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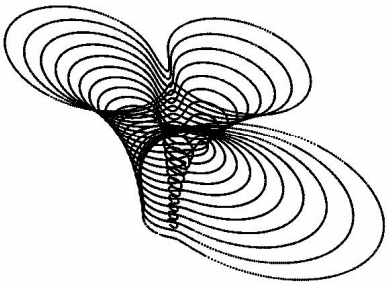

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SOLUTION METHODS FOR $\mathbb{R}$–LINEAR PROBLEMS IN $\mathbb{C}^n$

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Timo Eirola, Marko Huhtanen, Jan von Pfaler : *Solution methods for* $\mathbb{R}$*–linear problems in* $\mathbb{C}^n$ [1]; Helsinki University of Technology Institute of Mathematics Research Reports A454 (2003).

Abstract: *We consider methods, both iterative and direct, for solving an* $\mathbb{R}$*–linear system* $Mz + M_{\#}\overline{z} = b$ *in* $\mathbb{C}^n$ *with a pair of matrices* $M, M_{\#} \in \mathbb{C}^{n\times n}$ *and a vector* $b \in \mathbb{C}^n$*. Algorithms that avoid formulating the problem as an equivalent real linear system in* $\mathbb{R}^{2n}$ *are introduced. Conversely, this implies that real linear systems in* $\mathbb{R}^{2n}$ *can be solved with the methods proposed in this paper. Our study is motivated by Krylov subspace iterations with which using the real formulation can be disastrous in the standard linear case. Related matrix analysis and spectral theory are developed.*

AMS subject classifications: 15A04, 65F10

Keywords: $\mathbb{R}$–linear operator in $\mathbb{C}^n$, characteristic bivariate polynomial, isometry, LU–decomposition, QR–factorization, iterative methods, consimilarity

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[1] in electronic version figures 4 and 12 have been recalculated to squeeze the file size Oct 29, 2002.

1 Introduction

Consider solving, with a pair of square matrices $M, M_\# \in \mathbb{C}^{n\times n}$ and a vector $b \in \mathbb{C}^n$, the $\mathbb{R}$–linear system

$$Mz + M_\#\overline{z} = b. \tag{1}$$

Any standard linear system is a special case of this when either $M_\#$ or M is zero (linear and anti–linear, respectively). If both of these matrices are nonzero, we have a real linear operator in $\mathbb{C}^n$. This type of equations arise in certain engineering applications; see [21, 22, 18, 20]. See also [15, Chapters 4.15 and 5.25] and references therein.

In this paper we introduce direct and iterative methods for solving (1). Our study is originally motivated by iterative methods since the problem could readily be rewritten as an equivalent linear system of doubled size for its real and imaginary parts. Then any of the standard Krylov subspace methods could be executed. The usual linear case suggests, however, that this is not necessarily a good idea since the speed of convergence of iterations can be prohibitively slow; see [3, 4, 2]. It can also be regarded as a somewhat unnatural approach because $\mathbb{R}$ is not algebraically closed.

To avoid the real formulation with Krylov subspace methods, one option is to generate a matrix $Q_k \in \mathbb{C}^{n\times k}$ with orthonormal columns. To this end we employ the $\mathbb{R}$–linear operator corresponding to the left hand–side of (1) in an Arnoldi type of iteration. Then projecting the problem to $\mathbb{C}^k$, by using the Q_k computed, gives rise to a real linear system which can be solved with dense matrix techniques. This approach can be interpreted as a Galerkin approximation. Also minimal residual methods are devised.

It is also of interest to note that any real $2n$-by-$2n$ system can be written as (1). Therefore all the solution methods introduced in this paper apply to real linear systems in $\mathbb{R}^{2n}$ as well. This gives rise to new direct methods as well as novel non–symmetric iterations for real problems.

Clearly $\mathbb{R}$–linearity is a weaker assumption than $\mathbb{C}$–linearity. Therefore the problem considered involves two complex square matrices which makes the arising matrix analysis very interesting. A large part of the paper is devoted to these question. The spectrum of an $\mathbb{R}$–linear operator in $\mathbb{C}^n$ is introduced. We present various canonical forms, factorizations and respective solution formulas for the problem (1).

The paper is organized as follows. In section 2 we develop basic matrix analysis and spectral theory for $\mathbb{R}$–linear operators in $\mathbb{C}^n$. Direct methods for solving real linear systems are derived. In section 3 we introduce iterative methods for solving the corresponding problem and give numerical examples. In section 4 some preliminary ideas are considered for computing the spectrum numerically. Properties of the spectrum are illustrated with numerical experiments.

2 Properties of $\mathbb{R}$–linear operators in $\mathbb{C}^n$

When $\mathbb{C}^n$ is regarded as a vector space over $\mathbb{R}$, an $\mathbb{R}$–linear operator in $\mathbb{C}^n$ can be represented by a $2n$-by-$2n$ matrix. However, in this paper we consider $\mathbb{C}^n$ as a vector space over $\mathbb{C}$ with its usual complex valued inner product and associate with the system (1) an $\mathbb{R}$–linear mapping

$$\mathcal{M}(z) = Mz + M_{\#}\overline{z} \tag{2}$$

in $\mathbb{C}^n$. For the converse, when $\mathbb{C}^n$ is regarded as a vector space over $\mathbb{C}$, it is easy to verify that any real linear mapping in $\mathbb{C}^n$ can be represented in this form, after fixing a basis. We call M and $M_{\#}$ the linear and anti–linear parts of $\mathcal{M}$, respectively.

Aside from the system (1) one can consider its real form by using the matrices M and $M_{\#}$. To this end, write $z = x + iy$ and $b = c + id$. Then equating the real and imaginary parts gives rise to the linear system

$$\begin{bmatrix} \mathrm{Re}(M + M_{\#}) & -\mathrm{Im}(M - M_{\#}) \\ \mathrm{Im}(M + M_{\#}) & \mathrm{Re}(M - M_{\#}) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}. \tag{3}$$

We denote by $A \in \mathbb{R}^{2n\times 2n}$ the arising coefficient matrix. Conversely[2], this block structuring provides the conditions on reformulating a real $2n$-by-$2n$ linear system as an $\mathbb{R}$–linear problem in $\mathbb{C}^n$.

It is readily seen that if the pairs $(M, M_{\#})$ and $(N, N_{\#})$ correspond to the matrices A and B, respectively, then the real linear map

$$\mathcal{M}(\mathcal{N}(z)) = (MN + M_{\#}\overline{N_{\#}})\, z + (MN_{\#} + M_{\#}\overline{N})\, \overline{z} \tag{4}$$

corresponds to the matrix AB. Hence, under sufficient assumptions on invertibility (which are generically satisfied),

$$\mathcal{M}^{-1}(z) = (M - M_{\#}\overline{M}^{-1}\overline{M_{\#}})^{-1} z + (\overline{M_{\#}} - \overline{M}{M_{\#}}^{-1}M)^{-1}\overline{z}. \tag{5}$$

Further, the pair $(-iI, 0)$ corresponds to $J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$ and $(M^*, {M_{\#}}^T)$ to A^T.

Using these, some properties of A can be expressed via M and $M_{\#}$ as follows.

Proposition 1 *Let A be the coefficient matrix of (3). Then*

1. *A is (skew–) symmetric $\iff$ $M^* = (-)\, M$ and ${M_{\#}}^T = (-)\, M_{\#}$.*
2. *A is orthogonal $\iff$ $M^*M + {M_{\#}}^T\, \overline{M_{\#}} = I$ and $M^*M_{\#} + {M_{\#}}^T\, \overline{M} = 0$.*
3. *A is Hamiltonian $\iff$ $M^* = -M$ and ${M_{\#}}^T = M_{\#}$.*

[2]Using the notation of [3, Section 5.1], this corresponds to representing $A \in \mathbb{R}^{2n\times 2n}$ as the sum $A = M_{\star} + M_{\#_{\star\star}}$ in a unique way, i.e., we have an "$\mathbb{R}$–linear splitting" of A.

Since $M_{\#_{\star\star}}$ is similar to $-M_{\#_{\star\star}}$, its eigenvalues are symmetrically located with respect to the origin. Therefore, if $M_{\#_{\star\star}}$ dominates in this splitting, (3) can be very difficult to solve fast with iterative methods.

4. A is symplectic $\iff$ $M^*M - {M_\#}^T \overline{M_\#} = I$ *and* $M^*M_\# - {M_\#}^T \overline{M} = 0$.

(Hamiltonian means that $A^T = JAJ$ and symplectic that $A^T JA = J$.)

With the norm $\|\mathcal{M}\| = \max_{\|z\|=1} \|\mathcal{M}(z)\|$ the set of $\mathbb{R}$–linear operators in $\mathbb{C}^n$ is a Banach algebra over $\mathbb{R}$. However, $(M, M_\#) \mapsto (M^*, {M_\#}^T)$ is not an involution since $(\overline{\alpha}M^*, \alpha {M_\#}^T) \neq \overline{\alpha}(M^*, {M_\#}^T)$ for $\alpha \in \mathbb{C}\backslash\mathbb{R}$. In particular, we are not dealing with a $\mathbf{C}^*$–algebra.

2.1 The spectrum of an $\mathbb{R}$–linear operator in $\mathbb{C}^n$

For solvability of (1) it is natural to define the spectrum as follows.

Definition 2 *$\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{M} : \mathbb{C}^n \mapsto \mathbb{C}^n$, if the range of $\lambda I - \mathcal{M}$ is not $\mathbb{C}^n$. The set of eigenvalues of $\mathcal{M}$ is denoted by $\sigma(\mathcal{M})$.*

If $\lambda = \alpha + i\beta \in \mathbb{C}$, with $\alpha, \beta \in \mathbb{R}$, is an eigenvalue of $\mathcal{M}$, then there exists a vector $b \in \mathbb{C}^n$ such that the equation $(\lambda I - M)z - M_\#\overline{z} = b$ does not have a solution. Then its equivalent real formulation does not have a solution either. If A is the coefficient matrix in (3), this implies that

$$A(\alpha, \beta) = \alpha I - \beta J - A \tag{6}$$

is not invertible, i.e., $\det A(\alpha, \beta) = 0$. We call $\det A(\alpha, \beta)$ the characteristic bivariate polynomial of $\mathcal{M}$. Consequently, we have an algebraic criterion for finding the eigenvalues of $\mathcal{M}$. The following gives a geometric interpretation.

Proposition 3 *If $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{M}$, then there exists a nonzero vector $z \in \mathbb{C}^n$ such that $\mathcal{M}(z) = \lambda z$.*

It is now clear that $\lambda \notin \sigma(\mathcal{M})$ if and only if $\lambda I - \mathcal{M}$ is invertible.

Although an eigenvalue λ gives rise to an $\mathbb{R}$–linear invariant subspace for $\mathcal{M}$, we are actually dealing with a mildly nonlinear eigenproblem. More precisely, there need not be an invariant subspace associated with an eigenvector z of $\mathcal{M}$ when $\mathbb{C}^n$ is regarded as a vector space over $\mathbb{C}$. Indeed, with $\rho, \sigma \in \mathbb{R}$ we have

$$\mathcal{M}((\rho + i\sigma)z) = (\rho + i\sigma)\lambda z - i\,2\sigma M_\#\overline{z} \tag{7}$$

which belongs to $\operatorname{span}\{z, M_\#\overline{z}\}$, or equivalently, to $\operatorname{span}\{z, Mz\}$.

Proposition 4 *A subspace $V \subset \mathbb{C}^n$ is invariant for $\mathcal{M}$ if and only if it is simultaneously invariant for $z \mapsto Mz$ and $z \mapsto M_\#\overline{z}$.*

Proof It is clear that the latter implies the former. For the converse, assume that $\mathcal{M}(V) \subset V$. Then with $z \in V$ and $\beta \in \mathbb{C}$ we have $V \ni \beta\mathcal{M}(z) - \mathcal{M}(\beta z) = (\overline{\beta} - \beta)M_\#\overline{z}$, so that $M_\#\overline{V} \subset V$. Therefore, also $MV \subset V$.

In case V is an invariant subspace for $\mathcal{M}$, the spectrum of $\mathcal{M} : V \mapsto V$ is a subset of $\sigma(\mathcal{M})$, a property of fundamental importance in sparse matrix computations.

With an invertible $\mathbb{R}$–linear operator $\mathcal{T}$ in $\mathbb{C}^n$, consider a similarity transformation $\mathcal{T}^{-1} \circ \mathcal{M} \circ \mathcal{T}$ of $\mathcal{M}$. The spectrum of $\mathcal{M}$ remains invariant in this operation if the real form $B \in \mathbb{R}^{2n\times 2n}$ of $\mathcal{T}$ commutes with J. This is equivalent to having $\mathcal{T}(z) = Tz$ for an invertible $T \in \mathbb{C}^{n\times n}$. In this case we say that $\mathcal{T}^{-1} \circ \mathcal{M} \circ \mathcal{T}$ is a $\mathbb{C}$–linear similarity transformation of $\mathcal{M}$. The simplest such a $\mathcal{T}$ is $\mathcal{T}(z) = \lambda z$ with $\lambda \in \mathbb{C}\backslash\{0\}$. Then $\mathcal{T}^{-1} \circ \mathcal{M} \circ \mathcal{T}(z) = Mz + \frac{\overline{\lambda}}{\lambda} M_{\#} z$.

