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A.A. El-Sabbagh F.A. Abd El Salam K.El Nagaar

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

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Abstract: *Given a symmetric relation in a Hilbert space, then one can consider its selfadjoint extension as the Krein space. We show that selfadjoint Krein space extension plays a natural role in certain boundary value problems. We will show that boundary value problems with eigenvalues, depending boundary conditions are linearized.*

AMS subject classifications: 47A20

Keywords: Symmetric Operator, Symmetric Relation, Hilbert Space, Perturbing Case, Selfadjoint Extensions, Krein Space.

Correspondence

Department of Mathematics
Faculty of Engineering
Benha University, Shoubra, 108 Shoubra Street
Cairo, Egypt.

alysab1@hotmail.com.

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, FI-02015 TKK, Finland
email:math@tkk.fi http://math.tkk.fi/

1 Introduction

For a given symmetric linear relation S in a Hilbert space H, the selfadjoint extensions of S can be characterized as restrictions of the adjoint S^* of S, when S is the minimal relation associated with a formally symmetric ordinary differential expression in: L^2-function space, then the restrictions involve linear combinations of the boundary values of the elements in the domain $D(S^*)$of S^*. When the selfadjoint extensions are canonical within the space H , the coefficients of these combinations can be taken to be constants. In the case of selfadjoint extensions in inner product spaces larger than the given space H, they depend analytically on a parameter, see [3], [9], [11], [18] and [22]. We shall prove that every generalized resolvent $R(\ell)$ of S can be expressed in terms of a fixed generalized resolvent $G(\ell)$ of S and the Weyl coefficients $\Psi(\ell)$ of $R(\ell)$ relative to $G(\ell)$ which can be written as;

$$R(\ell)f = G(\ell)f + s(\ell)\Psi(\ell)[f, s(\ell)] \tag{1}$$

where $S(\ell)$ is a holomorphic basis for the null space $v(S^* - \ell)$ .One of the most important problems in the theory of operators is that of extension of symmetric operator to a selfadjoint one on a Hilbert space H. When we have a symmetric linear relation in Hilbert space, the spectral theorem can be constructed. Given a symmetric linear relation S in Hilbert space H^2 with equal defect numbers n, then there exists a selfadjoint extension for that relation in this Hilbert space. We define a minimal and maximal relation associated to S on a Hilbert space H^2. We may take this S to be: $S := T_{min} \cap C*$, where C is spanned by $\{\sigma, \zeta\}$,$\zeta -$ dimensional subspace in H^2, Tmin is the minimal relation in H^2. We shall construct the extensions of S related to the generalized resolvents and the selfadjoint extensions in H^2 ,we have mentioned above.

2 Preliminaries

Several observations in this section can be found in [5],[6]and [27]. If T, S are single-valued, then T, S become graph of linear operators. We shall use the following notations: $D(T) = \{x \mid \exists y, \{x, y\} \in T\}$, the domain, $R(T) = \{y | \exists\, x, \{x, y\} \in T\}$, the range, in particular

$T(0) = \{y | \{0, y\} \in T\}$, multivalued part,
$(T) = \{x | \{x, 0\} \in T\}$, nullspace
$T + S = \{\{x, y + z\} | \{x, y\} \in T\{x, z\} \in S\}$, sum
$ST = \{\{x, z\} | \{x, y\} \in T, \{y, z\} \in S\}$, product
$\mathbb{R} =$ Set of real numbers
$\mathbb{C} =$ Set of complex numbers
$C_0 = \mathbb{C}\backslash\mathbb{R}$
$T^\perp =$ orthogonal complement in H^2,
$T \Theta S = T \cap S$, see [10] and [13]

Now consider the symmetric relation which we interested in as $S := T_{min}C^*$ in a Hilbert space H, where: $C = \{\sigma, \zeta\}$, - dimensional subspace in H^2, Tmin defined as in Section 2 in [19].

$$T_{min} = \{\{f, g\}T_{max}|f(a) = 0^1_k,\ f(b) = 0^1_k\}.$$

$$T_{max} := \{\{f, g\} \in (L^2(\triangle\ dt))^2\}$$

with the property that $\underset{\sim}{f}$ contains an absolutely continuous function $\widetilde{f}$ such that from some $\widetilde{g} \in g : jf'(t) - H\widetilde{f}(t) = \triangle(t)\widetilde{g}(t)$, almost all $t \in [a, b]$.

