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Abstract: *We study the relation between the growth of sequences* $\|T^n\|$ *and* $\|(n+1)(I-T)T^n\|$ *for operators* $T \in \mathcal{L}(X)$ *satisfying weak variants of the Ritt resolvent condition* $\|(\lambda - T)^{-1}\| \leq \frac{C}{|\lambda - 1|}$ *for various sets of* $|\lambda| > 1$.

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1 Introduction

Let $T \in \mathcal{L}(X)$; a bounded linear operator on a (complex) Banach space X. It was R. K. Ritt who first studied the *Ritt resolvent condition*

$$\|(\lambda - T)^{-1}\| \leq \frac{C}{|\lambda - 1|} \tag{1}$$

for $|\lambda| > 1$. R. K. Ritt himself proved that if T satisfies (1) for $|\lambda| > 1$, then $\lim_{n\to\infty} \|T^n/n\| = 0$, see [13]. Clearly (1) implies that $\sigma(T) \subset \mathbb{D} \cup 1$, but in fact even $\sigma(T) \subset K_\delta^c \cap (\mathbb{D} \cup \{1\})$ for some $\delta > 0$, where

$$K_\delta := \{\lambda = 1 + re^{i\theta} : r > 0 \text{ and } |\theta| < \frac{\pi}{2} + \delta\}; \tag{2}$$

see O. Nevanlinna [10, Theorem 4.5.4] and Yu. Lyubich [6].

The following result was given by Y. Katznelson and L. Tzafriri in 1986: for *power bounded* operators T in the sense that $\sup_{n\geq 1} \|T^n\| < \infty$, we have $\sigma(T) \subset \mathbb{D} \cup \{1\}$ if and only if $\lim_{n\to\infty} \|(I - T)T^n\| = 0$, see [5]. Related to this, J. Zemánek asked in 1992 whether (1) implies $\lim_{n\to\infty} \|(I - T)T^n\| = 0$, too. This was answered in positive by O. Nevanlinna, and he also noted that if (1) hold in the larger set $K_\delta \cup \mathbb{D}^c$ for some $\delta > 0$, then T is power bounded, see [10, Theorem 4.5.4], [11] and [16].

It was then observed independently in 1998 by B. Nagy and J. Zemánek [9], O. Nevanlinna, and Yu. Lyubich [6] that if (1) holds for all $|\lambda| > 1$, then (1), indeed, holds for all $\lambda \in K_\delta \cup \mathbb{D}^c$ for some $\delta > 0$ (with another possibly larger constant $\tilde{C}$ in place for C). Hence, if T satisfies (1) for all $|\lambda| > 1$, then T is power bounded. The upper bound $\sup_{n\geq 1} \|T^n\| \leq (eC^2)/2$ was given by N. Borovykh, D. Drissi and M. N. Spijker, see [1]. A tighter estimate $\sup_{n\geq 1} \|T^n\| \leq C^2$ was shown by O. El-Fallah and T. Ransford in [2].

Much of these developments culminate in the following fundamental result connecting power boundedness, the Ritt resolvent condition and the *tauberian condition* (3):

Proposition 1. *The following are equivalent:*

(i) T satisfies (1) *for all* $|\lambda| > 1$,

(ii) $\sigma(T) \subset \mathbb{D} \cup \{1\}$ *and T satisfies* (1) *for all $\lambda \in K_\delta$ for some $\delta > 0$, and*

(iii) T is power bounded, and it satisfies the tauberian condition

$$\sup_{n\geq 1} (n + 1)\|(I - T)T^n\| \leq M \tag{3}$$

for some $M < \infty$.

Indeed, the equivalence (i) $\Leftrightarrow$ (ii) has already been discussed above. That (ii) $\Rightarrow$ (iii) is given in [10, Theorem 4.5.4], and we shall compute an estimate for M in (3) in Theorem 4. That (iii) implies (i) was reported in [11, Theorem

2.1]. The proof relies on the theory of analytic semigroups, and it follows closely [12, Theorem 5.2][1].

We further note that J. Esterle has pointed out in [3] that

$$\liminf_{n\to\infty} (n+1)\|(I-T)T^n\| \geq \frac{1}{96}$$

for a power-bounded T satisfying $\sigma(T) = \{1\}$; see also [8] (and references therein) for the determination of the optimal lower bound $1/e$ instead of $1/96$. Hence the stronger version $\lim_{n\to\infty}(n+1)(I-T)T^n = 0$ of the tauberian condition (3) cannot generally hold for T satisfying (1) for all $|\lambda| > 1$.