A general $\mathbb{R}$–linear similarity transformation in $\mathbb{C}^n$ need not preserve the spectrum except that the eigenvalues on the real axis remain invariant.

To quantify (7) more generally, consider the kernel of $\lambda I - \mathcal{M}$, i.e., the set $\{z \in \mathbb{C}^n : \lambda z - \mathcal{M}(z) = 0\}$. Denote by r its dimension as a subspace of $\mathbb{C}^n$ over $\mathbb{R}$ and let m be the dimension of the largest $\mathbb{C}$–linear subspace it contains. The resulting "multiplicity" index pair $(r/2, m)$ gives useful information regarding the eigenvalues of $\mathcal{M}$. Clearly, if the anti–linear part of $\mathcal{M}$ vanishes, then $r/2 = m$ for every eigenvalue.

Example 1 Let $\mathcal{M}$ be upper-triangular with $M = \left[\begin{smallmatrix} 2 & 1 \\ 0 & 4 \end{smallmatrix}\right]$ and $M_{\#} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then for any $\mu, \nu \in \mathbb{C}$ of modulus one we have the eigenvectors and eigenvalues

$$z = \left[\begin{smallmatrix} \mu \\ 0 \end{smallmatrix}\right] , \ \lambda = 2 + \overline{\mu}^2 \qquad \text{and} \qquad z = \left[\begin{smallmatrix} \frac{2\nu+\nu^3+\overline{\nu}}{4+2\nu^2+2\overline{\nu}^2} \\ \nu \end{smallmatrix}\right] , \ \lambda = 4 + \overline{\nu}^2 \ .$$

These are the only eigenvalues and eigenvectors (up to real multiples) of $\mathcal{M}$. Thus the spectrum of $\mathcal{M}$ consists of two circles having one common point $\lambda = 3$. For $\lambda \in \sigma(\mathcal{M})\backslash\{3\}$ we always have the index pair $(1/2, 0)$. At the intersection point we get $(1, 0)$.

The spectrum has the following algebro-geometric structure (see also [13]).

Theorem 5 *The spectrum of $\mathcal{M} : \mathbb{C}^n \mapsto \mathbb{C}^n$ is a bounded algebraic curve of degree $2n$ at most. The mapping $\lambda \mapsto (\lambda I - \mathcal{M})^{-1}$ is smooth for $\lambda \notin \sigma(\mathcal{M})$.*

Proof Since $\lambda I - \mathcal{M}$ is invertible if and only if (6) is, the spectrum of $\mathcal{M}$ consists of those points $(\alpha, \beta) \in \mathbb{R}^2$ for which $\det A(\alpha, \beta)$ is zero. This is clearly a bivariate polynomial in the real variables α and β of degree $2n$. That the arising algebraic curve must be bounded follows from Proposition 3 and the fact that $\mathcal{M}$ is a bounded operator.

For the second claim, for a fixed λ the mapping $\mathcal{R}(\lambda) = (\lambda I - \mathcal{M})^{-1}$ is also an $\mathbb{R}$–linear operator in $\mathbb{C}^n$. Therefore $(\lambda I - \mathcal{M})^{-1}(z) = R(\lambda)z + R_{\#}(\lambda)\overline{z}$ for matrices R and $R_{\#}$ depending on λ. The inverse of (6) is smooth at those points where the determinant is nonzero. Thus, R and $R_{\#}$ are smooth as well.

Although we do not have a general spectral mapping theorem, it is clear, for a fixed $\mu \in \mathbb{C}$, what are the spectra of $\mu I + \mathcal{M}$ and $\mu I \circ \mathcal{M}$ in terms of $\sigma(\mathcal{M})$.

The boundedness assertion of Proposition 5 imposes restrictions on those algebraic curves that can appear as the spectrum of an $\mathbb{R}$–linear operator in $\mathbb{C}^n$.

As we already saw in Example 1, the spectrum can contain circles. More generally:

Theorem 6 *Assume $S \in \mathbb{C}^{n\times n}$ is invertible and $\mathcal{R}(z) = Rz + R_{\#}\,\overline{z} = S^{-1}\mathcal{M}(Sz)$ is upper (lower) triangular. Then $\sigma(\mathcal{M})$ is the union of the circles*

$$\left\{\lambda \in \mathbb{C} : |r_{j,j} - \lambda| = |r^{\#}_{j,j}|\right\} \ , \quad j = 1, \ldots, n.$$

Proof Assume $\mathcal{R}$ is upper triangular. Clearly the spectra of $\mathcal{R}$ and $\mathcal{M}$ are the same.

If λ is not in the union of the circles, then the equations of type

$$(r_{k,k} - \lambda)w_k + r^{\#}_{k,k}\,\overline{w}_k = v_k$$

are uniquely solvable for w_k. Then $\lambda z - \mathcal{R}(z) = 0$ implies $z = 0$. Hence λ is not an eigenvalue of $\mathcal{R}$.

If λ is in the union, take the first j such that $|r_{j,j} - \lambda| = |r^{\#}_{j,j}|$. Set $w_j = \left(\frac{r^{\#}_{j,j}}{r_{j,j}-\lambda}\right)^{\frac{1}{2}}$ ($w = 1$ if $r^{\#}_{j,j} = 0$) and $w_k = 0$ for $k = j+1, \ldots, n$. Then $(r_{j,j} - \lambda)w_j - r^{\#}_{j,j}\,\overline{w}_j = 0$ and the equations for w_k, $k = j-1, \ldots, 1$ are uniquely solvable recursively to give an eigenvector of $\mathcal{R}$.

In case $M_{\#} = 0$ we may use a Schur decomposition of M and the circles reduce to points.

Remark. We can assume the diagonal entries of $R_{\#}$ to be nonnegative real (after performing a $\mathbb{C}$–linear diagonal unitary similarity transformation, if necessary). Hence we have a spectral mapping theorem in case $\mathcal{M}$ is triangularizable, i.e., by knowing only $\sigma(\mathcal{M})$ we can readily determine $\sigma\left(p(\mathcal{M})\right)$ for any polynomial p. To this end use (4) repeatedly.

In the situation of Theorem 6 there exists an increasing chain of nested invariant subspaces of $\mathcal{M}$ of dimension k for $k = 1, 2, \ldots$. The spectrum of $\mathcal{M}$ restricted to these subspaces consists of k circles corresponding to the first k pairs of the diagonal entries of R and $R_{\#}$. In this manner there arises a hierarchy among these circles since, unlike with the Schur decomposition (which exists if $M_{\#} = 0$), we cannot reorder the diagonal entries of R and $R_{\#}$ pairwise in general. To see this, it suffices to consider a 2-by-2 case. The circle corresponding to the $(1,1)$-entries always gives rise to an invariant subspace of dimension 1. The other circle need not have an invariant subspace associated with it. Consider, for example, $\mathcal{M}$ with $M = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $M_{\#} = \left[\begin{smallmatrix} 3 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. The invariant subspace of M corresponding to the eigenvalue $\lambda = 0$ of M is not invariant for $z \mapsto M_{\#}\overline{z}$. Thus, by Proposition 4, the order of the diagonal entries of M and $M_{\#}$ cannot be swapped.

Remark. Under the assumptions of Theorem 6, the characteristic bivariate polynomial of $\mathcal{M}$ factors as the product of 2nd degree bivariate polynomials. So one might consider using Krylov subspace techniques to locate just a few of these circles. However, the prescribed hierarchy can make this a very challenging problem.

If R and $R_\#$ are diagonal matrices, it is natural to say that $\mathcal{M}$ is diagonalizable in a $\mathbb{C}$–linear similarity transformation. Equivalently, $\mathcal{M}$ has n linearly independent eigenvectors which each give rise to an invariant subspace of $\mathcal{M}$. If the matrix S can be chosen unitary, we say that $\mathcal{M}$ is unitarily diagonalizable. Then M is normal while the condition on $M_\#$ means that the matrix is unitarily condiagonalizable, i.e., complex symmetric [8, Chapter 4.6]. See [8, Chapter 4.5] for a careful study and examples of the case in which M is additionally Hermitian. To this corresponds a symmetric coefficient matrix A in (3).

REMARK. If $\mathcal{M}$ is unitarily diagonalizable in a $\mathbb{C}$–linear similarity transformation, then its real form A lies in the unitary orbit of binormal matrices. For this, see [11].

The spectrum is not the union of circles in general.

Example 2 One readily verifies that with $M = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ and $M_\# = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 2 \end{smallmatrix}\right]$ the eigenvalues of $\mathcal{M}$ are given by those $\lambda \in \mathbb{C}$ that satisfy the equation $\lambda^2 - 2e^{-i2\theta}\lambda - e^{-i2\theta} = 0$ for some $\theta \in [0, 2\pi)$.

For a general $\mathcal{M}$ we can always have SMS^* in the upper triangular form with a unitary change of basis transformation S (use the Schur decomposition). Or, by performing Householder transformations in an obvious way, we can have a Hessenberg matrix $SM_\#S^T$.

To generalize the concept of unitary $\mathbb{C}$–linear similarity transformation, we call an $\mathbb{R}$–linear operator in $\mathbb{C}^n$ an *isometry* if it preserves the spectral norm. This gives us a group since, clearly, an isometry corresponds to an orthogonal matrix in $\mathbb{R}^{2n\times 2n}$. Hence if $\mathcal{U}(z) = Uz + U_\# \overline{z}$ is an isometry, we have

$$\mathcal{U}^{-1}(z) = U^*z + {U_\#}^T \overline{z}\,.$$

Example 3 If $Q \in \mathbb{C}^{n\times k}$ satisfies $\mathrm{Re}(Q^*Q) = I$, then $\mathcal{U}(z) = (I - QQ^*)z - QQ^T\overline{z}$ is an isometry (use Proposition 1 item *2*). In addition, it satisfies $\mathcal{U}^2 = I$. Note that the columns of Q need not be linearly independent, like $Q = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1-i & 1+i \\ 0 & 0 \end{smallmatrix}\right]$ illustrates. Moreover, if $Q \in \mathbb{C}^{n\times 1}$ is a unit vector, then $\mathcal{U}$ corresponds to a Householder transformation in $\mathbb{R}^{2n\times 2n}$.

With an isometry we preserve the lengths but not the angles, i.e., for $z, w \in \mathbb{C}^n$ the inner product (z, w) need not equal $(\mathcal{U}(z),\mathcal{U}(w))$ unless $\mathcal{U}$ is unitary. In connection with the QR–decomposition we need isometries which map an arbitrary pair of vectors to be parallel; see section 2.2.

Proposition 7 *Let $\mathcal{U}$ be an isometry. Then $\sigma(\mathcal{U})$ is either empty, a finite set on the unit circle, or the unit circle.*

Proof If λ is an eigenvalue of $\mathcal{U}$, then it must have modulus one. If $\sigma(\mathcal{U})$ is not finite, then the respective algebraic curve must be closed. Thereby it is the unit circle. To see that the spectrum can be empty, consider $\mathcal{U}(z) = \left[\begin{smallmatrix} 0 & 1 \\ i & 0 \end{smallmatrix}\right]\overline{z}$.

If $M^* = -M$ and ${M_\#}^T = -M_\#$, then $(\mathcal{M} + I) \circ (\mathcal{M} - I)^{-1}$ gives us an isometry, i.e., an analogy of the Cayley transform.

The following can be verified by a direct computation.

Proposition 8 *Let $\mathcal{U}$ be an isometry. If $\mathcal{M}(z) = Mz + M_\#\overline{z}$ with $M^* = M$ and ${M_\#}^T = M_\#$ (or $M^* = -M$ and ${M_\#}^T = -M_\#$), then $\mathcal{U}^{-1} \circ \mathcal{M} \circ \mathcal{U} = Nz + N_\#\overline{z}$ with $N^* = (-)\, N$ and ${N_\#}^T = (-)\, N_\#$.*

In a translation of an anti–linear operator we have $M = \kappa I$ with $\kappa \in \mathbb{C}$. This case is of particular importance in view of applications [21, 22, 18, 20] (with $\kappa = 0$ it arises in particle physics). It also appears after preconditioning the system (1) with the inverse of M from the left, under the assumption that M is readily invertible. We denote the corresponding operator by $\mathcal{M}_\kappa$, that is, $\mathcal{M}_\kappa(z) = \kappa z + M_\#\overline{z}$. This yields another instance where we encounter circles.

Proposition 9 *For $\mathcal{M}_\kappa$ the spectrum is the union of circles centered at κ.*

Proof Repeat the arguments of [8, p. 245] with the translation κ.

