider is the quadruple of function spaces

$$\mathcal{F}_1 \subseteq G_1 \subseteq G_2 \subseteq \mathcal{F}_2.$$

If for G_2 we take e.g. the Sobolev spaces $W^{l,p}(\Omega)$ on open subsets of $\mathbb{R}^n$ and keep $G_1 = \mathcal{F}_1 \equiv C^k(\mathbb{R}^n)$ fixed, $k = [l]$, $k < l$, we obtain the free interpolation problem for Sobolev functions by smooth functions. See [5], [7], [8] and, growing in number, some other rather recent papers on free interpolation in normed function spaces (F.-Ch. Liu, Lars I. Hedberg, Y. Netrusov and many others). This problem is sometimes also called the Luzin Approximation problem (for Sobolev functions etc.). The "massiveness" (volume, etc.) of the difference $\Omega \setminus \Omega_\varepsilon$, where Ω_ε is the coincidence set of functions in G_2 with globally "smoother" functions of G_1 is measured not only in terms of Lebesgue measure but also by Hausdorff measures or capacities, Riesz capacities, Bessel, etc. with various parameters. Free interpolation g_ε in normed spaces naturally poses the problem of interdependence of the norm $\|f - g_\varepsilon\|$ and the "massiveness" in capacity of the coincidence set $E_{f,g_\varepsilon} = \Omega_e$. In the case when the used capacities of any nonempty set are bounded from below by "absolute" positive constants, sufficiently "good" Luzin approximation implies that the coincidence set Ω_ε has to cover any compact subset of the space Ω and thus the Luzin approximation reduces to the theorems of embedding type as e.g. the embedding of Sobolev spaces $W^{l,p}(\mathbb{R}^n)$ into spaces of smooth functions $C^{k,\alpha}$, $0 \leq \alpha \leq 1$, for appropriate values of parameters (l, p, n).

The works of J. Marcinkiewicz and A. Zygmund on Whitney type theory on $\mathbb{R}$, [33], [34], were inspired by subtle problems of harmonic analysis, singular integrals, boundary behaviour of solutions of Laplace equations, etc. Some examples illustrating these applications are described in [33] and [34]. It is clear that analogous examples arise in analysis in $\mathbb{R}^n$. Moreover, the list of applications in [33], [34] may be considerably extended.

The Real Analysis literature on differentiation of functions on the real line is extremely rich and diversified (see [45] and the whole "backward chain" of references started therein, spread over the whole XX century). In the literature of the subject there is a clearly observed gulf separating the case of differentiation theory of functions on the real line and the pointwise differentiation of functions of several variables (on subsets of $\mathbb{R}^n$). Nevertheless, it seems remarkable that the concepts of approximate Peano differentiability and the Whitney–Luzin concept of k-quasismoothness match so well in the differentiation theory of functions on arbitrary measurable subsets P of $\mathbb{R}^n$, for arbitary n. From this broader perspective the Whitney–Luzin structure theorem appears as a paradigm in the mainstream of the differentiation theory of arbitrary measurable functions.

Let us remark also that arbitrary measurable subsets of $\mathbb{R}^n$ are the simplest examples of general measure metric spaces. It may be expected that understanding the differentiation theory of functions on these spaces may play some role in understanding the differentiable structure of general measure metric spaces.

Finally let us notice that there are many cases in the literature when the "massiveness" of the complement of the free interpolating set A is measured by, on the one hand, more crude set-theoretical concepts like (second) category, or more subtle concepts like capacities (Riesz capacities or Bessel capacities for Sobolev spaces, conformal capacities, see [5], [7], [8]), Hausdorff measures. Though some of the latter results are related with powerful and deep Sobolev embedding theorems, apparently none of these theories so far can claim the coherence of the Whitney–Luzin quasismoothness theory.

A more detailed systematic and comprehensive exposition of these and other related topics is under preparation. Since the related literature is rather huge, the scope of this work is not easy to handle.

To finish, let us notice that a challenging problem is an extension of Whitney's theory of differentiability to the context of noncommutative Lie groups and Lie algebras, typically Heisenberg groups, or subriemannian structures on manifolds modelled on them. However, at this moment, we leave this recently intensively growing area for a future discussion.

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