It is clear that $S \subset S^*$ and from von Neumann's identity $S^* = S + M_\mu$, direct sums, $\mu \in C_0$. As usual, there exists a selfadjoint extension A in Krein space K for S and then we have $H \subset A, S \subset A, \rho(A) \neq \phi$ where ρ is the resolvent set, we shall consider this special case as an example.

Consider the system $Jy' - \widetilde{H}y = \ell\triangle y + \triangle f$, where: $\widetilde{H}(t) = \widetilde{H}(t)^*$, $\triangle(t) = \triangle(t)^* \geq 0$,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$t \in [a,\ b)$, $H = L^2(\triangle dt)$, we assume that system is regular at a limit point at b, and we define,

$$T_{max} := \{\{f, g\} \in H^2 \mid\ Jf' - \widetilde{H}f = \triangle g\}.$$

We mean by

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

$$S = \{\{f, g\}T_{max}|f(a) = 0\} \subset S^* = T_{max}$$

and

$$\widetilde{Y}(\ell) = \begin{pmatrix} \widetilde{Y}_{11} & \widetilde{Y}_{12} \\ \widetilde{Y}_{21} & \widetilde{Y}_{22} \end{pmatrix} = (\widetilde{Y}_1(\ell), \widetilde{Y}_2(\ell)), \widetilde{Y}_1(a,\ \ell) = I,$$

the fundamental solution is

$$M(\ell) = -\lim_{t\to b} \lim \widetilde{Y}_{12}(t,\ \ell)^{-1}\widetilde{Y}_{11}(t,\ \ell)$$

is Q-function of S and

$$\widetilde{Y}_0(\ell) + \widetilde{Y}_2(\ell)M(\ell) = \widetilde{H}(t)$$

span $V(S^* - \ell)$, see [8], [9], [11] and [20].

3 Extension of the Perturbing Case

In this section, we considered extensions of symmetric subspaces, (Relations), in the same Hilbert space H. If, however, the deficiency indices of S are not equal, then there always exists a Hilbert space , such that H , and S has a selfadjoint extension in H. Such an extension is called finite dimensional if:

$$\dim(\widetilde{H}\Theta H) < \infty,$$

see [12], [15], [16] and [21].

Theorem 3.1 *The extension of S corresponds to the generalized boundary value problem with Stieltjes boundary conditions $\int_a^b f\, d\mu^* = 0$.*

Proof. For that when $S := T_{min} \cap C^*, (\{f, g\} \in T_{max}, \{\mu, 0\} \in C), T_{max} = T^*_{\min}$ and when

$$D\Omega(\ell)D^* : (n + r) \times (n + r),$$

which describes the extension problem we need, and then we get that result. We refer to [3], [14], [17] and [18].

The results presented in this section were given by Codington [6], but we give here an independent and much simpler proof of the results.

Let S be the graph of a Hermitian operator in H, i.e.,

$$[Sf, g] = [f, Sg] \quad \forall\, f,\, g \in\, D(S),$$

such that $D(S)$ is not necessarily dense in H but such that $R(S) \subseteq \overline{D(S)}$. Suppose that an extension of S, A say, exists. Then A is the graph of an operator selfadjoint in $\overline{D(A_S)}$. All the selfadjoint extensions of S in H can be described in this way, then we have the following, see [5], [24] and [27].

Lemma 3.1 *Let K be a closed subspace of H, and let A be the graph of an operator such that$\overline{D(A)} = K$and A is selfadjoint in K. In particular $R(A) \subseteq K$. Let:*

$$B := A\Theta\{(0, g)H^2 : gK^{\perp}\}, \tag{2}$$

Then B is a selfadjoint subspace in H^2.

Proof. First note that since K is closed, we can write $H = K\Theta K^{\perp}$ so that any element $f \in H$ can be written as $f = f_1 + f_2$,where $f_1 \in K, f_2 \in K^{\perp}$ and $[f_1, f_2] = 0$.

Now let $\{f, g\} \in B, \{f, g\}$ be of the form $\{h, Ah + g\}$, where $h \in D(A)$ and $g \in K^{\perp}$. From any element $\{h', Ah' + g\} \in B$, we have:

$$[Ah' + g_1', h] = [Ah', h] + [g_1', h] = [h', Ah] = [h', Ah + g_1].$$

since

$$[Ah',\, h] = [h', Ah],\ \{h,\, Ah\} \in A$$

which is selfadjoint. Hence $\{f, g\} \in B^*$, so that $B \subset B^*$.