We shall show in this paper that the conditions of Proposition 1 can be combined in a different way. Indeed, we shall prove the following tauberian theorem and discuss some of its consequences:

Theorem 1. *If $T \in \mathcal{L}(X)$ satisfies the the Ritt condition* (1) *for all $\lambda > 1$ and tauberian condition* (3)*, then T is power bounded.*

We also estimate $\sup_{n\geq 1} \|(n+1)(I-T)T^n\|$ for operators satisfying (1) for all $|\lambda| > 1$. Most of the results of this paper (in particular, the main result Theorem 2) were proved in 2002 in [15].

2 Equivalent conditions under the tauberian condition

Let us remind the results of the classical tauberian theorem in the scalar case. Let $\{a_n\}$ be a complex sequence and $s_n = a_0 + a_1 + \ldots + a_n$ for $n \geq 0$. A. Tauber proved in 1897 that if

(i) $\lim_{n\to\infty}(n+1)a_n = 0$, and

(ii) $\lim_{r\to 1_-} f(r) = s$, where $f(r) = \sum_0^\infty a_n r^n$ for $0 < r < 1$,

then $\lim_{n\to\infty} s_n = s$. It was J. E. Littlewood who later in 1910 showed that the tauberian condition (i) can in fact be replaced by the weaker tauberian condition $\sup_n n|a_n| < \infty$. As it is mentioned in [14, Chapter 9], the proof with this modification becomes considerable harder.

If we take $a_n = (I-T)T^n$, we see that the weaker tauberian condition is exactly (3). Now the corresponding partial sums are simply $s_n = I - T^{n+1}$. In this paper, we are not interested in the *limit behaviour* of $\{s_n\}$, but only in the *boundedness* of this sequence under the weaker tauberian condition (3). This will save us from the extra complications that would be required if we had to take advantage of Littlewood's variant of the classical tauberian theorem instead.

[1]However, the restrictive assumption $0 \in \rho(A)$ must be first removed from [12, Theorem 5.2] by a more careful analysis.

Theorem 2. *Assume that $T \in \mathcal{L}(X)$ satisfies tauberian condition* (3)*, and*

$$\|(\lambda - 1)(\lambda - T)^{-1}\| \leq C \tag{4}$$

for all $\lambda > 1$. Then T is power bounded with the estimates

$$\|T^n\| \leq 2 + C\|T\| + 2M \quad \textit{and}$$
$$\limsup_{n\to\infty} \|T^n\| \leq 2 + C\|T\| + (1 + 1/e)\, M.$$

Proof. Define

$$s_n := \sum_{j=0}^{n-1} (I - T)T^j = 1 - T^n,$$
$$f(r) := \sum_{j=0}^{\infty} (I - T)T^j r^j = (I - T)(1 - rT)^{-1}, \text{ and}$$
$$f_n(r) := \sum_{j=0}^{n-1} (I - T)T^j r^j.$$

Then for all $r \in (0, 1)$ and $n \geq 0$, we have

$$\|s_n\| \leq \|s_n - f_n(r)\| + \|f_n(r) - f(r)\| + \|f(r)\|. \tag{5}$$

Condition (4) implies $\sup_{0\leq r<1} \|f(r)\| \leq 1 + C\|T\|$, and the last term of the right hand side is bounded by $C_1 := 1+C\|T\|$. For the second term, we have

$$\|f_n(r) - f(r)\| = \|\sum_{j\geq n}(I - T)T^j r^j\| \leq \sum_{j\geq n} \frac{M}{j+1} r^j$$
$$= \frac{M}{n+1} \sum_{j\geq n} \frac{n+1}{j+1} r^j \leq \frac{M}{n+1} r^n (1 - r)^{-1}$$

by (3). From now on, we choose $r_n := 1 - 1/n$ in (5). Then

$$\frac{M}{n+1} r_n^n (1 - r_n)^{-1} = \frac{M}{n+1} \left(1 - \frac{1}{n}\right)^n n \quad \begin{cases} \to M/e \text{ as } n \to \infty; \\ \leq M \text{ for all } n \geq 1. \end{cases}$$

So the second term in (5) is bounded with this choice of $r = r_n$.

The first term of the right side of inequality (5) (when choosing $r = r_n$) we have

$$s_n - f_n(r_n) = \sum_{j=0}^{n-1} (I - T)T^j (1 - r_n^j).$$

By the mean value theorem, there exists $r_0^j \in [r_n, 1)$ for any $j > 0$, such that we can estimate

$$1 - r_n^j = j r_0^{j-1}(1 - r_n) \leq j(1 - r_n) = \frac{j}{n}.$$

This together with (3) yields

$$
\begin{aligned}
\|s_n - f_n(r_n)\| &\le \sum_{j=0}^{n-1} \frac{j}{n} \|(I-T)T^j\| \\
&\le \sum_{j=0}^{n-1} \frac{j}{n} \frac{M}{j+1} \le M \frac{1}{n} \sum_{j=0}^{n-1} 1 = M.
\end{aligned}
$$

So $\|s_n\|$ is uniformly bounded, which is equivalent to the power boundedness of T. This completes the proof. $\square$

If the tauberian condition (3) holds for T, then a number of conditions will be equivalent. The following theorem is analogous to [11, Theorem 2.1], except that now (3) is a standing assumption instead of power boundedness.