This case is not covered by Theorem 6 since $\mathcal{M}_\kappa$ may not have an upper triangular form under $\mathbb{C}$–linear similarity transformation ([8, Theorem 4.6.3] determines when this is possible). Moreover, the situation is fundamentally different now since there is an invariant subspace associated with each circle.

At least one circle appears in the following case.

Proposition 10 *Assume $\mathcal{M}(z) = Mz + \kappa\overline{z}$ with $\kappa \in \mathbb{C}$. If the intersection the null spaces of M and $\overline{M}$ is nontrivial, then $\sigma(\mathcal{M})$ contains the set $\{\lambda \in \mathbb{C} : |\lambda| = |\kappa|\}$.*

Proof If $Mv = M\overline{v} = 0$, then

$$\lambda(\alpha v + \alpha\overline{v}) - \mathcal{M}(\alpha v + \alpha\overline{v}) = (\lambda\alpha - \kappa\overline{\alpha})(v + \overline{v}) \ .$$

If $v + \overline{v} \neq 0$ we get eigenvalues $\lambda = \kappa\overline{\alpha}/\alpha$, i.e., all complex numbers with modulus $|\kappa|$. If $\overline{v} = -v$, use vector $v - \overline{v}$.

Note that if $\operatorname{rank}(M) < \frac{n}{2}$, then the assumptions of the proposition are satisfied. This is the case also if M is real and singular.

If $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{M}_\kappa$, then $M_\#\overline{z} = (\lambda - \kappa)z$ holds for a nonzero $z \in \mathbb{C}^n$. Therefore $\overline{M_\#}z = (\overline{\lambda} - \overline{\kappa})\overline{z}$, so that

$$M_\#\overline{M_\#}z = (\overline{\lambda} - \overline{\kappa})M_\#\overline{z} = |\lambda - \kappa|^2 z.$$

Consequently, a necessary condition for λ to be an eigenvalue of $\mathcal{M}_\kappa$ is that $M_\#\overline{M_\#}$ has $|\lambda - \kappa|^2$ as its eigenvalue. Since $M_\#\overline{M_\#}$ may have no real non-negative eigenvalues, we infer that the spectrum of $\mathcal{M}_\kappa$ can be empty. See also [8, Chapter 4].

The spectrum of an $\mathbb{R}$–linear operator in $\mathbb{C}^n$ is related to the eigenvalues of its real form (3) as follows.

Proposition 11 *Let $A \in \mathbb{R}^{2n\times 2n}$ be the real form of $\mathcal{M}$. Then $\lambda = \alpha + i\beta \in \sigma(\mathcal{M}) \setminus \{0\}$ if and only if $\alpha^2 + \beta^2 \in \sigma\big(\alpha A + \beta JA\big)$.*

Proof Assume $z = x + iy \in \mathbb{C}^n$ is an eigenvector corresponding to λ. Then rewriting the equality $\mathcal{M}(z) = \lambda z$ by using (6) we have

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \otimes I \begin{bmatrix} x \\ y \end{bmatrix}$$

which is equivalent to

$$\left(\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \otimes I \right) A \begin{bmatrix} x \\ y \end{bmatrix} = (\alpha^2 + \beta^2) \begin{bmatrix} x \\ y \end{bmatrix},$$

proving the claim.

Fix $\rho \in \mathbb{R}$ and assume $\mu \in \mathbb{R}$ and consider

$$A + \rho JA. \tag{8}$$

Then using Proposition 11 we obtain $\beta = \rho\alpha$ and $\alpha\mu = \alpha^2(1 + \rho^2)$, so that $\lambda = \frac{\mu}{1+\rho^2}(1 + i\rho)$ is an eigenvalue of $\mathcal{M}$. In other words, any real eigenvalue of (8), with $\rho \in \mathbb{R}$, gives rise to an eigenvalue of $\mathcal{M}$.

REMARK. If $M_\#$ is complex symmetric, then (8) is symmetric for $\mathcal{M}_0$, independently of $\rho \in \mathbb{R}$. Consequently, $\sigma(\mathcal{M}_\kappa)$ is nonempty.

Proposition 12 *For $\mathcal{M}(z) = Mz + M_\#\overline{z}$ let $\widetilde{\mathcal{M}}(z) = M^*z + {M_\#}^T\overline{z}$. Then $\sigma(\widetilde{\mathcal{M}}) = \overline{\sigma(\mathcal{M})}$.*

Proof The real form of $\widetilde{\mathcal{M}}$ is A^T, where A is the real form of $\mathcal{M}$. Since $(\beta J + A)^T = -\beta J + A^T$, an eigenvalue $\alpha + i\beta$ of $\widetilde{\mathcal{M}}$ gives rise to an eigenvalue $\alpha - i\beta$ of $\mathcal{M}$ and vice versa.

In particular, if $M^* = M$ and ${M_\#}^T = M_\#$, then the spectrum is symmetric relative to the real axis. If $M^* = -M$ and ${M_\#}^T = -M_\#$, then $\sigma(\mathcal{M})$ is on the imaginary axis although it need not be symmetrically located with respect to the origin.

Naturally all the eigenvalues of a real linear operator $\mathcal{M}$ in $\mathbb{C}^n$ lie inside the disk $\{\lambda \in \mathbb{C} : |\lambda| \leq ||\mathcal{M}||\}$. Also the field of values is defined in an obvious way. Geršgorin disks have an analogy with

$$\rho_l(M, M_\#) = |m^\#_{ll}| + \sum_{j=1, j\neq l}^{n} (|m_{lj}| + |m^\#_{lj}|).$$

Then a direct adaptation, e.g., of the proof of [8, Theorem 6.1.1] can be used to show that the eigenvalues of $\mathcal{M}$ are located in the union of disks

$$\bigcup_{l=1}^{n} \{z \in \mathbb{C} : |z - m_{ll}| \leq \rho_l(M, M_\#)\}. \tag{9}$$

An analogy of the Bauer–Fike theorem holds as well.

Proposition 13 *Assume $S \in \mathbb{C}^{n\times n}$ is invertible such that $S^{-1} \circ \mathcal{M} \circ S$ is diagonal. If $\mathcal{E}$ is $\mathbb{R}$–linear in $\mathbb{C}^n$ and λ is an eigenvalue of $\mathcal{M} + \mathcal{E}$, then*

$$\operatorname{dist}(\sigma(\mathcal{M}), \lambda) \leq \left|\left|S^{-1}\right|\right| \, ||S|| \, \|\mathcal{E}\| \, .$$

Proof For a diagonal $\mathbb{R}$–linear operator the norm of the resolvent is the reciprocal of the distance of λ to the spectrum. To see this it suffices to consider the scalar case. With fixed λ_1, $\lambda_2 \in \mathbb{C}$ the inverse of $z \mapsto (\lambda - \lambda_1)z - \lambda_2\overline{z}$ is

$$z \mapsto \frac{1}{|\lambda - \lambda_1|^2 - |\lambda_2|^2}\big((\overline{\lambda} - \overline{\lambda_1})z + \lambda_2\overline{z}\big). \tag{10}$$

Choosing z on the unit circle such that $(\overline{\lambda} - \overline{\lambda_1})z$ and $\lambda_2\overline{z}$ are parallel, the norm of (10) is $\left|\frac{1}{|\lambda-\lambda_1|-|\lambda_2|}\right|$, i.e., the reciprocal of the distance of λ to the spectrum. Hence we can mimic the proof of [8, Theorem 6.3.2] together with (15).

This is of use, e.g., if M is diagonalizable and $\|M_\#\| \ll \|M\|$.

2.2 Factorizations for an $\mathbb{R}$–linear operator in $\mathbb{C}^n$

Consider solving the system $\mathcal{M}(z) = b$ for $b \in \mathbb{C}^n$. If both M and $M_\#$ are upper (lower) triangular matrices, then we can use the formula (10) on a sequence of 1-by-1 systems together with back (forward) substitution to find the solution[3]. In this case we say that $\mathcal{M}$ is upper (lower) triangular.

For solving a general real linear system in $\mathbb{C}^n$ we need to factorize $\mathcal{M}$.

LU–decomposition. For given $M, M_\# \in \mathbb{C}^{n\times n}$ consider finding a lower triangular $\mathcal{L}(z) = Lz + L_\#\overline{z}$ and an upper triangular $\mathcal{U}(z) = Uz + U_\#\overline{z}$ such that

$$\mathcal{M}(z) = Mz + M_\#\overline{z} = \mathcal{L}(\mathcal{U}(z)) = (LU + L_\#\overline{U_\#})z + (LU_\# + L_\#\overline{U})\overline{z}$$

holds for every $z \in \mathbb{C}^n$, i.e., $\mathcal{M} = \mathcal{L} \circ \mathcal{U}$. We assume that all the diagonal entries of L equal to 1, and that $L_\#$ is strictly lower triangular.

We need appropriate elementary $\mathbb{R}$–linear operators in $\mathbb{C}^n$. The following is easy to check, where, for the sake of clarity, both row and column vectors are boldfaced.

Lemma 14

$$\textit{If}\quad \mathcal{L}(z) = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{l} & I \end{bmatrix} z + \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{l}^\# & 0 \end{bmatrix} \overline{z}\,, \qquad \textit{then} \qquad \mathcal{L}^{-1}(z) = \begin{bmatrix} 1 & \mathbf{0} \\ -\mathbf{l} & I \end{bmatrix} z + \begin{bmatrix} 0 & \mathbf{0} \\ -\mathbf{l}^\# & 0 \end{bmatrix} \overline{z}\,.$$

Assume $\mathcal{L}_1$ is of this type and partition M and $M_\#$ accordingly as

$$M = \begin{bmatrix} m_{1,1} & \mathbf{m}_{1,2}^T \\ \mathbf{m}_{2,1} & M_{2,2} \end{bmatrix} \text{ and } M_\# = \begin{bmatrix} m_{1,1}^\# & \mathbf{m}_{1,2}^{\#\,T} \\ \mathbf{m}_{2,1}^\# & M_{2,2}^\# \end{bmatrix}.$$

We need to determine $\mathbf{l}$ and $\mathbf{l}^\#$. The linear part of $\mathcal{L}_1 \circ \mathcal{M}$ is

$$\begin{bmatrix} m_{1,1} & \mathbf{m}_{1,2}^T \\ m_{1,1}\,\mathbf{l} + \mathbf{m}_{2,1} + \overline{m_{1,1}^\#}\,\mathbf{l}^\# & \mathbf{l}\,\mathbf{m}_{1,2}^T + \mathbf{l}^\#\,\overline{\mathbf{m}_{1,2}^\#}^T + M_{2,2} \end{bmatrix}$$

[3]Hence the Gauss–Seidel and the Jacobi method, as well as any other basic iterations, can be devised by splitting a given $\mathbb{R}$–linear operator in $\mathbb{C}^n$ in an obvious way.

while its anti-linear part is

$$\begin{bmatrix} m_{1,1}^{\#} & \mathbf{m}_{1,2}^{\#\,T} \\ m_{1,1}^{\#}\,\mathbf{l} + \mathbf{m}_{2,1}^{\#} + \overline{m_{1,1}}\,\mathbf{l}^{\#} & \mathbf{l}\,\mathbf{m}^{\#T}_{1,2} + \mathbf{l}^{\#}\,\overline{\mathbf{m}_{1,2}}^{T} + M_{2,2}^{\#} \end{bmatrix}.$$

In order to have zeros in the first columns of these below the diagonal we take

$$\begin{bmatrix} \mathbf{l} & \mathbf{l}^{\#} \end{bmatrix} = - \begin{bmatrix} \mathbf{m}_{2,1} & \mathbf{m}^{\#}{}_{2,1} \end{bmatrix} \begin{bmatrix} m_{1,1} & m_{1,1}^{\#} \\ m_{1,1}^{\#} & \overline{m_{1,1}} \end{bmatrix}^{-1} \tag{11}$$

Thus, we need to assume that $|m_{1,1}| \neq |m_{1,1}^{\#}|$.

This is then repeated with the blocks

$$\mathbf{l}\,\mathbf{m}_{1,2}^{T} + \mathbf{l}^{\#}\,\overline{\mathbf{m}_{1,2}^{\#}}^{T} + M_{2,2} \equiv \widetilde{M}$$

and

$$\mathbf{l}\,\mathbf{m}^{\#T}_{1,2} + \mathbf{l}^{\#}\,\overline{\mathbf{m}_{1,2}}^{T} + M_{2,2}^{\#} \equiv \widetilde{M_{\#}}$$

of size $(n-1)$-by-$(n-1)$. If no breakdown occurs, after $n-1$ steps we have an upper triangular $\mathcal{L}_{n-1} \circ \cdots \circ \mathcal{L}_1 \circ \mathcal{M}$. Or equivalently, by using Lemma 14 repeatedly, we have an LU–decomposition of $\mathcal{M}$ (since products of lower triangular $\mathbb{R}$–linear operators in $\mathbb{C}^n$ remain lower triangular). The product $\mathcal{L}_1^{-1} \circ \mathcal{L}_2^{-1} \circ \cdots \circ \mathcal{L}_{n-1}^{-1}$ does not involve any computations since the lower triangular parts of its linear and anti-linear part are obtained by collecting the vectors from each of its factors.

