Helsinki University of Technology, Institute of Mathematics, Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2006 A506

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Sergey Korotov: *Error control in terms of linear functionals based on gradient averaging techniques*; Helsinki University of Technology, Institute of Mathematics, Research Reports A506 (2006).

Abstract:

The work is devoted to a recent trend in a posteriori error estimation which is based on the concept of computational error control in terms of linear (goal-oriented) functionals in addition to the classical error control in global energy norms.

AMS subject classifications: 65N15, 65N30, 65N50

Keywords: a posteriori error estimation, quantity of interest, finite element method, differential equation of elliptic type, linear elasticity, superconvergence.

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ISBN 951-22-8316-6
ISSN 0784-3143

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1 Introduction

The work is devoted to a recent trend in a posteriori error estimation which is based on the concept of computational error control in terms of linear (goal-oriented) functionals in addition to the classical error control in global energy norms (see, e.g., [1], [2], [3], [9], [10], [12], [13], [14] and references therein). Error estimates of such a type are strongly motivated by practical needs, in which analysts are often interested not in the value of the overall error, but in errors over certain parts of the solution domain or relative to some interesting characteristics ("quantities of interest"). One way for estimating such errors is to introduce a linear functional ℓ associated with a particular subdomain of interest (or a quantity of interest) and to obtain an estimate for $< \ell, u - \bar{u} >$, where u is the exact solution and $\bar{u}$ is the approximate one. In the paper we show that popular gradient averaging procedures can be effectively used (even for strongly inhomogeneous meshes and without any additional regularity of the exact solutions) for such type of error estimation for linear elliptic problems and demonstrate the effectivity of the proposed estimators in several numerical tests.

2 General scheme

In this section we present in a compact form a general scheme (cf. [9]) for obtaining the required estimates. Let Y be Hilbert space with a scalar product $(\cdot,\cdot)$ and a norm $\|y\|_Y = (y,y)^{1/2}$, V be Banach space with a norm $\|\cdot\|_V$, $\Lambda \in \mathcal{L}(V,Y)$, $\mathcal{A} \in \mathcal{L}(Y,Y)$ and

$$c_1\|y\|_Y^2 \le (\mathcal{A}y, y) \le c_2\|y\|_Y^2 \quad \forall y \in Y, \quad c_3\|w\|_V \le \|\Lambda w\|_Y \quad \forall w \in V_0, \quad (1)$$

where V_0 is a subspace of V and c_1, c_2, c_3 are positive constants. Given $f \in V_0^*$, consider the following problem.

Primal Problem ($\mathcal{P}$): Find $u \in V_0$ such that

$$(\mathcal{A}\Lambda u, \Lambda w) \;= < f, w > \quad \forall w \in V_0, \quad (2)$$

where $< \cdot,\cdot >$ denotes the duality pairing of the spaces V_0 and V_0^*.

Let ℓ be another element of V_0^*. We want to effectively estimate the quantity $< \ell, u - \bar{u} >$ for an arbitrary element $\bar{u} \in V_0$ viewed as an approximation of u. For this purpose one usually emploies the so-called adjoint problem (cf. [1], [3], [11], [13]):

Adjoint Problem ($\mathcal{P}_a$): Find $v \in V_0$ such that

$$(\mathcal{A}^*\Lambda v, \Lambda w) \;= < \ell, w > \quad \forall w \in V_0, \quad (3)$$

where $\mathcal{A}^*$ is the operator adjoint to $\mathcal{A}$.

Remark 1: In the case of a selfadjoint operator ($\mathcal{A} \equiv \mathcal{A}^*$) both above problems are associated with a functional of the type $J(w) \;=\; \frac{1}{2}(\mathcal{A}\Lambda w, \Lambda w) + \;<$

$\mu, w >$, which is known to have a unique minimizer in V_0 for any $\mu \in V_0^*$ in view of conditions (1).

