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Abstract: *In this paper we study a refined asymptotic analysis for the quenching problem of the reaction diffusion equation* $u_t - u_{xx} = f(u)$ *with Cauchy-Dirichlet data, in the case where we have a logarithmic singularity, i.e.,* $f(u) = \ln(\alpha u)$, $\alpha \in (0, 1)$. *Our main goal is to give a precise asymptotic expression for the solution in a backward space-time parabola near a quenching point.*

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1 Introduction

We consider the nonlinear diffusion problem

$$\begin{aligned} u_t - u_{xx} = h(u), &\qquad x \in (-l, l), \quad t \in (0, T), \\ u(x, 0) = u_0(x), &\quad x \in [-l, l], \\ u(\pm l, t) = 1, &\quad t \in [0, T), \end{aligned} \tag{1.1}$$

where the initial function satisfies $0 < u_0(x) \leq 1$ and $u_0(\pm l) = 1$. Here T and l are positive constants. We assume that the reaction term $h(u)$ is singular at $u = 0$ in the sense that $\lim_{u \downarrow 0} h(u) = -\infty$. For $u > 0$ we take $h(u)$ to be smooth and to satisfy $(-1)^k h^{(k)}(u) < 0$; $k = 0, 1, 2$.

Originally the equation (1.1) was studied when $h(u) = -u^{-1}$, see [12]. This equation arises in the study of electric current transients in polarized ionic conductors (see [5, 12, 16] and references therein). The special interest taken in [12] for the equation (1.1) is motivated by the possibility that the solution $u(x, t)$ approaches zero in finite time. The reaction term then tends to infinity and the smooth solution ceases to exist. This phenomenon is called quenching. We say that a is a quenching point and T is a quenching time for $u(x, t)$, if there exists a sequence $\{(x_n, t_n)\}$ with $x_n \to a$ and $t_n \uparrow T$, such that $u(x_n, t_n) \to 0$ as $n \to \infty$.

After Kawarada's paper [12], the quenching problem for the equation (1.1) has been studied extensively by many authors, see for example the detailed review articles [13, 15]. In most of the papers that deal with the quenching problem for the equation (1.1), the reaction term is a power singularity, i.e., $h(u) = -u^{-p}, \quad p > 0$. The results concern existence and nonexistence of quenching points, qualitative properties of the quenching set, asymptotic behavior of the solutions in space and time near the quenching points, etc. For a power singularity it is now well-known that for sufficiently large l quenching occurs in finite time [1, 2, 14]. It is also known that the set of quenching points is finite [8].

In [17] the equation (1.1) was studied in the case where we have only a logarithmic singularity, i.e., $h(u) = \ln(\alpha u), \quad \alpha \in (0, 1)$. It was shown there that in spite of this weakening of the singularity, quenching still occurs for sufficiently large l and that the set of quenching points is finite. The main result in [17] concerns the asymptotic behavior of the solution in a neighborhood of a quenching point. It was assumed that

$$u_0''(x) + \ln(\alpha u_0(x)) \leq 0. \tag{1.2}$$

Then, it was shown that the quenching-rate satisfies

$$\lim_{t \uparrow T} \left(1 + \frac{1}{T - t} \int_0^{u(x,t)} \frac{d\tau}{\ln(\alpha \tau)} \right) = 0 \tag{1.3}$$

uniformly, when $|x - a| < C\sqrt{T - t}$ for every $C \in (0, \infty)$. Corresponding results for power singularities are well-known, see [3, 4, 8, 9, 10].

In this paper we refine the asymptotic result (1.3). The main result (Theorem 1.1) gives a precise asymptotic expression for the solution in a backward space-time parabola near a quenching point. The analysis is based on methods developed in [6, 7, 11]. These techniques were first developed for so-called blowup problems of reaction diffusion equations in [7, 11]. Then, for example, $h(u) = u^p$ or $h(u) = e^u$ in (1.1), and blowup means that $u(x,t) \to \infty$ in finite time. Subsequently these approaches were applied to quenching problems with a power singularity in [6].

Let us first explain briefly how (1.3) is proven in [17]. Without loss of generality we may assume that the quenching point is the origin. Define new variables:

$$y = \frac{x}{\sqrt{T-t}}, \quad s = -\ln(T-t).$$

Then the inverse transformations $x = x(y,s)$ and $t = t(s)$ are well defined. By these variables we define the function w:

$$w(y,s) = 1 + \frac{1}{T-t}\int_0^{u(x,t)} \frac{d\tau}{\ln(\alpha\tau)} = 1 + \frac{1}{T-t(s)}\int_0^{u(x(y,s),t(s))} \frac{d\tau}{\ln(\alpha\tau)}. \quad (1.4)$$

Then the equation (1.1) can be written in the form

$$w_s = w_{yy} - \frac{1}{2}yw_y + w + F, \quad (1.5)$$

where $F = \frac{u_x^2}{u(\ln(\alpha u))^2}$, and $(y,s) \in (-le^{\frac{1}{2}s}, le^{\frac{1}{2}s}) \times (-\ln T, \infty)$. Observe that the nonlinear effects are contained in the F-term. The result (1.3) can now be formulated in the form:

$$w(y,s) \to 0, \quad (1.6)$$

uniformly for bounded y, as $s \to \infty$.

In [17], the result (1.6) was derived in two steps. In the first step it was shown that $w(y,s) - w(0,s)(1 - \frac{1}{2}y^2) \to 0$ uniformly for bounded y and in the second step that $w(0,s) \to 0$.

The purpose of the present paper is to conclude how fast this limit value is reached and to determine the asymptotic form of w with respect to y.

In order to deduce an asymptotic form for $w(0,s)$, we first have to derive a corresponding ordinary differential equation for $w(0,s)$. However, there is a technical difficulty, because F cannot be expressed explicitly as a function of y, s and w. Therefore we first replace the transformation (1.4) by (1.7) and subsequently the equation (1.5) by (1.8). Define

$$\phi(y,s) = \frac{u(x,t)}{(T-t)(-\ln(T-t))} - 1. \quad (1.7)$$

In terms of the function ϕ, the equation (1.1) can be written in the form

$$\phi_s = \mathcal{L}\phi + \frac{1}{s}f(\phi) + g(s), \quad (1.8)$$

where

$$f(\phi) = \ln(1+\phi) - \phi, \tag{1.9}$$

$$g(s) = \frac{1}{s}(\ln(\alpha s) - 1) = \frac{\ln(s)}{s}(1+o(1)), \quad s \to \infty \tag{1.10}$$

$$\mathcal{L} = \frac{\partial^2}{\partial y^2} - \frac{y}{2}\frac{\partial}{\partial y} + 1. \tag{1.11}$$

Using Theorem 4.2 and Lemma 4.18 in [17], it can be concluded that (1.6) is equivalent to

$$\phi(y,s) \to 0, \tag{1.12}$$

uniformly for bounded y, as $s \to \infty$.