Theorem 3. *Assume that $T \in \mathcal{L}(X)$ satisfies the tauberian condition* (3)*. Then the following are equivalent:*

(i) T is power bounded,

(ii) T satisfies Kreiss resolvent condition for some constant C_K

$$
\|(\lambda - T)^{-1}\| \le \frac{C_K}{|\lambda| - 1}
$$

for $|\lambda| > 1$,

(iii) there exists $0 < \eta \le 1 \le C < \infty$ such that T satisfies the Ritt resolvent condition (1) *for all real $\lambda \in (1, 1+\eta)$,*

(iv) there exists $0 < \eta \le 1 \le C < \infty$ such that T satisfies the second order Ritt condition

$$
\|(\lambda - 1)^2 (\lambda - T)^{-2} T\| \le C,
$$

for all real $\lambda \in (1, 1+\eta)$,

(v) there exists $0 < \delta \le 1 \le C < \infty$ such that T satisfies the Ritt resolvent condition (1) *for all $\lambda \in K'_\delta := \{\lambda = 1 + re^{i\theta} | r > 0, |\theta| < \frac{\pi}{2} + \delta\}$, and*

(vi) $A := T - I$ generates an uniformly bounded, norm continuous, analytic semigroup $t \mapsto e^{At}$ of linear operators.

Proof. It is shown by estimating the von Neumann series that (i) $\Rightarrow$ (ii). It is trivial that (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i) by Theorem 2, noting that the resolvent condition is only used near point 1 in the proof.

It is trivial that (iii) implies (iv). Conversely, noting that because $\sigma(T) \subset \overline{\mathbb{D}}$ by the tauberian condition (3), we obtain for all $|r| < 1$

$$
(I - rT)^{-1} = \sum_{j \ge 0} (j+1)(I-T)T^j r^j + (1-r)(I-rT)^{-2} T.
$$

From this we conclude, by using (3) in the estimation, that

$$\begin{aligned}\|(1-r)(I-rT)^{-1}\| &\leq (1-r)\cdot M\sum_{j\geq 0} r^j + \|(1-r)^2(I-rT)^{-2}T\| \\ &= M + \|(1-r)^2(I-rT)^{-2}T\|\end{aligned}$$

for all $0 \leq r < 1$. Replacing $r = 1/\lambda$ shows now that (iv) $\Rightarrow$ (iii).

Claims (i) and (v) are equivalent by Proposition 1 and the extension result that can be found e.g. in [9]. By the classical theorem of E. Hille and K. Yoshida, claim (v) is equivalent (apart from the analyticity of the semigroup) to the existence of $C_{HY} < \infty$ such that for each integer $k \geq 1$

$$\|(\lambda - T)^{-k}\| \leq \frac{C_{HY}}{(\lambda-1)^k} \quad \text{for all} \quad \lambda > 1. \tag{6}$$

Setting $k = 1$ gives (iii). Conversely, (i) $\Rightarrow$ (vi) (apart from the analyticity) by the estimate

$$\|e^{tT}\| \leq \sum_{j\geq 0} \frac{\|T^j\| t^j}{j!} \leq \sup_{j\geq 0} \|T^j\| \cdot e^t$$

for all $t \geq 0$. Moreover, it is not difficult to see that $\|Ae^{tA}\| \leq Mt^{-1}(1-e^{-t})$ where $A := T - I$, if (3) holds. This implies that e^{tA} is analytic, by a slight generalization of [12, Theorem 5.2]. □

The implication (i) $\Rightarrow$ (ii) (with explicit constants) was first given by Z. Yuan by using a Cauchy integration argument, see [15]. We remark that the tauberian condition (3) implies $\|T^n\| = O(\ln n)$, and by [4, Theorem 3.3], [7], the growth can really be there for an operator in a Banach space. Condition (3) "almost" implies condition (iii) of Theorem 3, too. Indeed, as $(1-r)(I-rT)^{-1} = I - r(I-T)(I-rT)^{-1}$ for all $|r| < 1$, we obtain the estimate

$$\|(1-r)(I-rT)^{-1}\| \leq 1 + M\sum_{j\geq 0} \frac{|r|^{j+1}}{j+1} = 1 + M\ln\frac{1}{1-|r|}$$

for all $0 \leq |r| < 1$. Setting $r = 1/\lambda$ for $\lambda > 1$ gives now

$$\|(\lambda-1)(\lambda-T)^{-1}\| \leq 1 + M\ln\frac{\lambda}{\lambda-1}.$$

Hence $\|(\lambda - T)^{-1}\| = O\left((\lambda-1)\ln(\lambda-1)\right)$ as $\lambda \to 1+$. Again, the logarithmic term can really be present on the right hand side, see [7].