Proposition 1: Let u and v be solutions of problems $(\mathcal{P})$ and $(\mathcal{P}_a)$, respectively. Then for $\bar{u}, \bar{v} \in V_0$ we have

$$< \ell, u - \bar{u} > = \; E_0(\bar{u}, \bar{v}) + E_1(\bar{u}, \bar{v}), \tag{4}$$

where

$$E_0(\bar{u}, \bar{v}) \;= < f, \bar{v} > - \, (\mathcal{A}\Lambda\bar{u}, \Lambda\bar{v}), \quad E_1(\bar{u}, \bar{v}) \;=\; (\mathcal{A}\Lambda(u-\bar{u}), \Lambda(v-\bar{v})). \tag{5}$$

For the proof see [9, p. 36].

The error decomposition (4) is valid for any elements $\bar{u}$ and $\bar{v}$ from V_0. The term $E_0(\bar{u}, \bar{v})$ is explicitly computable, whereas $E_1(\bar{u}, \bar{v})$ contains unknown solutions of problems $(\mathcal{P})$ and $(\mathcal{P}_a)$. It is obvious that the term $E_0(\bar{u}, \bar{v})$ is dominating if $\bar{v}$ is "sufficiently close" to v, i.e., in such a case $E_0(\bar{u}, \bar{v})$ can serve as a directly computable estimator for the quantity $< \ell, u - \bar{u} >$. This situation can happen in concrete real-life problems if we have enough computer resources (time, memory) to compute the element $\bar{v}$ (later on always called as an approximation to v) be sufficiently accurate, i.e., close to v. Another situation is when one has certain limits for computing $\bar{v}$. In this case, it seems that a more natural way is to use an approximate solution of $(\mathcal{P}_a)$ having approximately the same quality as the approximate solution of $(\mathcal{P})$ (i.e., the element $\bar{u}$) and to try to recover the unknown Λu and Λv in the term E_1 by suitable post-processing techniques. We note also that the same approximation $\bar{v}$ can be used for different approximations for the primal problem, and even for another primal problems (e.g., with different f).

To be more precise, let V_h and V_τ be two finite-dimensional subspaces of V_0, and let $\bar{u} = u_h$, $\bar{v} = v_\tau$, where u_h and v_τ are solutions of the following problems

$$(\mathcal{A}\Lambda u_h, \Lambda w_h) = < f, w_h > \quad \forall w_h \in V_h, \quad (\mathcal{A}^*\Lambda v_\tau, \Lambda w_\tau) = < \ell, w_\tau > \quad \forall w_\tau \in V_\tau. \tag{6}$$

In a particular case of $V_h \equiv V_\tau$, the term $E_0(u_h, v_\tau)$ vanishes due to the orthogonality condition in (6). To the contrary, using non-coinciding subspaces V_h and V_τ one gets the estimate that has both terms presented. Further, let G_h and G_τ be suitable post-processing operators defined on V_h and V_τ, respectively. We replace $E_1(u_h, v_\tau)$ by the directly computable functional $\tilde{E}(u_h, v_\tau) := E_0(u_h, v_\tau) + \tilde{E}_1(u_h, v_\tau)$, where

$$\tilde{E}_1(u_h, v_\tau) \;:=\; (\mathcal{A}(G_h(\Lambda u_h) - \Lambda u_h), G_\tau(\Lambda v_\tau) - \Lambda v_\tau). \tag{7}$$

If the operators G_h and G_τ properly recover unknown elements Λu and Λv, then one can expect that the difference between $E_1(u_h, v_\tau)$ and $\tilde{E}_1(u_h, v_\tau)$ is reasonably small, and, thus, the latter quantity can succesfully be used instead of E_1.

3 Examples

In this section, the general error estimation scheme presented in Section 2, is used for a construction of reliable estimators for the error $< \ell, u - u_h >$ for two concrete model problems – scalar linear elliptic equation and linear elasticity problem. In addition, the constructed estimators are of the integral form and naturally suggest convenient mesh adaptivity procedures, which are shown to be quite effective for both types of problems.