We shall now discuss how the result (1.12) might be refined for $\phi(y,s)$. Because $\phi(y,s) \to 0$, it is evident that the linear part will eventually dominate in the equation (1.8). We will study the equation (1.8) as a dynamical system in the space $L^2_\rho(R)$, where $\rho(y) = \exp(-y^2/4)$. Therefore we expand the function $\phi(y,s)$ with respect to the eigenfunctions of $\mathcal{L}$ in that space, i.e., $\phi = \sum a_j(s)h_j(y)$. Here the functions $h_j(y)$ are scaled Hermite polynomials and form an orthonormal base on $L^2_\rho(R)$. The spectrum of this operator is $\{\lambda_j | \lambda_j = \frac{2-j}{2}, j = 0,1,2,..\}$. By projecting the equation (1.8) to the subspaces generated by the functions $h_j(y)$, we get the ordinary differential equations for $a_j(s)$:

$$a_j'(s) = (1-\frac{k}{2})a_j(s) + \langle \frac{f(\phi)}{s} + g, h_j\rangle_{L^2_\rho} \quad j = 0,1,2,... \tag{1.13}$$

By analogy with classical ODE theory, we expect that one term in the Fourier series is dominant, i.e., $\phi(y,s) \approx a_j(s)h_j(y)$, for some j as $s \to \infty$. Linearizing for the nonzero eigenvalues, we get $\phi(y,s) \approx c_j \exp(\frac{2-j}{2}s)h_j(y)$. The positive eigenvalues ($j = 0,1$) are incompatible with the result (1.12), and therefore the nonlinear part has to dominate the positive eigenspace in (1.13). For the zero eigenvalue ($j = 2$), we can see that the linear part vanishes and after some calculations that $a_2(s)$ satisfies

$$a_2'(s) = -c^*\frac{1}{s}(1+o(1))a_2(s)^2,$$

from which we obtain after integration that $\phi(y,s) \approx \frac{C^*}{\ln(s)}\left(2-y^2\right)$.

The goal in this paper is to give a proof for this formal argument. The presence of a nontrivial null space for the operator $\mathcal{L}$ suggests the use of center manifold theory. More precisely, we use the methods developed in [6, 7, 11] for the analysis of infinite dimensional dynamical systems. The main result of this paper gives a refined asymptotics of the quenching:

Theorem 1.1. *Assume that $h(u) = \ln(\alpha u)$ in (1.1) and that (1.2) holds. Furthermore assume that $u(x,t)$ quenches at $(0,T)$. Let $\phi(y,s)$ be defined by (1.7) and assume that $|a_2(s)| \geq M(\ln(s)/s)^2$ for some $M > 0$. Then for any $C > 0$ and $\varepsilon > 0$ there exists s_0 such that*

$$\sup_{|y|<C} \left|\phi(y,s) - \frac{(y^2-2)}{8\ln(s)}\right| = O(\frac{\varepsilon}{\ln(s)}) \tag{1.14}$$

when $s \geq s_0$.

Restated in terms of u, (1.14) becomes

$$\frac{u(x,t)}{(T-t)(-\ln(T-t))} \approx 1 + \frac{1}{8\ln(-\ln(T-t))}\left(\frac{x^2}{T-t} - 2\right), \tag{1.15}$$

in the sense that the difference is $o((\ln(-\ln(T-t)))^{-1})$ as $t \uparrow T$, uniformly in parabolas $|x|^2 \leq C(T-t)$.

2 Preliminaries

We shall study the equation (1.1) as a dynamical system in the space $L^2_\rho(R)$. Therefore we first extend the equation (1.1) to all $x \in R$. This extended equation has the same solution in the region $\{(x,t) \in R^2 | x \in (-l,l), t \in (0,T)\}$ as the equation (1.1). The technical construction below is done similarly as in [18] or in [8], see also [17]. Without loss of generality, we may assume $l = 1$ in the equation (1.1). So let $x \in R$ and following [18] define the kernels:

$$V(x,t) = \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right), \quad W(x,t) = \frac{x}{2\sqrt{\pi t^3}} \exp\left(-\frac{x^2}{4t}\right),$$

when $x \in R$ and $0 < t < \infty$.

Differentiating these, we can see that $V_x = -W$, $V_t = V_{xx}$ and $W_t = W_{xx}$.

Define the extension $\overline{u}$ of $u(x,t)$, when $x > 1$ and $t > 0$ by

$$\overline{u}(x,t) = (x-1)\int_0^t W(x-1, t-\tau) u_x(1,\tau) d\tau + 1. \tag{2.1}$$

Here $u_x(1,t)$ is obtained from the solution of the equation (1.1) ($u_x(1,t) = \lim_{z\uparrow 1} u_x(z,t)$).

Lemma 2.1. *The function* $\overline{u}$ *satisfies:*

$$\overline{u}_t - \overline{u}_{xx} = 2u_x(1,0)V(x-1,t) + 2\int_0^t V(x-1,t-\tau) u_{x\tau}(1,\tau) d\tau,$$

when $x > 1$.

Proof. See [17]. □

Correspondingly; in the extension of u to the left of $x = -1$ the term $u_x(1,t)$ in the equation (2.1) is replaced by the term $u_x(-1,t)$ and $V(x-1,t)$ by $V(x+1,t)$.

An extended equation is now defined

$$\tilde{u}_t - \tilde{u}_{xx} = f(\tilde{u}(x,t)); \qquad x \in R \backslash \{\pm 1\}, \quad 0 < t < T, \tag{2.2}$$

where

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & \text{when } |x| \leq 1, \\ \overline{u}(x,t), & \text{when } |x| > 1, \end{cases}$$

$$f(\tilde{u}) = \begin{cases} \ln(\alpha u), & \text{when } |x| \le 1, \\ \overline{g}(x,t), & \text{when } |x| > 1, \end{cases} \tag{2.3}$$

and

$$\overline{g}(x,t) = 2u_x(1,0)V(x-1,t) + 2\int_0^t V(x-1,t-\tau)u_{x\tau}(1,\tau)d\tau. \tag{2.4}$$

We can see that $\tilde{u} \in C^1(R)$ (fixed t), but f is not continuous at $x = \pm 1$, and therefore $\tilde{u}$ is not twice continuously differentiable.

Because $u(x,t)$ cannot quench at $x = \pm 1$, the functions $u_x(1,t)$ and $u_{xt}(1,t)$ are uniformly bounded for $t \in [0,T)$.