If $M_{\#} = 0$, then this gives us the standard LU–factorization of M.

REMARK. The 2-by-2 matrix in (11) is now the "pivot". There are $n-1$ pivot matrices in all. In particular, pivoting is straightforwardly incorporated with the scheme by performing pre/post-operations with $\mathcal{P}(z) = Pz$, where P is a permutation matrix. This is needed if the inversion in (11) is ill-conditioned.

Define the jth principal minor of $\mathcal{M}$ by extracting the upper left j-by-j blocks of M and $M_{\#}$ and compute the value of the corresponding characteristic bivariate polynomial at the origin. It is easy to see that if all the principal minors of $\mathcal{M}$ are nonzero, this LU–factorization exists.

Assuming no breakdown occurs, a `Matlab` [17] code is as follows.

```
function [L,La,U,Ua]=rl_lu(M,Ma)

% This computes lower triangular L (with unit diagonal),
% strictly lower triangular La, and
% upper triangular U and Ua such that
%
%    M=L*U+La*conj(Ua)   and   Ma=L*Ua+La*conj(U)

n=size(M,1); L=eye(n); La=zeros(n); U=M; Ua=Ma;
for k=2:n ,
```

```
    a=U(k-1,k-1);  b=Ua(k-1,k-1);
    w=[U(k:n,k-1),Ua(k:n,k-1)]/[a,b;b',a'];
    L(k:n,k-1)=w(:,1);  La(k:n,k-1)=w(:,2);
    z=zeros(n-k+1,1);  U(k:n,k-1)=z;  Ua(k:n,k-1)=z;
    U(k:n,k:n)=U(k:n,k:n)-w*[U(k-1,k:n);conj(Ua(k-1,k:n))];
    Ua(k:n,k:n)=Ua(k:n,k:n)-w*[Ua(k-1,k:n);conj(U(k-1,k:n))];
  end
```

This requires $\sim \frac{4}{3}n^3$ complex flops to compute an LU–factorization of $\mathcal{M} : \mathbb{C}^n \mapsto \mathbb{C}^n$. The actual execution time depends on how well complex arithmetic is implemented on a computer. In practice a pivoting strategy is also needed.

For more symmetry, let $u_{j,j}\,,\ u^{\#}_{j,j}\,,\ j = 1, \ldots, n$ be the diagonal entries of U and $U_{\#}$, respectively. Define a diagonal operator $\mathcal{D}(z) = Dz + D_{\#}\overline{z}$ according to $D = \mathrm{diag}(\overline{u_{j,j}})$ and $D_{\#} = \mathrm{diag}(-u^{\#}_{j,j})$. If $|u_{j,j}| \neq |u^{\#}_{j,j}|\,,\ j = 1, \ldots, n$, then $\mathcal{D}$ is invertible and $\mathcal{M} = \mathcal{L} \circ \mathcal{D} \circ \widetilde{\mathcal{U}}$ with an upper triangular $\widetilde{\mathcal{U}}(z) = \mathcal{D}^{-1}(\mathcal{U}(z)) = \widetilde{U}z + \widetilde{U_{\#}}\overline{z}$ such that all the diagonal entries of $\widetilde{U}$ equal to 1 while $\widetilde{U_{\#}}$ is strictly upper triangular. This gives us a "Cholesky factorization" if $M^* = M$ and ${M_{\#}}^T = M_{\#}$; see Proposition 1 item *1*. Namely then $L^* = \widetilde{U}$ and ${L_{\#}}^T = \widetilde{U_{\#}}$. This adds to the fact that this type of real linear operators have many special properties.

For further structure, when M and $M_{\#}$ are banded, the factors $\mathcal{L}$ and $\mathcal{U}$ inherit the (maximum) band structure.

A given $2n$-by-$2n$ real matrix can fail to have an LU–factorization (without pivoting) but has an LU–factorization as an $\mathbb{R}$–linear operator in $\mathbb{C}^n$.

Example 4 To the matrix $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$ corresponds $M = 1/2$ and $M_{\#} = -1/2 + i$. For this operator $L = 1$, $L_{\#} = 0$, $U = 1/2$, and $U_{\#} = -1/2 + i$.

The converse holds as well and, in fact, we obtain a curious class of $\mathbb{R}$–linear operators (and the corresponding matrices A). If all the entries of M and $M_{\#}$ have equal modulus (say 1, like in the Schur matrix), then pivoting, i.e., pre and post operations by permutations does not cure a breakdown. Hence an appropriate strategy to avoid a breakdown needs to be devised.

Remark. For large problems the above algorithm can be used as a starting point for devising "ILU–preconditioners" for $\mathcal{M}$. In particular, if $A \in \mathbb{R}^{2n\times 2n}$ is regarded as an $\mathbb{R}$–linear operator in $\mathbb{C}^n$, then this gives rise to new ILU–preconditioning techniques for solving linear systems in $\mathbb{R}^{2n}$.

QR–decomposition. Here we consider slightly more general real linear operators $\mathcal{M} : \mathbb{C}^p \to \mathbb{C}^n$ defined via (2) by two matrices $M, M_{\#} \in \mathbb{C}^{n\times p}$. Our aim is to transform $\mathcal{M}$ to upper triangular form by operating with isometries from the left. Clearly, the standard Householder transformations in $\mathbb{C}^n$ could be applied to make either the linear or the anti–linear part of $\mathcal{M}$ upper triangular, but we want them in this form simultaneously.

Theorem 15 *For a given $\mathbb{R}$–linear operator $\mathcal{M} : \mathbb{C}^p \to \mathbb{C}^n$ there exists an isometry $\mathcal{Q}$ (in $\mathbb{C}^n$) such that $\mathcal{R} = \mathcal{Q}^{-1} \circ \mathcal{M}$ is upper triangular*

This is proved by the construction that follows. For this purpose we need special elementary isometries.

For given $x, y \in \mathbb{C}^n$ we want a real linear isometry that maps x and y in the direction $e = [1\,0\,\cdots\,0]^T$. If x and y are linearly dependent, a standard Householder transformation in $\mathbb{C}^n$ will do. So, let us assume that x and y are linearly independent over $\mathbb{R}$. We look for an isometry in the form $\mathcal{H}(z) = z - UU^*z - UU^T\,\overline{z}$, where $U \in \mathbb{C}^{n\times 2}$ is such that $\mathrm{Re}(U^*U) = I$; see Example 3. We call this a real linear Householder transformation. Writing $V = [x\ y] \in \mathbb{C}^{n\times 2}$ gives us the equation

$$V - 2\,U\ \mathrm{Re}(U^*V) = e\,a^* \tag{12}$$

for some $a \in \mathbb{C}^2$. Hence U is of the form $U = (V - e\,a^*)R$, where $R \in \mathbb{R}^{2\times 2}$. By multiplying with U^* we obtain $\mathrm{Re}(U^*(V + e\,a^*)) = 0$. Therefore (12) holds, if $\mathrm{Re}((V - e\,a^*)^*(V + e\,a^*)) = 0$. Setting $w = V^*e$ this becomes

$$\mathrm{Re}(V^*V + w\,a^* - a\,w^* - a\,a^*) = 0\ .$$

Vector $c = \mathrm{Re}(V^*V)^{\frac{1}{2}}\ [\,\frac{1}{i}\,]$ satisfies $\mathrm{Re}(V^*V - c\,c^*) = 0$. We try $a = \eta\,c$, where $|\eta| = 1$, so that $\mathrm{Re}(V^*V - a\,a^*) = 0$. Equation $\mathrm{Re}(w\,a^* - a\,w^*) = 0$ amounts to

$$\mathrm{Re}(w_1\overline{\eta c_2} - \eta c_1\overline{w_2}) = 0\,. \tag{13}$$

This is equivalent to $q\eta + \overline{q\eta} = 0$, where $q = c_2\overline{w_1} - c_1\overline{w_2}$. Thus (13) is satisfied for $\eta = i\,\overline{q}/|q|$. Finally, we get U by orthonormalising the columns of $V - e\,a^*$ with respect to the inner product $\langle u, v\rangle = \mathrm{Re}(u^*v)$.

Since $\mathcal{H}^2 = I$ we also have $\mathcal{H}(e\,a^*) = V$.

The following `Matlab` code finds U.

```
function U=rl_H(x,y)

  V=[x,y]; n=length(x);
  c=real(V'*V)^(1/2)*[1;i];
  q=V(1,:)*[-c(2);c(1)];
  V(1,:)=V(1,:)-i*sign(q)*c';
  [Q,R]=qr([real(V);imag(V)],0);
  U=Q(1:n,:)+i*Q(m+1:2*n,:);
```

For the QR–decomposition we first want a real linear Householder transformation such that the first columns of

$$\widehat{M} = (I - UU^*)M - UU^T\,\overline{M_\#} \qquad \text{and} \qquad \widehat{M_\#} = (I - UU^*)M_\# - UU^T\,\overline{M}$$

are multiples of e. Let m and $m^\#$ be the first columns of M and $M_\#$. Then

$$p = (I - UU^*)m - UU^T\,\overline{m^\#} \qquad \text{and} \qquad q(I - UU^*)m^\# - UU^T\,\overline{m}$$

are both multiples of e if

$$\begin{aligned} p + q &= (I - UU^*)(m + m^\#) - UU^T\,\overline{(m + m^\#)} \qquad \text{and} \\ i(p - q) &= (I - UU^*)\,i(m - m^\#) - UU^T\,\overline{i(m - m^\#)} \end{aligned}$$

are such. Thus we take the real linear Householder transformation that maps $m + m^\#$ and $i(m - m^\#)$ to multiples of e. Then the first columns of $\widehat{M}$ and $\widehat{M_\#}$ have zeros below the first entries.

After this we continue similarly with the lower right $(n-1)$-by-$(p-1)$ blocks of $\widehat{M}$ and $\widehat{M_\#}$. Below is the `Matlab` code for this decomposition.

```
function [Q,Qa,R,Ra]=rl_qr(M,Ma)

  % This constructs a real linear isometry z -> Q*z+Qa*conj(z)
  % and  upper triangular R,Ra such that
  %     M = Q*R+Qa*conj(Ra)    and    Ma = Q*Ra+Qa*conj(R)

[n,p]=size(M); R=M; Ra=Ma; Q=eye(n); Qa=zeros(n);
for k=1:min(p,n-1) , kn=k:n; kp=k:p;
  x=R(kn,k); y=Ra(kn,k);
  U=rl_H(x+y,i*(x-y));
  W=U'*R(kn,kp)+conj(U'*Ra(kn,kp));
  R(kn,kp)=R(kn,kp)-U*W; Ra(kn,kp)=Ra(kn,kp)-U*conj(W);
  W=Q(:,kn)*U+Qa(:,kn)*conj(U);
  Q(:,kn)=Q(:,kn)-W*U'; Qa(:,kn)=Qa(:,kn)-W*transpose(U);
end
```

This implementation requires $\sim \frac{40}{3}\,n^2p$ complex flops. With back substitution this algorithm can be used to solve overdetermined $\mathbb{R}$–linear systems.

Remark. The prescribed real linear Householder transformations can also be used in computing an isometry $\mathcal{U}$ such that $\mathcal{U}^{-1} \circ \mathcal{M} \circ \mathcal{U}$ has its linear and anti–linear parts in Hessenberg form.

Schur decomposition. Here we consider bringing given real linear operator to triangular form under a real linear isometric similarity transform. The construction of this part proves the following.

Theorem 16 *For given $\mathbb{R}$–linear operator $\mathcal{M}$ there exists an isometry $\mathcal{U}$ such that $\mathcal{T} = \mathcal{U}^{-1} \circ \mathcal{M} \circ \mathcal{U}$ is upper triangular*

We need the following auxiliary result.

Lemma 17 *There exist vectors $x, y \in \mathbb{C}^n$, linearly independent over $\mathbb{R}$, and $B \in \mathbb{R}^{2\times 2}$ such that $\mathcal{M}([x\ y]) = [x\ y]\,B$.*

Proof Let $A \in \mathbb{R}^{2n\times 2n}$ correspond to $\mathcal{M}$. Take $u, v \in \mathbb{R}^{2n}$ either as

- two linearly independent real eigenvectors of A, or
- a real eigenvector u and vector v such that $Av - \lambda v = u$, or
- the real and imaginary parts of an eigenvector corresponding to a non–real eigenvalue.

Set $[x\ y] = [I\ iI]\,[u\ v]$.