3.1 Linear elliptic equation of the second order

Problem Formulation: Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 1, 2, ...)$ with a Lipschitz continuous boundary $\partial\Omega$. We set

$$Y = L_2(\Omega, \mathbb{R}^d), \quad V = H^1(\Omega), \quad V_0 = H_0^1(\Omega), \quad V_0^* = H^{-1}(\Omega),$$

define Λ as $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})^T$, and consider the problem: Find u such that

$$-\mathrm{div}(\mathrm{A}\nabla u) = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{8}$$

where $f \in L_2(\Omega)$, the matrix $\mathrm{A} = \{a_{ij}(x)\}_{i,j=1}^d$ is symmetric and is such that

$$a_{ij}(x) \in L_\infty(\Omega), \qquad \mathrm{A}(x)\xi \cdot \xi \geq c_4 \, \|\xi\|^2 \quad \forall \xi \in \mathbb{R}^d \quad \forall x \in \overline{\Omega}. \tag{9}$$

Problem $(\mathcal{P})$ consists of finding $u \in V_0$ such that

$$\int\limits_\Omega \mathrm{A}\nabla u \cdot \nabla w \; dx = \int\limits_\Omega f w \; dx \quad \forall w \in V_0. \tag{10}$$

Let $u_h \in V_0$ be a continuous piecewise affine finite element approximation of u computed on the *primal mesh* $\mathcal{T}_h$. For simplicity, we consider the following linear functional

$$< \ell, u - u_h > := \int\limits_\Omega \varphi(u - u_h) \; dx, \quad \text{where} \quad \mathrm{supp}\,\varphi := \omega \subseteq \Omega, \tag{11}$$

estimation of which gives a certain information about (local) behaviour of the error $u - u_h$ in subdomain ω.

Adjoint Problem $(\mathcal{P}_a)$ consists of finding $v \in V_0$ such that

$$\int\limits_\Omega \mathrm{A}\nabla v \cdot \nabla w \; dx \; = < \ell, w > \quad \forall w \in V_0. \tag{12}$$

If, in (11), the function $\varphi \in L_2(\Omega)$, then the adjoint problem (12) has a unique solution.

Construction of Estimator: Let $v_\tau \in V_0$ be a continuous piecewise affine finite element approximation of v computed on the *adjoint mesh* $\mathcal{T}_\tau$. Then value (11) is estimated by $\tilde{E}(u_h, v_\tau) = E_0(u_h, v_\tau) + \tilde{E}_1(u_h, v_\tau)$, with

$$E_0 = \int\limits_\Omega f v_\tau \, dx - \int\limits_\Omega \mathrm{A}\nabla v_\tau \cdot \nabla u_h \, dx, \tag{13a}$$

$$\tilde{E}_1 = \int\limits_\Omega \mathrm{A}(G_\tau(\nabla v_\tau) - \nabla v_\tau) \cdot (G_h(\nabla u_h) - \nabla u_h) \, dx, \tag{13b}$$

where the post-processing operator (usually called gradient averaging) G_h (or G_τ): $L_\infty(\Omega, \mathbb{R}^d) \rightarrow H^1(\Omega, \mathbb{R}^d)$ defines a vector-valued piecewise affine function by setting each nodal values as, e.g., the mean (or weighted mean) of values ∇u_h (or ∇v_τ) on all elements incident with the corresponding nodal point (cf. [6] and references therein).

Remark 2: In the above we consider problem (8) with a homogeneous Dirichlet boundary condition only for simplicity. A more general problem with non-homogeneous mixed boundary conditions can be treated in the same way.

Remark 3: For averaging procedures in higher dimensions we refer to [4]. The scheme proposed above is also valid for approximations obtained via finite elements of higher orders, the relevant averagings in this case are given, e.g., in [5] (and in references therein).

Remark 4: Successful replacement of unknown solutions by the averaged gradients is actually based on the phenomenon of the superconvergence [4, 5, 14], which is mathematically proved only in the presence of sufficient smoothness of the exact solutions and sufficient regularity of the meshes used in computations. The performance of the estimator for such special cases was demonstrated, e.g., in some tests in [9, 8]. However, the estimator surprisingly works very well even if we have no additional regularity of the solution and use in computations strongly inhomogeneous meshes. This is clearly shown in all below tests (also for the linear elasticity) in which we consider the solution domains with reentrant corners. In all the tests of the present paper we define averaged gradients using weighted mean of relevant values of gradients with respect to areas of triangles in each patch. This choice of averaging is used mainly due to the fact that triangles in meshes used can be of quite different sizes in what follows and, thus, simple mean does not seem to be reasonable. However, the tests performed showed that another averaging procedures give very similar results.