Lemma 2.2. *The functions $\overline{u}(x,t)$ and $\overline{g}(x,t)$ satisfy for some positive c_1 and c_2 that $1 \le \overline{u}(x,t) < c_1 < \infty$ and $0 \le \overline{g}(x,t) < c_2 < \infty$, when $|x| \ge 1$ and $0 \le t < T$.*

Proof. See [17]. □

We can now define:

$$\tilde{\phi}(y,s) = \frac{\tilde{u}(x,t)}{(T-t)(-\ln(T-t))} - 1, \tag{2.5}$$

where $x \in R$ and $t \in (0,T)$.

Differentiating (2.5), we get

$$\tilde{\phi}_s = \mathcal{L}\tilde{\phi} + \frac{1}{s}f(\tilde{\phi}) + g(s) + F \tag{2.6}$$

where f and g are defined by (1.9) and (1.10). Furthermore

$$F = \chi_B \cdot \Big(\frac{2u_x(1,0)V(x-1,t) + 2\int_0^t V(x-1,t-\tau)u_{x\tau}(1,\tau)d\tau}{-\ln(T-t)} + \\ + 1 - \frac{\tilde{u}(x,t)}{(\ln(T-t))^2(T-t)} - \frac{1}{s}(\ln(1+\tilde{\phi}) - \tilde{\phi}) - \frac{1}{s}(\ln(\alpha s) - 1)\Big),$$

where $B = \{x \in R \big| |x| > 1\}$.

It can be seen easily that

$$|F| \le C\frac{|\tilde{\phi}|}{s}, \tag{2.7}$$

when $|x| > 1$ for sufficiently large s. One can also verify the following estimates

$$|\tilde{\phi}(y,s)| \le C(y^2 + 1), \tag{2.8}$$

and

$$|\tilde{\phi}_y(y,s)| \le C(|y| + 1). \tag{2.9}$$

Consider now the extended equation (2.6) as a dynamical system in the space

$$L^2_\rho(R) = \{g \in L^2_{loc}(R) : \int_R g(y)^2\rho(y)dy < \infty\}, \tag{2.10}$$

where $\rho(y) = \exp(\frac{-y^2}{4})$. Note that (2.8) and (2.9) imply that $\tilde{\phi} \in L^2_\rho(R)$. For simplicity we from now on use the notation $\phi = \tilde{\phi}$. Then

$$\phi_s - \mathcal{L}\phi = \frac{1}{s} f(\phi) + g(s) + F. \tag{2.11}$$

The space L^2_ρ is a Hilbert space with an inner product

$$\langle f, g \rangle_{L^2_\rho} = \int_R f(y)g(y)\rho(y)dy.$$

Concerning the linear operator $\mathcal{L}$ it is known that it is selfadjoint [7], i.e., that

$$\langle \mathcal{L}f, g \rangle_{L^2_\rho} = \langle f, \mathcal{L}g \rangle_{L^2_\rho}. \tag{2.12}$$

with spectrum $\lambda_k = 1 - \frac{1}{2}k; \quad k = 0, 1, 2, ...$ The corresponding eigenfunctions are $h_k(y) = \alpha_k H_k(\frac{1}{2}y)$, where H_k are the (standard) Hermite polynomials and $\alpha_k = (\pi^{\frac{1}{2}} 2^{k+1} k!)^{-\frac{1}{2}}$. The first three eigenfunctions are

$$h_0 = \frac{1}{\sqrt{2}} \pi^{\frac{-1}{4}}, \quad h_1 = \frac{1}{2} \pi^{\frac{-1}{4}} y, \quad h_2 = \frac{1}{2} \pi^{\frac{-1}{4}} (\frac{1}{2} y^2 - 1). \tag{2.13}$$

The Fourier-expansion of ϕ with respect to this space is:

$$\phi(y, s) = \sum_{j=0}^{\infty} a_j(s) h_j(y), \tag{2.14}$$

where $\mathcal{L}$ has eigenmodes:

$$\phi_+(y, s) \stackrel{def}{=} a_0(s)h_0(y) + a_1(s)h_1(y), \tag{2.15}$$

$$\phi_0(y, s) \stackrel{def}{=} a_2(s)h_2(y), \tag{2.16}$$

$$\phi_-(y, s) \stackrel{def}{=} \sum_{j=3}^{\infty} a_j(s)h_j(y). \tag{2.17}$$

We have now presented the functional framework, which is needed in the proof of Theorem 3.1. The importance of the decomposition above will become clear in the next Section.

3 Refined asymptotics

In this Section we prove the main Theorem 1.1 of this paper. This is done by first concluding the claim in Theorem 3.1 in an L^2_ρ-sense. Then we extend this result by showing that the convergence is uniform on compact subsets. In this Section $\phi(y, s)$ is the solution of (2.11).

In the following we use the notation

$$\|h(\underline{y}, s)\|_{L^p_\rho} = \Big(\int_R |h(z, s)|^p \rho(z) dz \Big)^{1/p}.$$

We assume throughout this paper that

$$\|\phi_0(\underline{y},s)\|_{L^2_\rho} \geq C\frac{(\ln(s))^2}{s^2}, \tag{3.1}$$

for some $C > 0$.

Let us briefly comment on this assumption. Note that the equation (2.11) is now nonhomogenous because of the term $g(s)$ (unlike the corresponding situation in [6, 7, 11]). Due to the nonlinearity of the equation this difficulty cannot be avoided by a simple transformation. However we believe that the behavior in Theorem 3.1 is generic and that the cases where (3.1) does not hold are exceptional, but we are unable to prove it. Finally observe that by Theorem 3.1 the assumption (3.1) specifies the asymptotics of the solution in detail.

Theorem 3.1. *If* (3.1) *holds, then* $\|\phi(\underline{y},s) - \frac{(y^2-2)}{8\ln(s)}\|_{L^2_\rho} = o(\frac{1}{\ln(s)})$.

The first fundamental fact in the proof of Theorem 3.1 is Lemma 3.4. The proof of this Lemma is based on [7, Lemma 3.1, p.836]. We formulate this Lemma here as Lemma 3.3. Before that we introduce Lemma 3.2. This Lemma gives an asymptotic form of the function $a_0(s)$ (as $s \to \infty$), which is essentially a consequence of the term $g(s)$ (only dependent of s) in the equation (2.11).

Lemma 3.2. *Let* $a_0(s)$ *be as in* (2.14) *and* (2.15). *Then*

$$a_0(s) = -(1+o(1))\sqrt{2}\pi^{1/4}\frac{\ln(s)}{s},$$

as $s \to \infty$.