Conversely, let $\{f, g\} \in B^*$ so that

$$[Ah + g_1, f] = [h, g] \text{ forall } \{h, Ah + g_1\} \in B \tag{3}$$

where $\{h, Ah\} \in A, g_1 \in K^{\perp}$. also $B^* \subseteq A^*$ implies that

$$[Ah, f] = [h, g] \text{ forall } h \in D(A) \tag{4}$$

so that using (4) we get from (3)

$$[g_1, f] = 0 \text{ forall } g_1 \in K \tag{5}$$

$fD \in (A)$ if $f \models 0$, so that $\{f, Af\} \in A$. Suppose that $g_1' = g - Af$. To show that $g_1' \in K^{\perp}$ for $h \in D(A), [g_1', h] = [g - Af, h] = [g, h] - [Af, h]$ implies $[g, h] = [f, Ah + g_1]$ so that: $[g_1', h] = [f, Ah + g_1] - [Af, h] = [f, g] = 0$.

Hence, $g_1' \in K$. So any $\{f, g\} \in B^*$ is of the form $\{f, g\} = \{f, Af + g_1'\}$,where $g_1' \in K$, so that $\{f, g\} \in B$. Hence , $B^* \subseteq B$. This proves the lemma.

To find connection between possible extensions of $\overline{S \text{ in}}$ and the extensions in the Hilbert space H, we denote the adjoint of S in $\overline{D(S)}$ by S^*. Let D_+, D_-be the spaces defined by

$$D\pm := \{(g_1, S^* g_1) \in S^*, g_1 = \pm i g_1\}. \tag{6}$$

Then according to the extension theorem, S has a selfadjoint extension in $\overline{D(S)}$ if and only if : $dimD_+ = dimD_-$. Let

$$X\pm = \{\{h, \pm ih\} : h \in H\Theta D(S)\}. \tag{7}$$

Then we have the following results

$$\begin{aligned} M^+ &= D_+ \Theta X^+ \\ M- &= D - X - . \end{aligned} \tag{8}$$

We show first of these, whereas the second follows on exactly similar lines.

(i)D_+, X^+: are orthogonal: for any $\{g_1, ig_1\} \in D^+, g_1 \in D(S^*)$ and any $[h, ih\} \in X^+, h \in H\Theta D(S)$, we have $[\{h, ih\}, \{g, ig_1\}] = [h, g_1] + [h, g_1] = 0$ since $[h, g_1] = 0$.

(ii) For any $\{h, ih\} X^+, h \in H\Theta D(S)$, and $\overline{\{f, g\} \in S, [g, h] = [}f, ih] = 0$ for all: $\{f, g\} \in S$. This is because $R(S) \subseteq \overline{D(S)}$ and $h \perp \overline{D(S)}$; so that $X^+ \subseteq M^+$. The fact that $D^+ \subseteq M^+$ is clear.

(iii) Let $\{f, if\} \subseteq M^+$ where $f_1 \in H$, and so:

$$f_1 = g_1 + h, g_1 \in D(S), h \in H\Theta D(S) \tag{9}$$

so that $\{f, if\} \in S$ implies $[g, f_1] = [f, if_1]$ for $\{f, \overline{g\} \in} S$ or $[g, g_1 + h] = [f, i(g_1 + h)]$. But $[g, h] = [f, h] = 0$. since $g, f \in \overline{D(S)}$ and $h \in H\Theta D(S)$ hence $[g, g_1] = [f, ig_1]$ for all $\{f, g\} \in S$, and so $\{g_1, ig_1\} \in S^*$. This shows that every element $\{f_1, if_1\}$ of M^+ can be written as sum of an element of D^+ and an clement of X^+.

From (i), (ii), and (iii) follows that

$$M^+ = X^+ \Theta D_+. \tag{10}$$

From the above discussion we deduce the coming theorem, see [25], [26] and [30].

Theorem 3.2 *Let S be a densely defined closed symmetric operator in H with finite but unequal deficiency indices, and let H_1 be a Hilbert space such that $H \in H_1$and such that S has a selfadjoint extension in H_1.*

Then this extension is not finite.

Proof. We have $dimX^+ = dimX^- = dim(H\Theta D(S^*))$. Thus if $\dim D_+ = \dim D_-$, both being finite, we see that $\dim M^+ = \dim M^-$ is not possible unless $\dim(H\Theta D(S^*))$ is infinite, see [4], [7], [28], and [30].

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