Test 1a. Let Ω be a planar (i.e., $d = 2$) L-shaped domain with re-entrant corner at $(0, 0)$ obtained from a square $(-1, 1) \times (-1, 1)$, $\omega := (-0.2, 0) \times (-0.2, 0)$ (see Fig. 1 (left)), let $\varphi \equiv 1$ in ω and vanish outside of ω, A be the unit matrix, $f \equiv 10$ in Ω. The finite element solution u_h is calculated on $\mathcal{T}_h$ with 88 nodes, the error $< \ell, u - u_h >= 0.005346$.

The results of performance of the estimator for various choices of adjoint meshes (having 31, 42, ..., 2077 nodes, see Figure 1 (right)) are presented in Figure 2. The behaviour of the estimator $\tilde{E}$ and its terms, E_0 and $\tilde{E}_1$, demonstrated there is very typical for all other tests for this type of problems. To evaluate the quality of the estimator we use the so-called *effectivity index*

$$I_{eff} := \frac{\tilde{E}(u_h, v_\tau)}{< \ell, u - u_h >},$$

and clearly observe that: a) estimator gives reasonably good results ($0.72 \leq I_{eff} \leq 0.93$) for adjoint meshes which are considerably coarser than the primal meshes (in this case both terms in the estimator are quite comparable); b) estimator is asymptotically correct ($I_{eff} \to 1$ as $\tau \to 0$) (in this case, $\tilde{E}_1 \to 0$ and $\tilde{E}$ converges to the exact error). In this case, only the first term E_0 can be used, in fact, for the estimation.

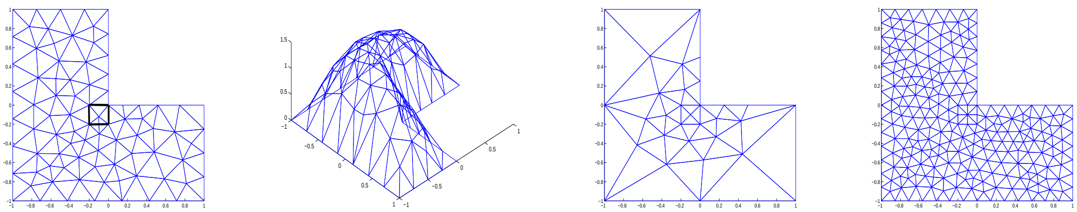

Figure 1: Ω, ω, $\mathcal{T}_h$ (88 nodes), u_h, and typical $\mathcal{T}_\tau$ (31 and 247 nodes) for Test 1a

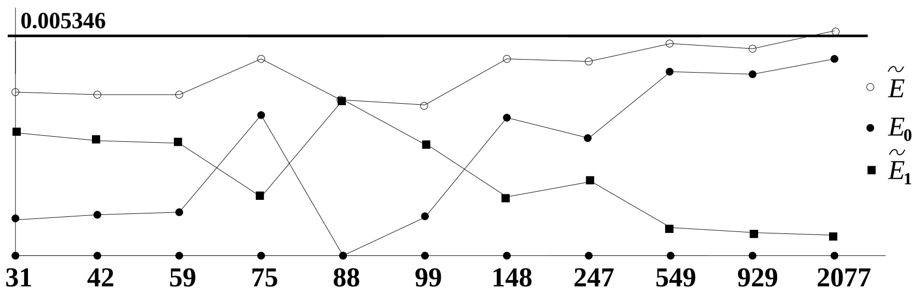

Figure 2: Behaviour of estimator $\tilde{E}$ and its parts, E_0 and $\tilde{E}_1$, for various choices of adjoint meshes in Test 1a. Mesh $\mathcal{T}_h$ has 88 nodes in all computations

Test 1b (adaptivity). The estimator $\tilde{E}$ is, in fact, an integral over Ω, i.e.,

$$\tilde{E}(u_h, v_\tau) := \sum_{T \in \mathcal{T}_h^{(i)}} I_T, \tag{14}$$

where each contribution I_T is a value of the integral taken over a particular element T of the current primal mesh $\mathcal{T}_h^{(i)}$. To construct the next primal mesh $\mathcal{T}_h^{(i+1)}$ in order to decrease the value (11), we propose the following adaptive procedure. First, we find the maximum among all modulus $|I_T|$'s and, secondly, mark up those elements T's which have their contributions larger than the “user-given threshold” θ ($\theta \in [0,1]$) times that maximum value. Refining the marked elements (and making the mesh conforming), we get $\mathcal{T}_h^{(i+1)}$.