Proof. Projecting the equation (2.11) to the subspace generated by h_0, we get by (2.12) and (2.13) that

$$a_0' = a_0 + \sqrt{2}\pi^{1/4}g(s) + \frac{1}{s}\langle f(\phi), h_0\rangle_{L^2_\rho} + \langle F, h_0\rangle_{L^2_\rho}.$$

Using (2.7) and Hölder's inequality, we derive that

$$|\langle F, h_0\rangle_{L^2_\rho}| \leq \frac{C}{s}\int_{e^{\frac{s}{2}}}^{\infty} |\phi(\eta,s)|e^{-\frac{\eta^2}{4}}d\eta \leq \frac{C}{s}\|\phi(\underline{y},s)\|_{L^2_\rho},$$

and from (1.9) that $|\langle f(\phi), h_0\rangle_{L^2_\rho}| \leq C\|\phi(\underline{y},s)\|_{L^2_\rho}$. Hence it follows from (1.12) and (2.8) that

$$a_0' = a_0+\sqrt{2}\pi^{1/4}\frac{\ln(s)}{s}(1+o(1))+\frac{1}{s}O(\|\phi(\underline{y},s)\|_{L^2_\rho}) = a_0+\sqrt{2}\pi^{1/4}\frac{\ln(s)}{s}(1+o(1)). \tag{3.2}$$

Furthermore, Parseval's formula and (1.12) imply that

$$\lim_{s\to\infty} a_0(s) = 0. \tag{3.3}$$

By the equation (3.2) it holds that:

$$(e^{-s}a_0(s))_s = (1+o(1))\sqrt{2}\pi^{1/4}e^{-s}\frac{\ln(s)}{s},$$

where the claim follows from a partial integration and from (3.3). □

Lemma 3.3. [7] *Let $x(t)$, $y(t)$ and $z(t)$ be absolutely continuous, real valued functions which are non-negative and satisfy: (a) $z' \geq c_0 z - \epsilon(x+y)$, (b) $|x'| \leq \epsilon(x+y+z)$, (c) $y' \leq -c_0 y + \epsilon(x+z)$, (d) $x, y, z \to 0$, as $t \to \infty$, where c_0 is any positive constant and ϵ is a sufficiently small positive constant. Then: either* **(i)** *$x, y, z \to 0$ exponentially fast, or else,* **(ii)** *there exists a time t_0 such that $z + y \leq b\epsilon x$ for $t \geq t_0$, where b is a positive constant depending only on c_0.*

Lemma 3.4. *Let ϕ_0 and ϕ_- be defined by* (2.16) *and* (2.17)*. If* (3.1) *holds, then*

$$\|a_1(s)h_1(\underline{y})\|_{L^2_\rho} + \|\phi_-(\underline{y}, s)\|_{L^2_\rho} = o(\|\phi_0(\underline{y}, s)\|_{L^2_\rho}).$$

Proof. We prove the inequalities (a), (b) and (c) in Lemma 3.3. Let $x(s) = \|\phi_0(\underline{y}, s)\|_{L^2_\rho}$, $y(s) = \|\phi_-(\underline{y}, s)\|_{L^2_\rho}$ and $z(s) = \|a_1(s)h_1(\underline{y})\|_{L^2_\rho}$. Furthermore, note that the condition (d) follows from (1.12) by Parseval's formula.

By (3.1) and Lemma 3.2, we get

$$|a_0(s)| = |(1+o(1))\sqrt{2}\pi^{1/4}\frac{\ln(s)}{s}| \leq \frac{Cs}{\ln(s)}\|\phi_0(\underline{y}, s)\|_{L^2_\rho}. \tag{3.4}$$

(a) The inequality for $\frac{\partial z}{\partial s} = \frac{\partial}{\partial s}\|a_1(s)h_1(\underline{y})\|_{L^2_\rho}$: Projecting the equation (2.11) to the subspace generated by h_1, it follows that

$$a_1' = \frac{1}{2}a_1 + \frac{1}{s}\langle f(\phi), h_1\rangle_{L^2_\rho} + \langle F, h_1\rangle_{L^2_\rho}.$$

Multiplying this by $sgn(a_1(s))$, and estimating the terms on the right-hand side as in the proof of Lemma 3.2, we get

$$|a_1(s)|_s \geq \frac{1}{2}|a_1(s)| - \frac{C}{s}\|\phi(\underline{y}, s)\|_{L^2_\rho}.$$

Because $\|\phi\|_{L^2_\rho} = \|a_0h_0 + a_1h_1 + \phi_0 + \phi_-\|_{L^2_\rho} \leq |a_0| + |a_1| + \|\phi_0\|_{L^2_\rho} + \|\phi_-\|_{L^2_\rho}$, then by the inequality (3.4)

$$|a_1(s)|_s \geq (\frac{1}{2} - \frac{C}{s})|a_1(s)| - (\frac{C}{\ln(s)} + \frac{C}{s})\|\phi_0(\underline{y}, s)\|_{L^2_\rho} - \frac{C}{s}\|\phi_-(\underline{y}, s)\|_{L^2_\rho}.$$

Therefore it holds for sufficiently large s that

$$\frac{\partial}{\partial s}\|a_1(s)h_1(\underline{y})\|_{L^2_\rho} \geq (\frac{1}{2} - \varepsilon)|a_1(s)| - \varepsilon\{\|\phi_0(\underline{y}, s)\|_{L^2_\rho} + \|\phi_-(\underline{y}, s)\|_{L^2_\rho}\}. \tag{3.5}$$

(b) The inequality for $\frac{\partial x}{\partial s} = \frac{\partial}{\partial s}\|\phi_0(\underline{y}, s)\|_{L^2_\rho}$: Projecting the equation (2.11) to the subspace generated by h_2, we get

$$a_2' = \frac{1}{s}\langle f(\phi), h_2\rangle_{L^2_\rho} + \langle F, h_2\rangle_{L^2_\rho}. \tag{3.6}$$

As in part (a), one can see that

$$||a_2(s)|_s| \leq \frac{C}{s}\|\phi(\underline{y},s)\|_{L^2_\rho} \leq \frac{C}{s}\{|a_0(s)|+|a_1(s)|+\|\phi_0(\underline{y},s)\|_{L^2_\rho}+\|\phi_-(\underline{y},s)\|_{L^2_\rho}\}.$$

Thus by (3.4)

$$|\frac{\partial}{\partial s}\|\phi_0(\underline{y},s)\|_{L^2_\rho}| \leq \varepsilon\{\|a_1(s)h_1(\underline{y})\|_{L^2_\rho} + \|\phi_0(\underline{y},s)\|_{L^2_\rho} + \|\phi_-(\underline{y},s)\|_{L^2_\rho}\}. \tag{3.7}$$

(c) The inequality for $\frac{\partial y}{\partial s} = \frac{\partial}{\partial s}\|\phi_-(\underline{y},s)\|_{L^2_\rho}$: Projecting (2.11) to the negative eigenspace (2.17), we get

$$\frac{\partial}{\partial s}\phi_- = \mathcal{L}\phi_- + \pi_-(\frac{1}{s}f(\phi) + F). \tag{3.8}$$

Multiplying by $\phi_-\rho$ and integrating with respect to y, we obtain

$$\begin{aligned}\int_R (\frac{\partial}{\partial s}\phi_-)\phi_-\rho = \int_R &(\mathcal{L}\phi_-)\phi_-\rho + \\ &\frac{1}{s}\int_R \pi_-(f(\phi))\phi_-\rho + \int_R \pi_-(F)\phi_-\rho.\end{aligned} \tag{3.9}$$

We now estimate the terms on the right-handside of the equation (3.9).