Finally, the tauberian condition (3) "almost" implies condition (vi) of Theorem 3. Indeed, as $\|Ae^{tA}\| \leq Mt^{-1}(1-e^{-t})$ where $A := T - I$, and the function $t \mapsto t^{-1}(1-e^{-t})$ is decreasing for $t \geq 0$, it follows that

$$\|e^{tA}\| \leq 1 + \int_0^t \|Ae^{tA}\|\, dt \leq 1 + M + M(1-e^{-1})\ln t \quad \text{for all} \quad t \geq 1.$$

3 An upper bound for $\|(n+1)(I-T)T^n\|$

Assume that $T \in \mathcal{L}(X)$ satisfies the Ritt resolvent condition (1) for all $|\lambda| > 1$. Then $\sup_{n\geq 1} \|T^n\| \leq C^2$, as shown in [2] as a particular case of a much more general result. The earlier upper bound $\sup_{n\geq 1} \|T^n\| \leq (eC^2)/2$ was given in [1]. We proceed to give a common upper bound for the operators $n(I-T)T^n$ appearing in the tauberian condition (3).

Theorem 4. *Assume that $T \in \mathcal{L}(X)$ satisfies (1) for all $|\lambda| > 1$. Then*

$$\sup_{n\geq 1} (n+1)\|(I-T)T^n\| \leq 2 \sup_{n\geq 2} \|T^n\| + eC^3. \tag{7}$$

Proof. Recall that we have by the Cauchy interal

$$(I-T)T^n = \frac{1}{2\pi i} \int_\Gamma \lambda^n (1-\lambda)(\lambda - T)^{-1} d\lambda,$$

where Γ is an arbitrary positively oriented circle $|\lambda| = r > 1$. By partially integrating twice, we obtain

$$\begin{aligned} (I-T)T^n &= \frac{1}{\pi i (n+1)(n+2)} \int_\Gamma \lambda^{n+2}(1-\lambda)(\lambda-T)^{-3} d\lambda \\ &+ \frac{1}{n+1} \cdot \frac{2}{\pi i (n+2)(n+3)} \int_\Gamma \lambda^{n+3} (\lambda - T)^{-3} d\lambda. \end{aligned}$$

By partially integrating twice the Cauchy integral representation, we get

$$T^{n+1} = \frac{1}{\pi i (n+2)(n+3)} \int_\Gamma \lambda^{n+3} (\lambda - T)^{-3} d\lambda.$$

So we have for all $n \geq 1$

$$(n+1)(I-T)T^n - 2T^{n+1} = \frac{1}{\pi i (n+2)} \int_\Gamma \lambda^{n+2} (1-\lambda)(\lambda-T)^{-3} d\lambda.$$

By the Ritt resolvent condition (1) we get $\|(1-\lambda)(\lambda-T)^{-3}\| \leq C^3 |1-\lambda|^{-2}$ and hence for all $r > 1$

$$\|(n+1)(I-T)T^n - 2T^{n+1}\| \leq \frac{r^{n+2} C^3 J}{(n+2)\pi}, \tag{8}$$

where after computations

$$J = \int_{-\pi}^{\pi} \frac{r dt}{|re^{it} - 1|^2} = \frac{2\pi r}{r^2 - 1}.$$

Inserting the above expression for J into (8), we get

$$\|(n+1)(I-T)T^n - 2T^{n+1}\| \leq 2C^3 F(n,r), \tag{9}$$

where $F(n,r) := \frac{r^{n+3}}{(n+2)(r^2-1)}$ for all $r > 1$ and $n \geq 1$. Moreover,

$$\min_{r>1} F(n,r) = F\left(n, \sqrt{1+\frac{2}{n+1}}\right) = \frac{n+3}{2(n+2)}\left(1+\frac{2}{n+1}\right)^{\frac{n+1}{2}}$$

and after rather long computations that we omit here, we get finally $\sup_{n\geq 1} F\left(n, \sqrt{1+2/(n+1)}\right) = e/2$. These together with (9) prove the claim. $\square$

By letting $|\lambda| \to \infty$, it is easy to see that necessarily $C \geq 1$ in (1). Using this together with the bounds $\sup_{n\geq 1} \|T^n\| \leq C^2$ and (7) gives a more simple upper bound $\sup_{(n+1)\geq 1} (n+1)\|(I-T)T^n\| \leq (2+e)C^3$. In fact, the proof of Theorem 4 shows that the boundedness of sequences $\{T^n\}$ and $\{(n+1)(I-T)T^n\}$ is equivalent, whenever T satisfies only the "third order" Ritt condition

$$\sup_{|\lambda|>1} \|(1-\lambda)^3(\lambda - T)^{-3}\| < \infty.$$

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