This procedure has been tested for the same problem as in Test 1a, with initial $\mathcal{T}_h^{(0)}$ having 59 nodes and with $\mathcal{T}_\tau$ taken the same (42 nodes) for all 4 refinements steps (threshold $\theta := 0.7$ in all the steps for simplicity). The results are reported in Table 1.

We observe that the exact error (6-th column in Table 1) and the corresponding estimator values (5-th column in Table 1) monotonically decrease. The values of I_{eff} are very close to 1 in all computations, also in the case when the primal mesh is, e.g., 4.5 times more dense than the corresponding adjoint mesh (see the last raw in Table 1). Several corresponding meshes are presented in Figure 3.

Table 1: The results for adaptivity in Test 1b

Prim	Adj	E_0	$\tilde{E}_1$	$\tilde{E}$	$\int_\omega \varphi\,(u-u_h)\,dx$	I_{eff}
$\mathcal{T}_h^{(0)}(59)$	42	0.000785	0.003433	0.004218	0.005428	0.78
$\mathcal{T}_h^{(1)}(72)$	42	0.000586	0.001774	0.002360	0.002739	0.86
$\mathcal{T}_h^{(2)}(90)$	42	0.000582	0.001217	0.001799	0.001804	0.99
$\mathcal{T}_h^{(3)}(149)$	42	0.000470	0.000939	0.001409	0.001540	0.91
$\mathcal{T}_h^{(4)}(186)$	42	0.000420	0.000824	0.001244	0.001338	0.93

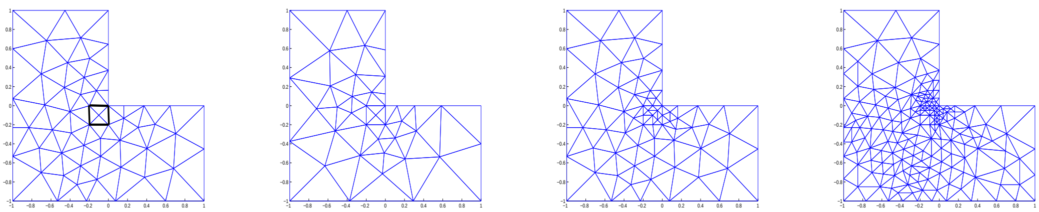

Figure 3: Meshes $\mathcal{T}_h^{(0)}$ (59 nodes), $\mathcal{T}_\tau$ (42 nodes), $\mathcal{T}_h^{(1)}$ (72 nodes) and $\mathcal{T}_h^{(4)}$ (186 nodes) for Test 1b

3.2 The linear elasticity problem

Problem Formulation: The classical formulation consists of finding the displacement $\mathbf{u}$ and the stress σ in a bounded elastic body $\Omega \subset \mathbb{R}^d$ such that

$$\text{div}\sigma + \mathbf{f} = 0 \quad \text{in} \quad \Omega\ , \ \sigma = \mathbb{L}\,\epsilon, \quad \epsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)\ , \tag{15}$$

$$\mathbf{u} = \mathbf{u^0} \quad \text{on} \quad \Gamma_1\ , \quad \sigma\, n = \mathbf{g} \quad \text{on} \quad \Gamma_2\ , \tag{16}$$

where n is the unit outward normal to the boundary $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, $\mathbf{f}$ and $\mathbf{g}$ are the given volume and surface loads, and $\mathbf{u^0}$ prescribes the displacement on Γ_1.