Now, let x, y be as in the previous lemma and take a real linear Householder transformation such that $\mathcal{H}([x\ y]) = e\,a^*$. Then also $\mathcal{H}(e\,a^*) = [x\ y]$. Consider

$$\widehat{\mathcal{M}}(z) = \mathcal{H}(\mathcal{M}(\mathcal{H}(z))) = \widehat{M}z + \widehat{M_\#}\,\overline{z}\,.$$

and let $\widehat{m}$ and $\widehat{m}^\#$ be the first columns of $\widehat{M}$ and $\widehat{M_\#}$. We have

$$\begin{bmatrix}\widehat{m} & \widehat{m}^\#\end{bmatrix}\begin{bmatrix}a^*\\ a^T\end{bmatrix} = \widehat{\mathcal{M}}(e\,a^*) = \mathcal{H}(\mathcal{M}([x\ y])) = \mathcal{H}([x\ y]\,B) = e\,a^*\,B\,,$$

since B is real. Here $\det\left[\begin{smallmatrix}a^*\\ a^T\end{smallmatrix}\right] = \overline{a_1}\,a_2 - a_1\,\overline{a_2} \neq 0$ unless $\overline{a_1}\,a_2 \in \mathbb{R}$. But the latter would imply that x and y are linearly dependent over $\mathbb{R}$ – a contradiction. Hence both $\widehat{m}$ and $\widehat{m}^\#$ are multiples of e.

Continue similarly with the lower right $(n-1)$-by-$(n-1)$ blocks of $\widehat{M}$ and $\widehat{M_\#}$ to obtain $\mathcal{U}$ as a composition of real linear Householder transformations.

Due to Proposition 8, the Schur decomposition $\mathcal{U}^{-1}\circ\mathcal{M}\circ\mathcal{U}$ of $\mathcal{M}$ is diagonal in case $M^* = M$ and ${M_\#}^T = M_\#$.

REMARK. Proposition 1 items *1* and *2* hence give us very special real linear operators. In view of this, to the polar decomposition of $A \in \mathbb{R}^{2n\times 2n}$ corresponds $\mathcal{M} = \mathcal{U}\circ\mathcal{S}$, where $\mathcal{U}$ is an isometry and $\mathcal{S}(z) = Sz + S_\#\overline{z}$ with $S^* = S$ and ${S_\#}^T = S_\#$.

With a small rank $M_\#$ the operator $\mathcal{M}$ can be regarded as "almost" $\mathbb{C}$–linear. So can its inverse in the following sense.

Proposition 18 *Let $\mathcal{M}(z) = Mz + M_\#\overline{z}$ be invertible with $\mathcal{M}^{-1}(z) = Rz + R_\#\overline{z}$. If M is invertible as well, then* $\operatorname{rank}(R - M^{-1}) \leq \operatorname{rank}(M_\#)$ *and* $\operatorname{rank}(R_\#) = \operatorname{rank}(M_\#)$.

Proof We have $\mathcal{M}^{-1}(\mathcal{M}z) = (RM + R_\#\overline{M_\#})z + (RM_\# + R_\#\overline{M})\overline{z}$. For this to be the identity we obtain the conditions

$$R_\# = -RM_\#\overline{M}^{-1} \text{ and } R = (M - M_\#\overline{M}^{-1}\overline{M_\#})^{-1}. \tag{14}$$

With R we can use the Sherman-Morrison formula to have the claims.

In this case one option is to use standard algorithms with (14) and the Sherman-Morrison formula to find the inverse of $\mathcal{M}$.

Not all the matrix factorizations have a particularly interesting analogue for $\mathbb{R}$–linear operators in $\mathbb{C}^n$. For instance, assume the real form $A \in \mathbb{R}^{2n\times 2n}$

of $\mathcal{M}$ is nonderogatory so that $A = T^{-1}\widetilde{C}T$ with a companion matrix $\widetilde{C}$. Since all the factors are real, we have $\mathcal{M} = \mathcal{T}^{-1} \circ \mathcal{C} \circ \mathcal{T}$ with $\mathcal{C}(z) = Cz + C_{\#}\overline{z}$ such that C is a companion matrix while $C_{\#}$ is a rank-1 matrix with one nonzero column. Since the spectrum of an $\mathbb{R}$–linear operator in $\mathbb{C}^n$ is not preserved under a general $\mathbb{R}$–linear similarity transformation, this factorization may not be very useful (aside from giving a very structured factor $\mathcal{C}$). In general T cannot be found such that the corresponding $\mathcal{T}$ would be a change of basis transformation.

2.3 Miscellaneous remarks

We also have a Neumann type of series expansion for the inverse. Consider first the operator $\mathcal{M}_\kappa$.

Theorem 19 *Assume $M_{\#} \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}$ is such that $||M_{\#}|| < |\lambda|$ holds. Then $(\lambda I - \mathcal{M}_0)^{-1}(z) = R(\lambda)z + R_{\#}(\lambda)\overline{z}$ with*

$$R(\lambda) = \sum_{j=0}^{\infty} \frac{(M_{\#}\overline{M_{\#}})^j}{\lambda|\lambda|^{2j}} \text{ and } R_{\#}(\lambda) = \sum_{j=0}^{\infty} \frac{(\overline{M_{\#}}M_{\#})^j M_{\#}}{|\lambda|^{2(j+1)}}.$$

Proof By making an ansatz

$$z = \frac{b}{\lambda} + \frac{M_{\#}\overline{b}}{|\lambda|^2} + \frac{M_{\#}\overline{M_{\#}}b}{\lambda|\lambda|^2} + \frac{M_{\#}\overline{M_{\#}}M_{\#}\overline{b}}{|\lambda|^4} + \frac{M_{\#}\overline{M_{\#}}M_{\#}\overline{M_{\#}}b}{\lambda|\lambda|^4} + \cdots,$$

it is straightforward to verify that z converges and solves the equation $\lambda z - M_{\#}\overline{z} = b$ for any $b \in \mathbb{C}^n$. Separating the linear and anti–linear terms (that is, the matrices multiplying b and $\overline{b}$, respectively) from this sequence gives $R(\lambda)$ and $R_{\#}(\lambda)$.

For a general $\mathbb{R}$–linear operator $\mathcal{M}$ we have

$$(rI - \mathcal{M})^{-1} = \sum_{j=0}^{\infty} \frac{\mathcal{M}^j}{r^{j+1}}, \tag{15}$$

whenever $r \in \mathbb{R}$ and $||\mathcal{M}|| < |r|$. Assume $\lambda \in \mathbb{C}$. Since solvingă $\lambda z - Mz - M_{\#}\overline{z} = b$ is equivalent to solving $z - \frac{M}{\lambda}z - \frac{M_{\#}}{\lambda}\overline{z} = \frac{b}{\lambda}$, we can employ (15) with this latter problem. A substitution to (15) gives a series expansion for the linear and anti–linear parts of $\mathcal{R}(\lambda) = (\lambda I - \mathcal{M})^{-1}$ as

$$R(\lambda) = \frac{I}{\lambda} + \frac{M}{\lambda^2} + \frac{M^2}{\lambda^3} + \frac{M_{\#}\overline{M_{\#}}}{\lambda|\lambda|^2} +$$

$$+\frac{M^3}{\lambda^4} + \frac{MM_{\#}\overline{M_{\#}}}{\lambda^2|\lambda|^2} + \frac{M_{\#}\overline{M}^2}{\overline{\lambda}^2|\lambda|^2} + \frac{M_{\#}\overline{MM_{\#}}}{|\lambda|^4} + \frac{M_{\#}\overline{M_{\#}}M}{\lambda^2|\lambda|^2} + \cdots$$

and

$$R_{\#}(\lambda) = \frac{M_{\#}}{|\lambda|^2} + \frac{MM_{\#}}{\lambda|\lambda|^2} + \frac{M_{\#}\overline{M}}{\overline{\lambda}|\lambda|^2} + \frac{M^2M_{\#}}{\lambda^2|\lambda|^2} + \frac{MM_{\#}\overline{M}}{|\lambda|^4} + \frac{M_{\#}\overline{M_{\#}}M_{\#}}{|\lambda|^4} + \cdots.$$

REMARK. Since the set of $\mathbb{R}$–linear operators in $\mathbb{C}^n$ is a normed algebra over $\mathbb{R}$, $\lim_{j\to\infty} ||\mathcal{M}^j||^{1/j}$ exists and gives the spectral radius of the real form of $\mathcal{M}$. However, its connection with the spectrum of $\mathcal{M}$ is not obvious (except when $M_\# = 0$) since $\sigma(\mathcal{M})$ can be even empty.

Regardless of the size of the spectrum, the minimal polynomial an $\mathbb{R}$–linear operator in $\mathbb{C}^n$ is well defined.

Theorem 20 *Let $\mathcal{M}$ be an $\mathbb{R}$–linear operator in $\mathbb{C}^n$. Then there exists a monic polynomial p of degree at most $2n$ such that $p(\mathcal{M}) = 0$.*

Proof Take p to be the minimal polynomial of $A \in \mathbb{R}^{2n\times 2n}$ corresponding to the real formulation (3) of $\mathcal{M}$. Since p has only real coefficients, $p(\mathcal{M})$ is also zero.

For iterative methods, the following is of interest. $2n$

Corollary 21 *If $\mathcal{M}$ is invertible, then $\mathcal{M}(q(\mathcal{M})) = I$ for a polynomial q of degree $2n-1$ at most.*

Proof Take $p(\lambda)/p(0) = \lambda q(\lambda) - 1$ which clearly has real coefficients. Therefore the equivalent real operator in $\mathbb{R}^{2n}$ gives the identity.

Example 5 In the context of forming polynomials in an $\mathbb{R}$–linear mapping in $\mathbb{C}^n$ many interesting classes of operators arise. In [18] there is an operator considered, the so–called Friedrichs operator, whose square is a $\mathbb{C}$–linear mapping in $\mathbb{C}^n$. Generalizing this, it is an interesting problem to find, for a given $\mathcal{M}$, a monic polynomial of the lowest possible degree such that $p(\mathcal{M})$ is $\mathbb{C}$–linear (or $\mathbb{C}$–anti–linear).

Rank-1 matrices are fundamental for matrix computations. In fact, let $M = m_1 m_2^*$ and $M_\# = n_1 n_2^*$ be of rank 1 both. Then there are three possibilities for the multiplicity indexes (see Example 1) of eigenvalue 0 of $\mathcal{M}$, i.e., we can have four different types of real linear low rank operators listed in Table 2.1.

	$\dim(\mathrm{span}\{m_1, n_1\}) = 2$	$\dim(\mathrm{span}\{m_1, n_1\}) = 1$
$\dim(\mathrm{span}\{m_2, \overline{n}_2\}) = 2$	$(\frac{2n-2}{2}, n-2)$	$(\frac{2n-1}{2}, n-2)$
$\dim(\mathrm{span}\{m_2, \overline{n}_2\}) = 1$	$(\frac{2n-1}{2}, n-1)$	$(\frac{2n-1}{2}, n-1)$

Table 1: Options for $\mathbb{R}$–linear operators in $\mathbb{C}^n$ with rank-1 matrices M and $M_\#$.

Example 6 Let $\sigma_1 \left[\begin{smallmatrix} u_a \\ u_b \end{smallmatrix}\right] \left[\begin{smallmatrix} v_a^* & v_b^* \end{smallmatrix}\right] \in \mathbb{R}^{2n\times 2n}$ the best rank-1 approximation to $A \in \mathbb{R}^{2n\times 2n}$ from its SVD, i.e., σ_1 is the largest singular value of A and $u_j, v_j \in \mathbb{R}^n$ for $j = a, b$. For the corresponding real linear operator in $\mathbb{C}^n$ the respective approximation is $\frac{\sigma_1}{2}(u_a + iu_b)\big((v_a^* - iv_b^*)z + (v_a^* + iv_b^*)\overline{z}\big) = \sigma_1\, u\, \mathrm{Re}(v^* z)$ with $u = u_a + iu_b$ and $v = v_a + iv_b$. In the classification of Table 2.1 this is in the lower right corner.

Repeating the construction of the preceding example with each rank-1 term in the SVD of A, we obtain an expansion

$$\mathcal{M}(z) = \textstyle\sum_{j=1}^{2n} \sigma_j \, u_j \, \mathrm{Re}(v_j^* z)$$

for $\mathcal{M}$. Although this is a potentially useful representation of $\mathcal{M}$, at this point we are not sure whether it should be called the SVD of $\mathcal{M}$.

Let $V_1, V_2 \subset \mathbb{C}^n$ be two subspaces of dimension k (over $\mathbb{C}$ as usual) and let $I_j : V_j \mapsto \mathbb{C}^k$ be an isometric isomorphism, for $j = 1, 2$. Define $\widetilde{\mathcal{P}}$ via

$$\begin{array}{ccc} V_1 & \xrightarrow{\widetilde{\mathcal{P}}} & V_2 \\ \downarrow I_1 & & \uparrow I_2^{-1} \\ \mathbb{C}^k & \xrightarrow{\mathcal{U}} & \mathbb{C}^k \end{array} , \tag{16}$$

where $\mathcal{U}$ is an $\mathbb{R}$–linear isometry in $\mathbb{C}^k$. Then $\mathcal{P} = \widetilde{\mathcal{P}} \oplus 0$ gives an $\mathbb{R}$–linear *partial isometry* in $\mathbb{C}^n$, i.e., $\|\mathcal{P}(z)\| = \|z\|$ for $z \in V_1$ while $\mathcal{P}(z) = 0$ for $z \in V_1^{\perp}$.