Because the greatest negative eigenvalue of $\mathcal{L}$ is $-\frac{1}{2}$, we conclude

$$\begin{aligned}\int_R (\mathcal{L}\phi_-)\phi_-\rho = \int_R &(\mathcal{L}(\sum_{j\geq 3} a_j h_j))(\sum_{j\geq 3} a_j h_j)\rho = \\ &\int_R (\sum_{j\geq 3}\frac{2-j}{2}a_j h_j))(\sum_{j\geq 3} a_j h_j)\rho \leq -\frac{1}{2}\|\phi_-\|^2_{L^2_\rho}.\end{aligned} \tag{3.10}$$

For the two nonlinear terms in (3.9) we have

$$|\int_R \pi_-(f(\phi))\phi_-\rho| \leq (\int_R (\pi_-(f(\phi)))^2\rho)^{\frac{1}{2}}\|\phi_-\|_{L^2_\rho} \leq C\|\phi\|_{L^2_\rho}\|\phi_-\|_{L^2_\rho}, \tag{3.11}$$

and

$$|\int_R \pi_-(F)\phi_-\rho| \leq (\int_R (\pi_-(F))^2\rho)^{\frac{1}{2}}\|\phi_-\|_{L^2_\rho} \leq \frac{C}{s}\|\phi\|_{L^2_\rho}\|\phi_-\|_{L^2_\rho}. \tag{3.12}$$

Substituting the inequalities (3.10), (3.11) and (3.12) into the equation (3.9), and using the inequality (3.4), we get for sufficiently large s that

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial s}\|\phi_-(\underline{y},s)\|^2_{L^2_\rho} \leq -\frac{1}{2}\|\phi_-(\underline{y},s)\|^2_{L^2_\rho} + \varepsilon\|\phi_-(\underline{y},s)\|_{L^2_\rho}\Big(&\|\phi_0(\underline{y},s)\|_{L^2_\rho} + \\ &+ \|\phi_-(\underline{y},s)\|_{L^2_\rho} + \|a_1(s)h_1(\underline{y})\|_{L^2_\rho}\Big),\end{aligned}$$

and therefore

$$\frac{\partial}{\partial s}\|\phi_-(\underline{y},s)\|_{L^2_\rho} \leq (-\frac{1}{2}+\varepsilon)\|\phi_-(\underline{y},s)\|_{L^2_\rho} + \varepsilon\Big(\|\phi_0(\underline{y},s)\|_{L^2_\rho} + \|a_1(s)h_1(\underline{y})\|_{L^2_\rho}\Big) \tag{3.13}$$

An application of Lemma 3.3 to the inequalities (3.5), (3.7) and (3.13), and recalling (1.12) and (3.1), gives the claim. □

The second crucial ingredient in the proof of Theorem 3.1 is to derive the ordinary differential equation for $a_2(s)$. This is done in Lemma 3.8. In order to obtain that Lemma, we need the estimate

$$\|\phi(\underline{y}, s)\|_{L^q_\rho} \leq C(q)\|\phi(\underline{y}, s)\|_{L^2_\rho},$$

for any $q > 2$. This inequality is a consequence of Lemmas 3.6 and 3.7. Before we prove these two Lemmas, we introduce some background from [11].

Let $S(t)$ be the linear semigroup corresponding to the heat equation in the strip $S = [0, T) \times R$. Take $u_0(x) \in L^1_{loc}(R)$ satisfying suitable growth conditions as $|x| \to \infty$, so that $(S(t)u_0(x))$ makes sense in S. Define now $w(y, s) = S(t)u_0(x)$, where $y = x(T-t)^{-\frac{1}{2}}$ and $s = -\ln(T-t)$. Then $w(y, s)$ satisfies

$$\begin{aligned} w_s = w_{yy} - \frac{1}{2}yw_y, \qquad y \in R, \quad s > 0, \\ w(y, 0) = w_0(y) \equiv u_0(x), \quad s = 0. \end{aligned} \tag{3.14}$$

Then we have

Lemma 3.5. **[11]** *Let $w(y, s)$ be the solution of (3.14). Then for any $r > 1$, $q > 1$ and $L > 0$, there exist $s_0^*(q, r) > 0$ and $C(q, r, L) > 0$ such that*

$$\|w(\underline{y}, s + s_0)\|_{L^r_\rho} \leq C\|w(\underline{y}, s)\|_{L^q_\rho},$$

for every $s > 0$ and every $s_0 \in [s_0^*, s_0^* + L]$.

This Lemma is used for our purposes below. More precisely, we estimate the first term at the right-hand side in the inequality (3.19) to take the final step in the proof of Lemma 3.6.

Lemma 3.6. *Let $r > 1$ and $L > 0$. Then there exist $s_0^*(r) > 0$ and $C(r, L) > 0$ such that $\|\phi(\underline{y}, s+s_0)\|_{L^r_\rho} \leq C\|\phi(\underline{y}, s)\|_{L^2_\rho}$, for every $s > 0$ and $s_0 \in [s_0^*, s_0^* + L]$.*

Proof. Multiplying the equation (2.11) by the function $sgn(\phi)$ and using Kato's inequality ($\Delta g \cdot sgn(g) \leq \Delta(|g|)$ in $\mathcal{D}'(R^N)$), we get by (2.7) that

$$\frac{\partial}{\partial s}|\phi(y, s)| \leq \mathcal{A}|\phi(y, s)| + (1 + c)|\phi(y, s)| + g(s), \tag{3.15}$$

where $\mathcal{A} = \frac{\partial^2}{\partial y^2} - \frac{y}{2}\frac{\partial}{\partial y}$. We now replace s by τ and multiply (3.15) by the function $e^{(1+c)(s+s_0-\tau)}e^{(s+s_0-\tau)\mathcal{A}}$, where

$$e^{s\mathcal{A}}z(y, s_0) = \int_R \frac{exp\{-\frac{(\eta - ye^{-\frac{1}{2}s})^2}{4(T-e^{-s})}\}}{\sqrt{4\pi(T - e^{-s})}} z(\eta, s_0)d\eta,$$

to obtain the inequality (3.15) in the form

$$\frac{\partial}{\partial \tau}\{e^{(1+c)(s+s_0-\tau)}e^{(s+s_0-\tau)\mathcal{A}}|\phi(y, \tau)|\} \leq e^{(1+c)(s+s_0-\tau)}e^{(s+s_0-\tau)\mathcal{A}}g(\tau). \tag{3.16}$$