In the above, $\mathbb{L} = \mathbb{L}(x) = (L_{ijkl}(x))_{i,j,k,l=1}^{d}$ denotes the fourth order tensor of the elastic coefficients, which satisfy the symmetry condition

$$L_{jikl} = L_{ijkl} = L_{klij} \ , \quad i,j,k,l = 1,\ldots,d, \tag{17}$$

and the condition that there exists a positive constant c_5 such that

$$\mathbb{L}(x)\,\tau : \tau \geq c_5 |\tau|^2 \quad \forall \tau \in \mathbb{M}_s^{d\times d}\,. \tag{18}$$

In the above, $\mathbb{M}_s^{d\times d}$ denotes the space of symmetric tensors of the second order and the symbol ":" stands for the scalar product of symmetric tensors, i.e.,

$$\tau : \varkappa := \sum_{i,j=1}^{d} \tau_{ij}\varkappa_{ij}, \quad \mathbb{L}\tau : \varkappa := \sum_{i,j,k,l=1}^{d} L_{ijkl}\tau_{ij}\varkappa_{kl}, \quad |\tau|^2 := \tau : \tau,$$

for $\tau, \varkappa \in \mathbb{M}_s^{d\times d}$.

Further, let

$$L_{ijkl} \in L_\infty(\Omega),\ \mathbf{f} \in L_2(\Omega, \mathbb{R}^d),\ \mathbf{g} \in L_2(\Gamma_2, \mathbb{R}^d),\ \mathbf{u^0} \in H^1(\Omega, \mathbb{R}^d), \tag{19}$$

and let (17)–(18) hold almost everywhere in Ω. In what follows we set

$$\mathbf{Y} = L_2(\Omega, \mathbb{M}_s^{d\times d}),\ \mathbf{V} = H^1(\Omega, \mathbb{R}^d),\ \mathbf{V_0} = \{\mathbf{v} \in H^1(\Omega, \mathbb{R}^d) \mid \mathbf{v} = 0 \text{ on } \Gamma_1\}\ ,$$

identify $\Lambda \mathbf{v}$ with $\epsilon(\mathbf{v})$ and apply the scheme proposed in Section 2.

Problem $(\mathcal{P})$ reads: Find $\mathbf{u} \in \mathbf{V_0} + \mathbf{u^0}$ such that

$$\int_\Omega \mathbb{L}\epsilon(\mathbf{u}) : \epsilon(\mathbf{w})\,dx \ = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,dx + \int_{\Gamma_2} \mathbf{g}\cdot\mathbf{w}\,ds \quad \forall \mathbf{w} \in \mathbf{V_0}. \tag{20}$$

Let $\mathbf{u_h} \in \mathbf{V_0} + \mathbf{u^0}$ be a continuous piecewise affine finite element approximation of $\mathbf{u}$ computed on (primal) mesh $\mathcal{T}_h$. Similarly to Section 3.1, we can wish to estimate the quantity

$$< \ell, \mathbf{u} - \mathbf{u_h} >:= \int_\Omega \mathbf{\Phi}\cdot(\mathbf{u} - \mathbf{u_h})\,dx, \text{ where } \ \mathrm{supp}\mathbf{\Phi} := \omega \subseteq \Omega, \tag{21}$$

providing with information on the error $\mathbf{u} - \mathbf{u_h}$ in (local) subdomain ω.

Corresponding Problem $(\mathcal{P}_a)$: Find $\mathbf{v} \in \mathbf{V_0}$ such that

$$\int_\Omega \mathbb{L}\epsilon(\mathbf{v}) : \epsilon(\mathbf{w})\,dx \ = < \ell, \mathbf{w} > \quad \forall \mathbf{w} \in \mathbf{V_0}. \tag{22}$$

Problem (22) is uniquely solvable provided $\mathbf{\Phi} \in L_2(\Omega, \mathbb{R}^d)$.

Construction of Estimator: Let $\mathbf{v}_\tau \in V_0$ be a continuous piecewise affine finite element approximation of $\mathbf{v}$ computed on (adjoint) mesh $\mathcal{T}_\tau$. The value (21) is evaluated then by $\widetilde{E}(\mathbf{u_h}, \mathbf{v}_\tau) = E_0(\mathbf{u_h}, \mathbf{v}_\tau) + \widetilde{E}_1(\mathbf{u_h}, \mathbf{v}_\tau)$, where

$$E_0(\mathbf{u_h}, \mathbf{v}_\tau) = \int_\Omega \mathbf{f} \cdot \mathbf{v}_\tau \, dx + \int_{\Gamma_2} \mathbf{g} \cdot \mathbf{v}_\tau \, ds - \int_\Omega \mathbb{L} \, \epsilon(\mathbf{u_h}) : \epsilon(\mathbf{v}_\tau) \, dx, \qquad (23a)$$