3 Iterative methods for solving $\mathbb{R}$–linear problems in $\mathbb{C}^n$

Assume $Q_k \in \mathbb{C}^{n\times k}$ with orthonormal columns has been generated. Then a low dimensional approximation to the problem (1) is given by

$$\mathcal{M}^{(k)}(w_k) = Q_k^* M Q_k w_k + Q_k^* M_{\#} \overline{Q}_k \overline{w}_k = Q_k^* b \,, \tag{17}$$

and $z_k = Q_k w_k$ gives the corresponding Galerkin approximation. The arising $\mathbb{R}$–linear mapping in $\mathbb{C}^k$ can also be used in approximating the spectrum of $\mathcal{M}$ via a Ritz type construction to have "Ritz curves". In particular, the subspace spanned by the columns of Q_k is invariant for $\mathcal{M}$ if and only if

$$(I - Q_k Q_k^*) M Q_k = 0 \;\text{ and }\; (I - Q_k Q_k^*) M_{\#} \overline{Q}_k = 0.$$

If this holds, then $\sigma(\mathcal{M}^{(k)}) \subset \sigma(\mathcal{M})$. Otherwise good approximations (in some sense) can be expected when the matrices on the left hand side are small in norm.

3.1 The case of $\mathcal{M}_\kappa$

To compute Q_k with an iterative method, consider first the simplest case involving the operator $\mathcal{M}_\kappa$ for $\kappa \in \mathbb{C}$. Then we can use a minimal residual approach which corresponds to replacing Q_k^* in (17) with Q_{k+1}^* and solving the arising low order problem with the least squares methods.

Now the Arnoldi method [1] is well defined in the sense that an application of $\mathcal{M}_\kappa$ to a starting vector $b \in \mathbb{C}^n$ gives

$$\kappa b + M_{\#} \overline{b}.$$

Orthogonalizing this against b yields $\alpha_1^1 b + \alpha_1^2 M_\# \overline{b}$ with $\alpha_1^1, \alpha_1^2 \in \mathbb{C}$. Applying $\mathcal{M}_\kappa$ to this vector gives

$$\kappa\alpha_1^1 b + (\kappa\alpha_1^2 + \overline{\alpha_1^1}) M_\# \overline{b} + \overline{\alpha_1^2} M_\# \overline{M_\#} b.$$

Orthogonalizing this against b and $\alpha_1^1 b + \alpha_1^2 M_\# \overline{b}$ yields a vector which is a linear combination of the vectors b, $M_\#\overline{b}$ and $M_\#\overline{M_\#}b$. An application of $\mathcal{M}_\kappa$ to this vector and then performing an orthogonalization yields a linear combination of the vectors b, $M_\#\overline{b}$, $M_\#\overline{M_\#}b$ and $M_\#\overline{M_\#}M_\#\overline{b}$. Continuing this inductively proves the following.

Theorem 22 *Let $\kappa \in \mathbb{C}$, $M_\# \in \mathbb{C}^{n\times n}$ and $b \in \mathbb{C}^n$. Then the Arnoldi method with $\mathcal{M}_\kappa$ gives an orthonormal basis $\{q_1, q_2, \dots\}$ of the Krylov subspace*

$$\operatorname{span}\{b, M_\#\overline{b}, M_\#\overline{M_\#}b, M_\#\overline{M_\#}M_\#\overline{b}, \dots\}. \tag{18}$$

REMARK. Solving $\kappa z + M_\#\overline{z} = b$ with a direct method is naturally equivalent to solving $\overline{M_\#}z + \overline{\kappa z} = \overline{b}$. However, an execution of the Arnoldi method with the complex conjugate $\overline{\mathcal{M}_\kappa}$ of $\mathcal{M}_\kappa$ using the starting vector $\overline{b}$ does not seem to generate a subspace with a simple spanning set like that of (18) unless simplifying assumptions are made.

By inspecting its spanning set, we can view (18) as a block–Krylov subspace generated with the matrix $M_\#\overline{M_\#}$ by using the starting vectors $\{b, M_\#\overline{b}\}$. In particular, a matrix $M_\# \in \mathbb{C}^{n\times n}$ is congruence normal if $M_\#\overline{M_\#}$ is normal; see [6] and references therein. In this case the ideas of [9, 13] can be used for generating this subspace with a recurrence whose length grows very slowly.

If $\deg(M_\#\overline{M_\#})$, the degree of the minimal polynomial of $M_\#\overline{M_\#}$, is moderate, then we have a nontrivial invariant subspace of $\mathcal{M}_\kappa$ with (18).

Corollary 23 *The dimension of* (18) *is* $\min\left\{\operatorname{rank}(M_\#) + 1, 2\deg(M_\#\overline{M_\#})\right\}$ *at most.*

Proof The claim follows by rewriting (18) as the sum of two subspaces as

$$\mathcal{K}(M_\#\overline{M_\#}; b) + M_\#\overline{\mathcal{K}(M_\#\overline{M_\#}; b)}, \tag{19}$$

where $\mathcal{K}(M_\#\overline{M_\#}; b) = \operatorname{span}\left\{b, M_\#\overline{M_\#}b, (M_\#\overline{M_\#})^2 b, \dots\right\}$.

In view of iterative methods, this illustrates how the bound of Theorem 20 can be pessimistic.

REMARK. Any invariant subspace of $\mathcal{M}_\kappa$ is necessarily invariant for $M_\#\overline{M_\#}$. For the converse, $\mathcal{K}(M_\#\overline{M_\#}; b)$ is an invariant subspace of $M_\#\overline{M_\#}$ for any vector $b \in \mathbb{C}^n$. Hence (19) is the smallest invariant subspace of $\mathcal{M}_\kappa$ containing $\mathcal{K}(M_\#\overline{M_\#}; b)$. For instance, if b is an eigenvector of $M_\#\overline{M_\#}$, then the dimension of (19) is either 1 or 2. Both cases are possible.

We denote by W_k the subspace spanned by the first k vectors in (18). Clearly, $\mathcal{M}_\kappa(W_k) \subset W_{k+1}$. This implies that the resulting canonical form (17) consists of a diagonal and a Hessenberg matrix for the linear and anti–linear parts of $\mathcal{M}_\kappa$, respectively. Writing $Q_k = [q_1\ q_2\ \dots\ q_k]$ we get:

Theorem 24 *The Arnoldi method with $\mathcal{M}_\kappa$ gives a Hessenberg matrix $Q_k^* M_\# \overline{Q}_k$ for $k = 1, 2, \dots$.*

Proof If W_j denotes the subspace spanned by the first j vectors in (18), then $M_\#$ maps $\overline{W}_j$ into W_{j+1}, for every $j > 0$.

If no breakdown occurs, with $k = n$ we have performed a consimilarity transformation of $M_\#$; see [7].

With iterative methods one is always interested in the length of the recurrence to have less expensive steps.

Theorem 25 *If ${M_\#}^T = cM_\#$ for $c = \pm 1$, then the Arnoldi method with $\mathcal{M}_\kappa$ is realizable with a 3–term recurrence.*

Proof Let $q_0, \ldots, q_{j-1}$ denote the orthonormal basis of W_j generated with the Arnoldi method. Then

$$(\mathcal{M}_\kappa(q_{j-1}), q_l) = (\kappa q_{j-1}, q_l) + (\overline{q}_{j-1}, {M_\#}^* q_l),$$

where the first inner product is zero for $j-l > 1$. Hence we have $(\overline{q}_{j-1}, {M_\#}^* q_l) = \overline{(q_{j-1}, {M_\#}^T \overline{q}_l)} = c\overline{(q_{j-1}, M_\# \overline{q}_l)} = 0$ for $j - l > 2$.

Under these assumptions the matrices $Q^*\kappa I Q$ and $Q^* M_\# \overline{Q}$ are diagonal and tridiagonal, respectively. With $c = -1$ the diagonal entries of $Q^* M_\# \overline{Q}$ equal zero, i.e., we then get a real skew–symmetric matrix. Hence the eigenvalues of the matrices $M_\#$ and $Q^* M_\# \overline{Q}$ can differ dramatically even though the eigenvalues of the mappings $z \mapsto M_\# \overline{z}$ and $z \mapsto Q^* M_\# \overline{Q} \overline{z}$ are the same.

REMARK. Since Q is unitary, the singular values of the matrix $M_\#$ equal those of $Q^* M_\# \overline{Q}$. Therefore, under the assumptions of Theorem 25, the singular values of $M_\#$ can be approximated with an iterative method relying on a 3–term recurrence.

The following is of use for preconditioning the problem (1) with the inverse of M from the left.

Proposition 26 *Let $\mathcal{M}(z) = Mz + M_\# \overline{z}$ be diagonalizable by a unitary change of basis transformation. If M is invertible, then $M^{-1} M_\#$ is complex symmetric.*

We have $M_\# \, \overline{Q_k} \, \overline{w}$ in the range of Q_{k+1} for all $w \in \mathbb{C}^k$. Hence

$$\begin{aligned} \left\| \kappa Q_k w_k + M_\# \overline{Q}_k \overline{w}_k - b \right\| &= \left\| \kappa Q_{k+1}^* Q_k w_k + Q_{k+1}^* M_\# \overline{Q}_k \overline{w}_k - Q_{k+1}^* b \right\| \\ &= \left\| \kappa \widetilde{I}_k w_k + \widetilde{H}_k \overline{w}_k - \|b\| \, e_1 \right\|, \end{aligned} \tag{20}$$

where $\widetilde{I}_k \in \mathbb{C}^{(k+1)\times k}$ is the identity matrix augmented with the row of zeros and $\widetilde{H}_k$ is a $(k+1)$-by-k Hessenberg matrix. Hence, solving the system $\kappa z + M_\# \overline{z} = b$ approximately with the corresponding minimal residual approach amounts to finding the minimum of the last expression in (20), e.g. by the real linear QR-decomposition.

A `Matlab` implementation of the method is as follows.

```
function x=rl_GMRES(kappa,Ma,b,tol)

nb=norm(b); Q=b/nb; H=[]; Ha=[]; eb=nb; err=1; j=0;
while err>tol ,j=j+1;
  r=Ma*conj(Q(:,j));
  for l=1:j,
    h=Q(:,l)'*r; r=r-Q(:,l)*h;
    Ha(l,j)=h; end
  nr=norm(r); Q=[Q,r/nr];
  jj=j:j+1; H(jj,j)=[kappa;0]; Ha(j+1,j)=nr; eb(j+1,1)=0;
  for l=1:j-1, U=UM{l};  ll=l:l+1;
    W=U'*H(ll,j)+conj(U'*Ha(ll,j));
    H(ll,j)=H(ll,j)-U*W; Ha(ll,j)=Ha(ll,j)-U*conj(W); end
  x=H(jj,j); y=Ha(jj,j);
  U=rl_H(x+y,i*(x-y)); W=U'*H(jj,j)+conj(U'*Ha(jj,j));
  H(jj,j)=H(jj,j)-U*W; Ha(jj,j)=Ha(jj,j)-U*conj(W);
  eb(jj)=eb(jj)-2*U*real(U'*eb(jj)); UM{j}=U;
  err=abs(eb(j+1))/nb;
end
w=rl_ut_solve(H(1:j,:),Ha(1:j,:),eb(1:j));
z=Q(:,1:j)*w;
```

Note that (similarly to the standard implementation of GMRES) the Hessenberg matrix is transformed isometrically to an upper triangular form while it is being built.

The work and storage needed with this method (as a function of the number of steps) are comparable with GMRES [19]. Further, we have:

Proposition 27 *The method above is at least as fast as the standard GMRES method applied to the real form* $A\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathrm{Re}(b) \\ \mathrm{Im}(b) \end{smallmatrix}\right]$ *of the problem.*

Proof Minimizing (20) amounts to finding the minimum of $\|\mathcal{M}_\kappa(z) - b\|$ for $z \in W_k$. GMRES applied to the real form minimizes the same but only in the set of real linear combinations of $b, \mathcal{M}_\kappa(b), \ldots, \mathcal{M}_\kappa^{k-1}(b)$ which is a subset of W_k.

The number of steps needed for the exact solution is bounded according to Corollary 23.

With $\kappa = 0$ we have a "conjugate GMRES" algorithm for solving $M_\# \overline{z} = b$ (which is equivalent to solving $\overline{M_\#} z = \overline{b}$ when direct methods are used). It coincides with GMRES if and only if $M_\#$ and b are real.

The case of $\kappa = 0$ and ${M_\#}^T = M_\#$ has been studied in [2].

Remark. As a curiosity, because of (5) we could also consider separately the linear and anti–linear parts of $\mathcal{M}_\kappa^{-1}$. This amounts to solving, after multiplying the second system with $M_\#$ from the left, two standard $\mathbb{C}$–linear

systems in $\mathbb{C}^n$ involving translations of $M_\#\overline{M_\#}$. In particular, if $M_\#$ is congruence normal, then the 3–term recurrence of [10] can be employed to this end.