Integrating (3.16) with respect to τ, from s to $s+s_0$, we conclude that

$$|\phi(y,s+s_0)| \le e^{(1+c)s_0} e^{s_0\mathcal{A}} |\phi(y,s)| + \int_s^{s+s_0} e^{(1+c)(s+s_0-\tau)} e^{(s+s_0-\tau)\mathcal{A}} g(\tau) d\tau. \tag{3.17}$$

Because g depends only on time, we have $e^{(s+s_0-\tau)\mathcal{A}} g(\tau) = cg(\tau)$. Furthermore, $s_0 \le L$, and thus

$$\int_s^{s+s_0} e^{(1+c)(s+s_0-\tau)} e^{(s+s_0-\tau)\mathcal{A}} g(\tau) d\tau \le C e^{(1+c)s_0} g(s).$$

Therefore the inequality (3.17) yields

$$|\phi(y,s+s_0)| \le C e^{s_0\mathcal{A}} |\phi(y,s)| + Cg(s). \tag{3.18}$$

Applying now Minkowski's inequality to (3.18), we deduce

$$\|\phi(\underline{y},s+s_0)\|_{L^r_\rho} \le C \| e^{s_0\mathcal{A}} |\phi(\underline{y},s)| \|_{L^r_\rho} + Cg(s). \tag{3.19}$$

Finally, by using Lemma 3.5 (take $q=2$ there) to the first term on right-hand side of (3.19), and Lemma 3.2 to the last term, we get the claim. □

Lemma 3.7. *Let $L > 0$. If (3.1) holds, then there exists $C = C(L)$ such that*

$$\|\phi(\underline{y},s)\|_{L^2_\rho} \le C \|\phi(\underline{y},s+L)\|_{L^2_\rho},$$

for all s.

Proof. Define $I_1 = \langle \frac{1}{s}(f(\phi)), h_2 \rangle_\rho$ and $I_2 = \langle F, h_2 \rangle_\rho$. By Hölder's inequality, we can see that

$$|I_j| \le \frac{C}{s} \|\phi\|_{L^2_\rho}, \tag{3.20}$$

when $j = 1, 2$. Multiplying the equation (3.6) by $sgn(a_2)$, we conclude by (3.20) that

$$|a_2(s)|_s \ge -\frac{C}{s} \|\phi\|_{L^2_\rho}. \tag{3.21}$$

On the other hand, by Lemma 3.4 it holds that

$$\|\phi(\underline{y},s)\|_{L^2_\rho} \le C(|a_0(s)| + |a_2(s)|). \tag{3.22}$$

Combining the inequalities (3.21), (3.22) and (3.4), we obtain

$$|a_2(s)|_s \ge -(\frac{C}{\ln(s)} + \frac{C}{s}) |a_2(s)| \ge -C|a_2(s)|. \tag{3.23}$$

An integration with respect to s yields

$$|a_2(s+L)| \ge e^{-cL} |a_2(s)|. \tag{3.24}$$

Using the relations (3.22), (3.24) and Lemma 3.2, we get

$$\|\phi(\underline{y},s+L)\|_{L^2_\rho} \ge C \max\Big\{ |a_2(s+L)|, |a_0(s+L)| \Big\} \ge \max\Big\{ Ce^{-cL}|a_2(s)|, C(L)|a_0(s)| \Big\} \ge C(L) \|\phi(\underline{y},s)\|_{L^2_\rho}.$$

□

Lemma 3.8. *The function $a_2(s)$ satisfies $a_2(s) \to 0$ and the ordinary differential equation ($c_i > 0, i = 1, 2$):*

$$sa_2' = -c_1(1 + o(1))a_2^2 + c_2(1 + o(1))\frac{\ln(s)}{s}a_2 + O[(\frac{\ln(s)}{s})^3], \tag{3.25}$$

as $s \to \infty$.

Proof. We first conclude by Parseval's formula and by (1.12) that $a_2(s) \to 0$ as $s \to \infty$.

Then write the equation (3.6) in the form

$$a_2'(s) = \frac{1}{s}\langle f(\phi), h_2\rangle_{L^2_\rho} + \langle F, h_2\rangle_{L^2_\rho} \equiv \frac{1}{s}I + J. \tag{3.26}$$

We estimate the term J. It can be seen by (2.7) that for s sufficiently large it holds: $|F| \le \frac{C}{s}|\phi(y, s)|^6$, and then by Lemmas 3.6 and 3.7 we obtain

$$|J| \le \frac{C}{s}\|\phi(\underline{y}, s)\|^6_{L^{12}_\rho} \le \frac{C}{s}\|\phi(\underline{y}, s)\|^6_{L^2_\rho}. \tag{3.27}$$

Define

$$\tilde{f}(\phi) = f(\phi) + \frac{1}{2}\phi^2. \tag{3.28}$$

Then we get:

(i) When $|\phi| < 1$, then, by Taylor's expansion, $|\tilde{f}(\phi)| \le C|\phi|^3$.

(ii) When $|\phi| \ge 1$, then $|\tilde{f}(\phi)| \le C\phi^2 \le C|\phi|^3$. Therefore

$$|\tilde{f}(\phi)| \le C|\phi|^3. \tag{3.29}$$

Using the formulas (3.28), (3.29), Hölder's inequality, Lemmas 3.6 and 3.7, we conclude (as $s \to \infty$)

$$I = -\frac{1}{2}\int_R \phi^2 h_2\rho + \int_R \tilde{f}(\phi)h_2\rho = -\frac{1}{2}\int_R \phi^2 h_2\rho + O(\|\phi(\underline{y}, s)\|^3_{L^2_\rho}) \tag{3.30}$$

Let: $I_1 = -\frac{1}{2}\int_R \phi^2 h_2\rho$, $I_2 = I - I_1$ and $\phi_r(y, s) = a_1(s)h_1(y) + \sum_{j=3}^\infty a_j(s)h_j(y)$. Therefore

$$\begin{aligned} I_1 = &-\frac{1}{2}\int_R a_0^2h_0^2h_2\rho - \frac{1}{2}\int_R a_2^2h_2^3\rho - \frac{1}{2}\int_R \phi_r^2h_2\rho - \\ &- a_0a_2\int_R h_0h_2^2\rho - a_0\int_R h_0h_2\phi_r\rho - a_2\int_R h_2^2\phi_r\rho \equiv \sum_{j=1}^6 P_j. \end{aligned} \tag{3.31}$$