$$\widetilde{E}_1(\mathbf{u_h}, \mathbf{v}_\tau) = \int_\Omega \mathbb{L} \left(\mathbf{G_h}(\epsilon(\mathbf{u_h})) - \epsilon(\mathbf{u_h}) \right) : \left(\mathbf{G}_\tau(\epsilon(\mathbf{v}_\tau)) - \epsilon(\mathbf{v}_\tau) \right) dx \, , \qquad (23b)$$

$$\mathbf{G_h}(\epsilon(\mathbf{u_h})) := \frac{1}{2} \left(\mathbf{G_h}(\nabla \mathbf{u_h}) + (\mathbf{G_h}(\nabla \mathbf{u_h}))^T \right), \qquad (24a)$$

$$\mathbf{G}_\tau(\epsilon(\mathbf{v}_\tau)) := \frac{1}{2} \left(\mathbf{G}_\tau(\nabla \mathbf{v}_\tau) + (\mathbf{G}_\tau(\nabla \mathbf{v}_\tau))^T \right). \qquad (24b)$$

In (23)–(24), the gradient averaging operators $\mathbf{G_h}$ and $\mathbf{G}_\tau$ are similar to those used in Section 3.1 (cf. [6]).

Test 2a. We solve the plane stress problem (i.e., $d = 2$) in domain Ω with re-entrant corner at $(0, 0)$ obtained from a square $(-1, 1) \times (-1, 1)$, $\omega := (-0.2, 0.0) \times (-0.1, 0.1)$ (see Fig. 4 (left)). The Young modulus $E = 10^6$, Poisson ratio $\nu = 0.3$, $\mathbf{f} \equiv (0, 0)$, the Dirichlet boundary condition $\mathbf{u^0} = (\mathbf{0}, \pm\mathbf{1})$ is prescribed on the upper and lower parts of $\partial\Omega$, respectively. In the remaining portions of $\partial\Omega$ the homogeneous Neumann condition is imposed. The finite element solution $\mathbf{u_h}$ is calculated on $\mathcal{T}_h$ with 120 nodes, the error $< \ell, \mathbf{u} - \mathbf{u_h} >= 0.002067$ for $\mathbf{\Phi} = (1, 1)$ in ω and $\mathbf{\Phi} = (0, 0)$ outside of ω. The results of computations for various choices of adjoint meshes are presented in Figure 5.

As in Test 1a, we see that the estimator gives reasonably good values ($0.61 \leq I_{eff} \leq 0.79$) for the case when adjoint meshes are coarser than the primal ones. Similarly to a scalar elliptic problem, we clearly observe the asymptotic convergence of the estimator as the adjoint meshes used are becoming more and more dense.

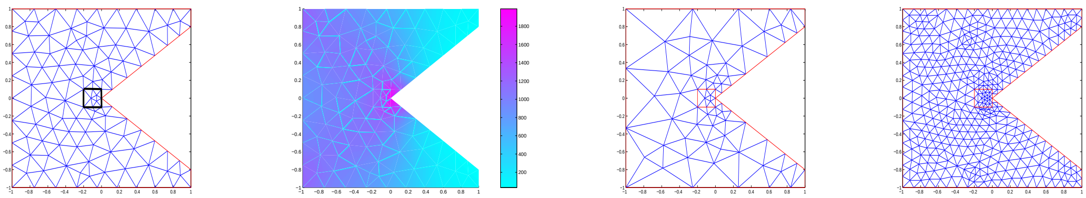

Figure 4: Ω, ω, $\mathcal{T}_h$ (120 nodes), Von Mises stress distribution, and typical $\mathcal{T}_\tau$ (75 and 435 nodes) for Test 2a

Test 2b (adaptivity). The adaptivity procedure is performed as proposed in Test 1b with the threshold $\theta = 0.6$ in all 4 refinement steps. The results are presented in Table 2. Similarly to Test 1b, we observe that the estimator values and the corresponding exact error values monotonically decrease. Several corresponding meshes are presented in Figure 6.