3.2 The general case

To compute Q_k for (17) with a general $\mathbb{R}$–linear operator by using an Arnoldi type of iteration is straightforward by orthogonalizing $\mathcal{M}(q_j)$ against the vectors $q_1, \ldots, q_j$ computed so far, for $j = 1, \ldots, k-1$.

As opposed to the case $\mathcal{M}_\kappa$, this iteration is less satisfactory since Hessenberg matrices do not arise. Not even with $\mathcal{M}(z) = Mz + \kappa\overline{z}$, for $\kappa \in \mathbb{C}$, any particular structure seems to appear. In particular, we do not have the property $\mathcal{M}(W_k) \subset W_{k+1}$, where W_k denotes the span of the vectors generated after $k-1$ steps.

For the number of steps we have the following analogy of [14, Proposition 2.6].

Proposition 28 *The Arnoldi method with $\mathcal{M}(z) = Mz + M_\#\overline{z}$ generates at most $\deg(M)(\mathrm{rank}(M_\#) + 1)$ linearly independent vectors.*

Proof Denote by $\mathcal{K}_j(M; q_0) = \mathrm{span}\{q_0, Mq_0, \ldots, M^{j-1}q_0\}$, where $q_0 = b/\|b\|$. If $v_1, \ldots, v_k$ is a basis of the range of $M_\#$, then

$$q_2 = \alpha_1^2(Mq_0 + M_\#\overline{q}_0) - \alpha_2^2 q_0 \in \mathcal{K}_2(M; q_0) + \mathrm{span}\{v_l\}_{l=1,\ldots,k}$$

with $\alpha_1^2, \alpha_2^2 \in \mathbb{C}$. Similarly, $q_3 \in \mathcal{K}_3(M; q_0) + \mathrm{span}\{M^j v_l\}_{\substack{j=0,1\\ l=1,\ldots,k}}$ so that the induction step becomes clear. Since $\mathcal{K}_n(M; q_0) + \mathrm{span}\{M^j v_l\}_{\substack{j=0,\ldots,n\\ l=1,\ldots,k}}$ has dimension at most $\deg(M)(\mathrm{rank}(M_\#) + 1)$, the assertion is proved.

This also implies that if $\deg(M)(\mathrm{rank}(M_\#) + 1) < n$, then $\mathcal{M}$ always has an invariant subspace.

Instead of a minimal residual approximation to the solution of the system $\mathcal{M}(z) = b$, we compute a Galerkin approximation by using (17).

For a minimum residual method we should augment Q_k with (typically) k extra orthonormal vectors such that the resulting span would include the range of $\mathcal{M}_{|W_k}$. This seems to become rather uneconomical.

3.3 Cost, restarting and related remarks

For an iterative method to be preferred over a direct method, typically the crucial bottleneck is the cost of matrix-vector products. Here all the standard ideas, like using the FFT techniques, apply in an obvious way.

Like with GMRES, restarting may be needed to save storage. In connection with this, there is now the additional possibility of solving the conjugated problem $\overline{\mathcal{M}(z)} = \overline{M_\#}z + \overline{M}\overline{z} = \overline{b}$ instead of the original system $\mathcal{M}(z) = b$. Either of these two options can be chosen before every new restart.

The Krylov subspace methods suggested above were based on an iterative generation of orthogonal projectors. These are very particular type of partial isometries (16). Hence the possibility of iteratively computing more general real linear partial isometries and using them in solving linear systems approximately needs to be studied further.

We have only considered methods that consume storage linearly. Devising a quasi–minimal residual type of iteration [4] is another alternative to save memory.

3.4 Numerical experiments

Next we consider iteratively solving a system $\mathcal{M}(z) = b$. In each experiment either `rl_GMRES` or `rl_Gal` applied to $\mathcal{M}$ is compared with GMRES applied to the equivalent real formulation of the problem. Here `rl_Gal` refers to the method of Section 3.2. To save storage, we also compare their restarted versions `rl_GMRES`(k), `rl_Gal`(k) and GMRES(k) restarted after every k steps. The residual at the jth step (defined similarly for the real formulation) is denoted by $r_j = \mathcal{M}(z_j) - b$,

The computations were performed with `Matlab` whose syntax we use.

Example 7 This family of $\mathbb{R}$–linear systems arose in connection with the inverse problem of reconstructing an unknown electric conductivity in the unit disc from boundary measurements; see [21, 22]. To this end one needs to solve repeatedly the system $\mathcal{M}_\kappa(z) = z + M_\# \overline{z} = \mathbf{1}$ resulting from a discretization of a weakly singular Fredholm integral equation of the second kind depending on various parameters. More precisely, $\mathcal{M}_\kappa$ depends on the measured current on the unit circle as well as on the point in the unit disc the reconstruction is being computed for. The right hand–side is the constant vector with ones. Due to the size $n = 2^{16}$ of the system, the matrices are not represented explicitly.

The problem was iteratively solved by using the simulated boundary data on the unit circle used in [21, Problem 4] with the initial guess $z_0 = \mathbf{1}$. We executed `rl_GMRES` and GMRES as well as their restarted versions with $k = 30, 60$. Since $M_\#$ is the product of a Toeplitz matrix and a diagonal matrix, matrix–vector products could be computed fast by using the FFT.

After fixing values for the parameters, we compared the relative residuals $\|r_k\| \,/\, \|r_0\|$ in the $\log_{10}$ scale for all the six iterations; see Figure 1.

The experiments were repeated by varying the parameters and each time `rl_GMRES` outperformed GMRES (see Proposition 27). Quantitatively we had approximately 30% shorter execution times in a typical case illustrated in Figure 1. There we also see that for the restarted iterations the difference could be even more drastic: `rl_GMRES`(60) converged whereas GMRES(60) stagnated. For $k = 30$ both methods stagnated.

Example 8 Here we illustrate the Galerkin approximation of Section 3.2 for iteratively solving the system $\mathcal{M}(z) = b$ with restarts. The computations

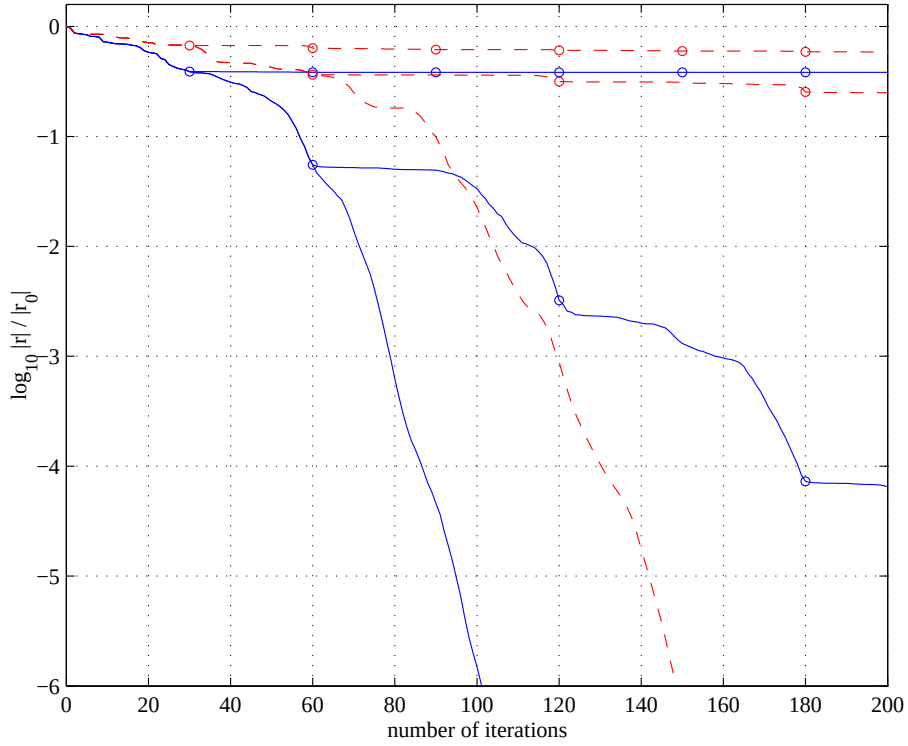


Figure 1: The convergence of the relative residuals in the $\log_{10}$ scale in Example 7. `rl_GMRES` and `rl_GMRES`(k) are depicted with a solid line and GMRES and GMRES(k) with a dashed line, $k = 30,\ 60$. The restart points are marked with 'o'.

were carried out with `Matlab` whose syntax is used. Below we denote by $R_{n,m} = \texttt{randn}\,(n, m) \in \mathbb{R}^{n\times m}$ a normally distributed random matrix which has been regenerated each time encountered. So no two matrices $R_{n,m}$ are the same. In this somewhat articifial problem $M = (20 + 10i)I + R_{n,n} + i\,R_{n,n}$, $M_{\#} = (I + R_{n,n} + i\,R_{n,n})/10$, and $b = R_{n,1} + i\,R_{n,1}$ with $n = 150$. By using the initial guess $z_0 = 0$, we executed `rl_Gal`(k) and GMRES(k) with $k = 2,\ 6,\ 10$.

Short restarting frequency seems to be optimal in this problem for both methods. See Figure 2 for the behavior of the relative residuals in the $\log_{10}$ scale.

4 Computing the spectrum of an $\mathbb{R}$–linear operator in $\mathbb{C}^n$

In this final section we consider some ideas for locating the spectrum of a real linear operator $\mathcal{M}$.

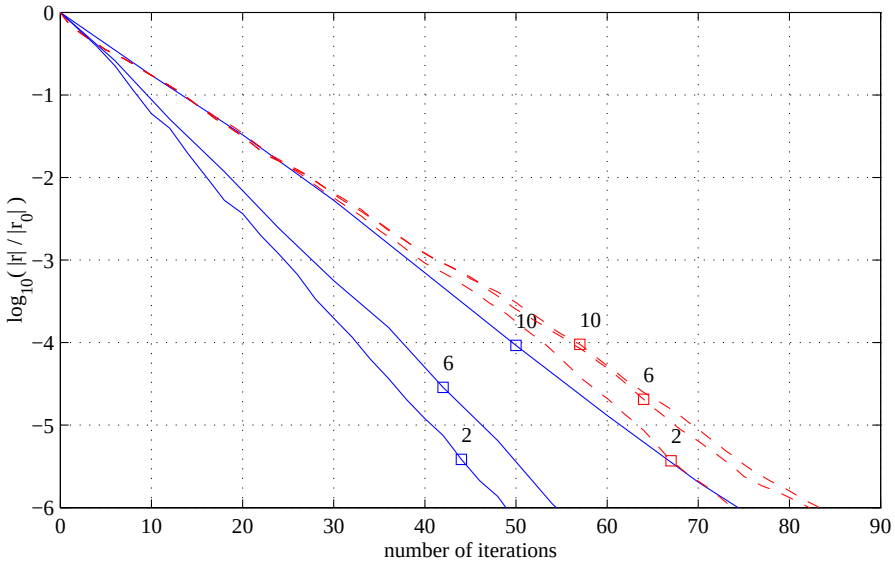


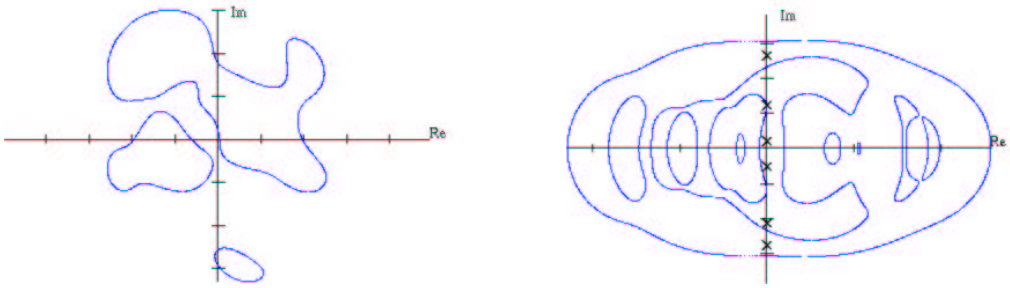
Figure 2: The relative residuals in the $\log_{10}$ scale in Example 8. `rl_Gal`(k) is depicted with a solid line and GMRES(k) with a dashed line. The labels above the squares refer to the restarting frequency k.

In the one-parameter family of real matrices $A(0,\beta) = -\beta J - A$ in (6), every real eigenvalue α of $A(0,\beta)$ corresponds to an eigenvalue $\alpha + i\beta$ of $\mathcal{M}$. Hence a brute force method to find the spectrum of $\mathcal{M}$ is to compute the eigenvalues of $A(0,\beta)$ for those β that are of interest. Since the spectrum is bounded by the norm, we need to consider only the interval $\{\beta \in \mathbb{R} : |\beta| \leq \|\mathcal{M}\|\}$ which can be further reduced (to be possibly non–convex) by using the Geršgorin disks (9). There are also many alternatives to benefit from Geršgorin disks if we perform change of basis transformations in a clever way.