Because the base $\{h_j\}_{j=0}^\infty$ is orthogonal and h_0 is constant, then

$$P_1 = 0 \quad \text{and} \quad P_5 = 0. \tag{3.32}$$

Furthermore we can verify that $\frac{1}{2}\int_R h_2^3\rho = c\int_R(y^2-2)^3\exp(-\frac{y^2}{4})dy = c_1 > 0$, therefore

$$P_2 = -c_1a_2^2. \tag{3.33}$$

Correspondingly $\int_R h_0 h_2^2 \rho = c_2 > 0$, thus by Lemma 3.2

$$P_4 = c_2 a_2 \frac{\ln(s)}{s}(1 + o(1)). \tag{3.34}$$

Using Hölder's and Minkowski's inequalities, and also Lemma 3.2, we obtain

$$\begin{aligned} |P_3 + P_6| &= |\int_R -\frac{1}{2}\phi_r(\phi_r + 2a_2h_2)h_2\rho| \\ &\le C\|\phi_r(\underline{y}, s)\|_{L^2_\rho}\Big\{\|\phi_r(\underline{y}, s)h_2(y)\|_{L^2_\rho} + c_4|a_2(s)|\Big\}, \end{aligned} \tag{3.35}$$

from which follows, by Hölder's inequality and Lemma 3.4, that (as $s \to \infty$)

$$|P_3 + P_6| \le o(|a_2(s)|)\Big\{\|\phi_r(\underline{y}, s)\|_{L^4_\rho} + c_4|a_2(s)|\Big\}. \tag{3.36}$$

Next we show that (as $s \to \infty$)

$$|P_3 + P_6| \le o(1)|a_2(s)| \max\Big\{C\frac{\ln(s)}{s}, |a_2(s)|\Big\}. \tag{3.37}$$

If $\|\phi_r(\underline{y}, s)\|_{L^4_\rho} \le C \max\Big\{C\frac{\ln(s)}{s}, |a_2(s)|\Big\}$, as $s \to \infty$, then (3.37) follows immediately from (3.36). On the other hand, if there exist sequences $c_i \to \infty$ and $s_i \to \infty$ such that

$$\|\phi_r(\underline{y}, s_i)\|_{L^4_\rho} \ge c_i \max\Big\{C\frac{\ln(s_i)}{s_i}, |a_2(s_i)|\Big\} \tag{3.38}$$

then, by Lemmas 3.2 and 3.4, one concludes: $\|\phi_r(\underline{y}, s_i)\|_{L^4_\rho} \ge Cc_i\|\phi(\underline{y}, s_i)\|_{L^2_\rho}$. Applying Lemmas 3.6 and 3.7 to this, and also making use of the triangle inequality, we can see that

$$\|\phi_r(\underline{y}, s_i)\|_{L^4_\rho} \ge Cc_i\|\phi(\underline{y}, s_i)\|_{L^4_\rho} \ge Cc_i\Big\{\|\phi_r(\underline{y}, s_i)\|_{L^4_\rho} - C\frac{\ln(s_i)}{s_i} - |a_2(s_i)|\Big\}$$

The assumption (3.38) now yields: $\|\phi_r(\underline{y}, s_i)\|_{L^4_\rho} \ge \tilde{C}_i\|\phi_r(\underline{y}, s_i)\|_{L^4_\rho}$, where $\tilde{C}_i \to \infty$ and $s_i \to \infty$. This is a contradiction, and so (3.38) does not hold and (3.37) is true.

Combining items (3.31), (3.32), (3.33), (3.34) and (3.37), we can conclude that (as $s \to \infty$)

$$I_1 = -c_1(1 + o(1))a_2(s)^2 + c_2(1 + o(1))\frac{\ln(s)}{s}a_2(s). \tag{3.39}$$

Invoke the formulas (3.27) and (3.30), and also (1.12) to get: $s(\frac{1}{s}I_2 + J) = O\{\|\phi(\underline{y}, s)\|^3_{L^2_\rho}\}$. Finally note that the formulas (3.26), (3.30) and (3.39) imply (as $s \to \infty$)

$$sa_2'(s) = -c_1(1+o(1))a_2(s)^2+c_2(1+o(1))\frac{\ln(s)}{s}a_2(s)+O\{\|\phi(\underline{y}, s)\|^3_{L^2_\rho}\}. \tag{3.40}$$

By Lemmas 3.2 and 3.4 we know that

$$\|\phi(\underline{y},s)\|_{L^2_\rho} \leq C\sqrt{(C\frac{\ln(s)}{s})^2 + |a_2(s)|^2} \leq C\Big(|C\frac{\ln(s)}{s}| + |a_2(s)|\Big).$$

Hence the equation (3.40) can be written in the form

$$sa_2'(s) = -c_1(1+o(1))a_2(s)^2 + c_2(1+o(1))\frac{\ln(s)}{s}a_2(s) + O\Big[(\frac{\ln(s)}{s})^3\Big]. \quad (3.41)$$

□

Lemma 3.9. *The solution of the equation in Lemma 3.8 is* $a_2(s) = \frac{c^*+o(1)}{\ln(s)}$*, as* $s \longrightarrow \infty$*.*

Proof. We show that the term $(-c_1(1+o(1))a_2(s)^2)$ is dominant on the right-handside of the equation (3.41).

From the equation (3.41) we get the estimate

$$p_1(a_2,s) \leq sa_2' \leq p_2(a_2,s), \quad (3.42)$$

where $(C > 0)$

$$\begin{cases} p_1(a_2,s) = -c_1(1+o(1))a_2(s)^2 + c_2(1+o(1))\frac{\ln(s)}{s}a_2(s) - C(\frac{\ln(s)}{s})^3 \\ p_2(a_2,s) = -c_1(1+o(1))a_2(s)^2 + c_2(1+o(1))\frac{\ln(s)}{s}a_2(s) + C(\frac{\ln(s)}{s})^3. \end{cases}$$

We shall determine under what conditions $a_2(s)$ is increasing or decreasing for large s. Therefore we solve the equations: $p_i = 0$. An elementary calculus yields

$$a_2(s) = \frac{1}{2}\cdot\frac{c_2}{c_1}(1+o(1))\frac{\ln(s)}{s}\Big[1 \pm \sqrt{1 - 4(\pm C)\frac{c_1}{c_2^2}(1+o(1))\frac{\ln(s)}{s}}\Big],$$

and thus

$$\begin{cases} a_2(s) = \frac{1}{2}\cdot\frac{c_2}{c_1}(1+o(1))\frac{\ln(s)}{s}\Big[1 \pm \Big\{1 - 2C\frac{c_1}{c_2^2}(1+o(1))\frac{\ln(s)}{s}\Big\}\Big], & \text{when } p_1 = 0 \\ a_2(s) = \frac{1}{2}\cdot\frac{c_2}{c_1}(1+o(1))\frac{\ln(s)}{s}\Big[1 \pm \Big\{1 + 2C\frac{c_1}{c_2^2}(1+o(1))\frac{\ln(s)}{s}\Big\}\Big], & \text{when } p_2 = 0. \end{cases} \quad (3.43)$$