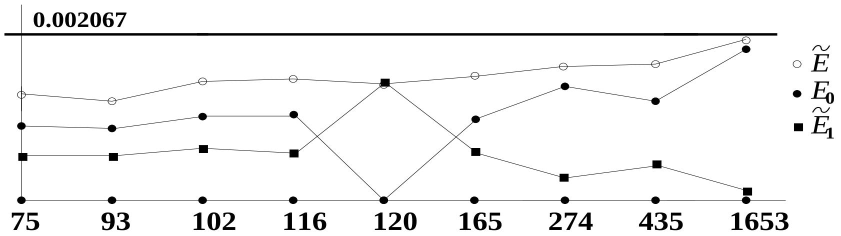

Figure 5: Behaviour of estimator $\tilde{E}$ and its parts, E_0 and $\tilde{E}_1$, for various choices of adjoint meshes in Test 2a. Mesh $\mathcal{T}_h$ has 120 nodes in all computations

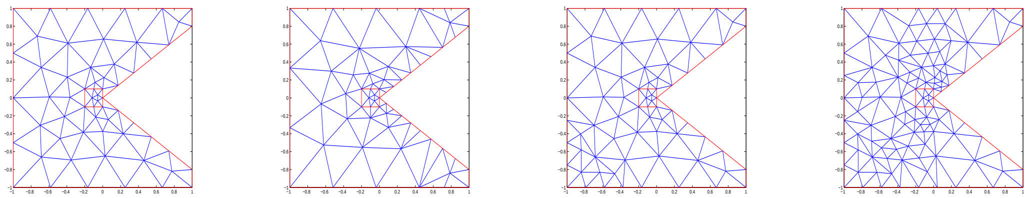

Figure 6: The meshes $\mathcal{T}_h^{(0)}$ (61 nodes), $\mathcal{T}_\tau$ (62 nodes), $\mathcal{T}_h^{(1)}$ (70 nodes) and $\mathcal{T}_h^{(4)}$ (111 nodes) for Test 2b

Table 2: The results for adaptivity in Test 2b

Prim	Adj	E_0	$\tilde{E}_1$	$\tilde{E}$	$\int\limits_\omega \mathbf{\Phi} \cdot (\mathbf{u} - \mathbf{u_h}) dx$	I_{eff}
$\mathcal{T}_h^{(0)}(61)$	62	0.001448	0.000543	0.001991	0.002898	0.69
$\mathcal{T}_h^{(1)}(70)$	62	0.001417	0.000482	0.001899	0.002644	0.72
$\mathcal{T}_h^{(2)}(90)$	62	0.001222	0.000531	0.001753	0.002433	0.72
$\mathcal{T}_h^{(3)}(103)$	62	0.001147	0.000514	0.001661	0.002334	0.71
$\mathcal{T}_h^{(4)}(111)$	62	0.001006	0.000498	0.001504	0.002211	0.68

4 Final Comments

1. Our approach is different from those proposed in [1, 3], where it is assumed that the primal and adjoint problems are always solved in the same space. Using our technique one can obtain reliable estimates also for the case when the number of nodes in the adjoint mesh is considerably smaller than the number of nodes in the primal mesh. The estimator still works very well even if we have no additional regularity of the solution and use in computations strongly inhomogeneous meshes.

2. The effectivity of the proposed technique, strongly increases when one is interested not in a single solution of the primal problem for a concrete data, but analyzes a series of approximate solutions for a certain set of boundary conditions and various right-hand sides (which is typical in the engineering

design when it is necessary to model the behavior of a construction for various working regimes). In this case, the adjoint problem can be solved *only once* for each linear functional (e.g., on a very dense adjoint mesh), and this solution can be further used in testing the accuracy of approximate solutions of various primal problems.

3. Our tests demonstrate the high effectivity of the adaptive procedure proposed in the paper, see also [8].

4. Estimates for the error measured in terms of specially constructed linear functionals can also be used for reliable evaluating local integral norms of the error, see [7] and [9] for details.

5. The approach proposed in this paper has been recently successfully tested for evaluating the nonlinear J-integral (see [12] for definitions), which serves as a fracture criterion in linear elastic fracture mechanics, in the work [11].

6. It seems to be possible to use the same ideology also for approximations obtained, e.g., by the hp-FEMs.

Acknowledgment

The author was supported by the Academy Research Fellowship no. 208628 from the Academy of Finland.

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ISBN 951-22-8316-6
ISSN 0784-3143