Remark. For a fixed $\mu \in \mathbb{C}$, the spectra of $\mathcal{M}$ and $\mu I \circ \mathcal{M}$ are related in an obvious way. However, for computations a multiplication by a scalar makes a difference. For instance, if $\sigma(\mathcal{M})$ is locally tangential to the real axis, then the prescribed approach is numerically less stable. The choice $\mu = i$ rotates the spectrum by $\pi/2$ and removes the problem in that particular neighborhood.

To get a rough picture of $\sigma(\mathcal{M})$, one option is to use a coarse grid for β over an interval of interest. The grid can then be refined in those areas where the spectrum appears to be changing rapidly while β varies. However, with a coarse tracking of the spectrum we face the risk of missing entire isolated subsets of $\sigma(\mathcal{M})$. For example, if $M_\# = 0$, then the spectrum consists of isolated points which are missed almost certainly. Also in a nearly $\mathbb{C}$–linear case with $\|M_\#\| \ll \|M\|$ the isolated subsets of the spectrum can be very small and could thus be overlooked.

To locate tiny subsets better, we employ the information also in the non–



(a) (b)

Figure 3: The spectra of the $\mathbb{R}$-linear operators of Examples 9–10. (a) The spectrum of $\mathcal{M}$ of Example 9. (b) The spectra of "symmetric" $\mathcal{M}_1$ (dotted line) and "anti–symmetric" $\mathcal{M}_2$ (crosses) of Example 10.

real eigenvalues and the corresponding eigenvectors of $A(0,\beta)$. To this end, set $\phi(w) = \frac{i\,w^*Jw}{w^*w}$ for $w \in \mathbb{C}^{2n}$ with $w \neq 0$. This satisfies $\phi(\mu u) = \phi(u) \in \mathbb{R}$ for any nonzero $u \in \mathbb{C}^{2n}$ and $\mu \in \mathbb{C}$. Also $|\phi(w)| \leq \|J\| = 1$.

With ϕ define the set valued function

$$\Phi(\beta) = \{\phi(w)\ \mathrm{Im}\,\lambda\ :\ A(0,\beta)w = \lambda w\,,\ w \neq 0\} \subset \mathbb{R}.$$

Obviously, if $A(0,\beta)$ has a real eigenvalue, then $0 \in \Phi(\beta)$.

Lemma 29 *If $|\beta| > 2\|A\|$, then $\beta\,\Phi(\beta) \subset \mathbb{R}_+$. If for all β the non–real eigenvalues of $A(0,\beta)$ are simple, then Φ is continuous.*

Proof For the first claim, assume $|\beta| > 2\|A\|$ and that $A(0,\beta)w = \lambda w\,,\ w \neq 0$. Since J is normal, $\min|\lambda \pm i\beta| \leq \|A\|$ by the Bauer–Fike theorem. Hence $|\mathrm{Im}\,\lambda| > \|A\|$ with $\mathrm{Im}\,\lambda$ having the same sign as β. If $\mathrm{Im}\,\lambda \geq 0$, then, since $\beta\,\phi(w) \in \mathbb{R}$

$$\beta\,\phi(w) = \frac{i\,w^*\beta Jw}{w^*w} = \frac{i\,w^*(-\lambda + A)w}{w^*w} = \frac{w^*(\mathrm{Im}\,\lambda + \frac{1}{2}(iA + (iA)^*))w}{w^*w} > 0.$$

In the case $\mathrm{Im}\,\lambda < 0$ we get $\beta\,\phi(w)z < 0$. Hence, $\mathrm{Im}\,\lambda\,\beta\,\phi(w) > 0$ in both cases.

Eigenvalues depend continuously on β and eigenvectors corresponding to simple eigenvalues can be chosen continuous. Hence the assumptions imply that the numbers $\phi(w)\ \mathrm{Im}\,\lambda$ depend continuously on β for $\mathrm{Im}\,\lambda \neq 0$. Further, these tend to zero when λ approaches $\mathbb{R}$ since $|\phi(w)| \leq 1$.

Since $\beta\,\Phi(\beta) \subset \mathbb{R}_+$ far from the origin, we see that all the elements of $\Phi(\beta)$ cross the origin as β runs over $\mathbb{R}$. In fact, the vanishing elements (often) seem to take opposite signs as we step over a isolated subset of spectrum. If we order Φ along decaying $\mathrm{Re}\,\lambda$, we may use simple bisection method to refine the grid on every change (of sign or length) of the vector Φ. If, at

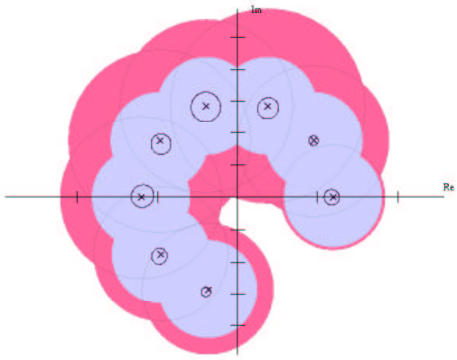


Figure 4: The spectrum of an $\mathbb{R}$–linear operator $\mathcal{M}$, where M is almost diagonal. The Bauer–Fike bound of Proposition 13 has been colored light blue by regarding $z \mapsto M_{\#}\overline{z}$ as the perturbation $\mathcal{E}$. The Geršgorin disks of $\mathcal{M}$ are shown in darker red color. See Example 11.

a change of sign in Φ, $\operatorname{Im}\lambda$ stays away from zero, we skip the interval. If $\operatorname{Im}\lambda$ crosses origin, we look for a subset of the spectrum. This way we have been able to locate isolated points and horizontal parts of the spectrum more accurately while, simultaneously, execution times have decreased. Of course, there is no guarantee that this tool manages to pick every isolated subset of the spectrum.

Using the lemma we can save computational work by refining the β–grid only on intervals of interest, but this still is a rather tedious way to visualize the spectrum. The same technique, with very coarse grid, can also be used to only locate a point on each isolated subset of the spectrum. These points, in turn, can be extended to find the corresponding piece of the curve $\{\alpha+i\beta \,:\, \det A(\alpha,\beta)=0\}$ using standard continuation techniques (see [16]).

Once sufficiently many points of $\sigma(\mathcal{M})$ have been computed accurately, one can also use the information to find the coefficient of the characteristic bivariate polynomial approximately. To this end, for example, the ideas of [12, Section 4.1] can be employed.

Next we consider numerical examples. All our matrices are artificially constructed and small since we only aim at illustrating various aspects of the spectrum. $R_{n,n}$ is defined as in Example 8.

Example 9 The spectrum of an $\mathbb{R}$–linear operator can be profuse and very arresting. We illustrate this with $\mathcal{M}:\mathbb{C}^{10}\mapsto\mathbb{C}^{10}$ having $M = R_{10,10}+i\,R_{10,10}$ and $M_{\#} = R_{10,10}+i\,R_{10,10}$. See Figure 3(a).

Example 10 To illustrate Proposition 12, we take $\mathcal{M}_1$ and $\mathcal{M}_2$ with the real forms $A_1 = R+R^T$ and $A_2 = R-R^T$ with $R = R_{20,20}$. The spectrum

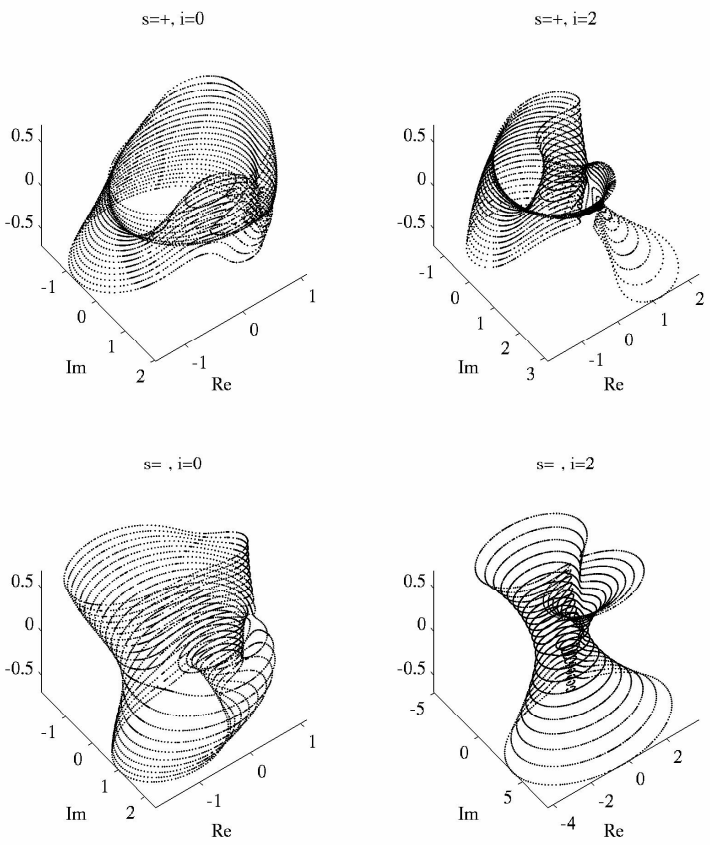


Figure 5: The spectrum of an $\mathbb{R}$-linear operator under $\mathbb{R}$–linear, but not $\mathbb{C}$–linear, similarity transformations. The horizontal copies of the complex plane correspond to the values of the parameter t. See Example 12.

of $\mathcal{M}_1$ is symmetric relative to the real axis. The spectrum of $\mathcal{M}_2$ consists of at most $2n$ isolated points. See Figure 3(b).

Example 11 To see how the spectrum varies, let $M = M_1 + \frac{1}{20}M_2$ such that $M_1 \in \mathbb{C}^{10\times 10}$ is a diagonal matrix having the eigenvalues $z_j = 6e^{i\theta_j}$, with $\theta_j = \frac{2\pi}{10}j$ for $j = 0, \ldots, 9$, and $M_2 = \frac{1}{2}\,(R_{10,10} + i\,R_{10,10})$. The anti–linear part is $M_\# = \frac{1}{4}\,(R_{10,10} + i\,R_{10,10})$. In Figure 4 we have plotted $\sigma(M)$ and $\sigma(\mathcal{M})$ together with the Geršgorin disks. The Bauer–Fike bound of Proposition 13 is also plotted by regarding $z \mapsto M_\#\overline{z}$ as the perturbation $\mathcal{E}$ of the $\mathbb{C}$–linear operator $\widetilde{\mathcal{M}}(z) = Mz$. Rounding to four digits, we had $\|S^{-1}\|\,\|S\| = 2.01$ and $\|\mathcal{E}\| = 4.05$.

Example 12 We illustrate the fact that the spectrum is not preserved, in general, in a $\mathbb{R}$–linear similarity transformation. We take $\mathcal{M} : \mathbb{C}^2 \mapsto \mathbb{C}^2$ where $M = \frac{1}{10}\left[\begin{smallmatrix} 0 & 11 \\ 3i & 10i \end{smallmatrix}\right]$ and $M_\# = \left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]$. The spectrum is a curve encircling the origin. Let

$$E_0 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right],\ E_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right],\ E_2 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right],\ \text{and } E_3 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right].$$

Then we consider $\mathcal{M}_{s,i,t} = \mathcal{T}_{i,t,s}^{-1} \circ \mathcal{M} \circ \mathcal{T}_{i,t,s}$, where the real forms of $\mathcal{T}_{i,t,+}$ and $\mathcal{T}_{i,t,-}$ are $\exp\left(\left[\begin{smallmatrix} tE_i & 0 \\ 0 & -tE_i \end{smallmatrix}\right]\right)$ and $\exp\left(\left[\begin{smallmatrix} 0 & tE_i \\ tE_i & 0 \end{smallmatrix}\right]\right)$, respectively. Unless $t = 0$, $\mathcal{T}_{i,t,+}$ and $\mathcal{T}_{i,t,-}$ are not $\mathbb{C}$–linear. In Figure 5 we have plotted $\sigma(\mathcal{M}_{i,t,s})$. The spectrum is shown for four pairs (s, i), $s = 0, 2$ and $i = \pm$, each on a separate plot. On each plot, the horizontal copies of the complex plane correspond to values of $t = -0.7 \ldots 0.7$. Note that the two real eigenvalues remain invariant here as in any similarity transformation.

5 Conclusions

Matrix analysis for $\mathbb{R}$–linear operators in $\mathbb{C}^n$ has been studied. Although we are dealing with a weaker assumption than $\mathbb{C}$–linearity, a large part of the familiar theory could be recovered. In particular, most of the matrix factorizations aimed at solving linear systems can be regarded as special cases of our more general results.

Basics of the spectral theory for $\mathbb{R}$–linear operators in $\mathbb{C}^n$ were developed together with some preliminary computational ideas for finding the spectrum.

Initial motivation for our study being Krylov subspace methods, we have introduced new iterative schemes that avoid using an equivalent real formulation.

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