Define the regions:

$$A_1 = \Big\{(a_2,s):\ s > s_0,\ a_2(s) > \frac{c_2}{c_1}(1+\varepsilon)\frac{\ln(s)}{s}\Big\}$$

$$A_2 = \Big\{(a_2,s):\ s > s_0,\ \frac{C}{c_2}(1+\varepsilon)(\frac{\ln(s)}{s})^2 < a_2(s) < \frac{c_2}{c_1}(1-\varepsilon)\frac{\ln(s)}{s}\Big\}$$

$$A_3 = \Big\{(a_2,s):\ s > s_0,\ a_2(s) < -\frac{C}{c_2}(1+\varepsilon)(\frac{\ln(s)}{s})^2\Big\}$$

$$B_1 = \Big\{(a_2, s):\ s > s_0,\ \frac{c_2}{c_1}(1-\varepsilon)\frac{\ln(s)}{s} < a_2(s) < \frac{c_2}{c_1}(1+\varepsilon)\frac{\ln(s)}{s}\Big\}$$

$$B_2 = \Big\{(a_2, s):\ s > s_0,\ -\frac{C}{c_1}(1+\varepsilon)(\frac{\ln(s)}{s})^2 < a_2(s) < \frac{C}{c_2}(1+\varepsilon)(\frac{\ln(s)}{s})^2\Big\}$$

Using the formulas (3.42) and (3.43), we can see that

$$\begin{cases} a_2(s) \text{ is increasing in the region } A_2 \\ a_2(s) \text{ is decreasing in the regions } A_1 \text{ and } A_3. \end{cases} \tag{3.44}$$

Because we know by Lemma (3.8) that: $a_2(s) \to 0$, as $s \to \infty$, then by (3.44) we can conclude that for sufficiently large s, $(s,a_2(s))$ cannot belong to the region A_3. Furthermore it follows from (3.1) that $(s, a_2(s)) \notin \bar{B}_2$. Hence for s sufficiently large it holds: $(s, a_2(s)) \in A_1 \cup \bar{B}_1 \cup A_2$. By the fact (3.44) we can see that for s sufficiently large there exists $k > 0$ such that

$$a_2(s) > k\frac{\ln(s)}{s}. \tag{3.45}$$

We prove next that for s suficiently large there exists $\beta > 0$ such that

$$a_2(s) > \beta\frac{(\ln(s))^2}{s}. \tag{3.46}$$

Define

$$z(s) = a_2(s) - \beta\frac{(\ln(s))^2}{s}, \tag{3.47}$$

where $\beta > 0$ is a constant, which will be determined later. Differentiating with respect to s the equation (3.47) and using the equation (3.41), we get

$$sz' = \beta\frac{(\ln(s))^2}{s}\Big\{1 - \frac{2}{\ln(s)} + c_2(1+o(1))\frac{\ln(s)}{s} - c_1(1+o(1))\cdot \\ (2z + \beta\frac{(\ln(s))^2}{s})\Big\} + O\Big[(\frac{\ln(s)}{s})^3\Big] + z\cdot c_2(1+o(1))\frac{\ln(s)}{s} - z^2c_1(1+o(1)). \tag{3.48}$$

By the inequality (3.45) one may deduce that $z(s) \geq -\beta\frac{(\ln(s))^2}{s}$, thus from the equation (3.48) it follows that

$$sz' \geq \beta\frac{(\ln(s))^2}{s}\Big\{1 - \frac{2}{\ln(s)} + c_2\cdot o(1)\frac{\ln(s)}{s} - \\ c_1(1+o(1))(2z + \beta\frac{(\ln(s))^2}{s})\Big\} - C(\frac{\ln(s)}{s})^3 - z^2c_1(1+o(1)). \tag{3.49}$$

Because we know that $a_2(s) \to 0$, then also by (3.47) it holds that $z \to 0$, as $s \to \infty$. Therefore from the ineqality (3.49) we obtain

$$sz' \geq \frac{1}{2}\beta\frac{(\ln(s))^2}{s} - C(\frac{\ln(s)}{s})^3 - 2c_1z^2, \tag{3.50}$$

for s sufficiently large.

We shall now conclude the claim (3.46) from the inequalities (3.45) and (3.50), and from the definition (3.47).

Let $\beta = \frac{k}{2\ln(s^*)}$, and choose s^* large enough. Then

(i) $z(s^*) > 0$,

(ii) $z'(s) \geq 0$, when $s \geq s^*$ and $z \in [0, \varepsilon\frac{(\ln(s))^2}{s}]$.

The claim (3.46) follows from the items (i) and (ii).

Substituting (3.46) in the equation (3.41), we obtain: $sa_2'(s) = -c_1(1 + o(1))a_2(s)^2$. Integrating this it follows that

$$\int_{a_2(s_0)}^{a_2(s)} \frac{dz}{z^2} = \int_{s_0}^{s} (-c_1(1 + o(1)))\frac{d\tau}{\tau},$$

and further

$$a_2(s) = \frac{1 + o(1)}{c_1 \ln(s)} = \frac{C^*(1 + o(1))}{\ln(s)}.$$

□

Proof of Theorem 3.1. We can write: $\phi(y, s) = a_0(s)h_0(y) + a_1(s)h_1(y) + \phi_0(y, s) + \phi_r(y, s)$, and then by Lemmas 3.2, 3.4 and 3.9 we get

$$\|\phi(\underline{y}, s) - \frac{C^*}{\ln(s)}h_2(\underline{y})\|_{L^2_\rho} = \|a_0h_0 + a_1h_1 + \phi_r + \phi_0 - \frac{C^*}{\ln(s)}h_2(\underline{y})\|_{L^2_\rho} \leq C\frac{\ln(s)}{s} + o(\frac{1}{\ln(s)}) \leq o(\frac{1}{\ln(s)}). \tag{3.51}$$

Finally we determine the constant C^*. By the equations (2.13), (3.31) and (3.33) it holds that

$$c_1 = \frac{1}{2}\int_R h_2^3\rho = \frac{\pi^{-3/4}}{16}\int_R \left(y^2/2 - 1\right)^3 \exp(-y^2/4)dy.$$

A straighforward calculation yields $c_1 = 2/\pi^{1/4}$ and $C^* = \pi^{1/4}/2$. Furthermore by (2.13) we get $C^*h_2(y) = (y^2 - 2)/8$. □

From this Theorem and from (2.9) we can conclude that the convergence is uniform, and the claim of Theorem 1.1 follows.

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