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ON QUENCHING WITH LOGARITHMIC SINGULARITY

Timo Salin

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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Abstract: *In this paper we study the quenching problem for the reaction diffusion equation* $u_t - u_{xx} = f(u)$ *with Cauchy-Dirichlet data, in the case where we have a logarithmic singularity, i.e.,* $f(u) = \ln(\alpha u)$, $\alpha \in (0,1)$. *We show that for sufficiently large domains of* x *quenching occurs, and that under certain assumptions on the initial function, the set of quenching points is finite.*
The main result of this paper concerns the asymptotical behavior of the solution in a neighborhood of a quenching point. This result gives the quenching rate for the problem. We also obtain new blow-up results for the equation $v_t - v_{xx} = \alpha v e^v - v_x^2$. *These concern the occurence of blow-up, the blow-up set and the asymptotics in a neighborhood of a blow-up point. The analysis is based on the equivalence between the quenching and the blow-up for these two equations.*

AMS subject classifications: 35K55, 35K60, 35B40

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tsalin@cc.hut.fi

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Institute of Mathematics, Helsinki Univ. of Tech., Espoo, 2001

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

1 Introduction

Consider the nonlinear diffusion problem

$$\begin{aligned} u_t - u_{xx} = -u^{-p}, \qquad x \in (-l, l), \quad t \in (0, T), \\ u(x, 0) = u_0(x), \quad x \in [-l, l], \\ u(\pm l, t) = 1, \quad t \in [0, T). \end{aligned} \tag{1.1}$$

Here p, l and T are positive constants, the initial function satisfies $0 < u_0(x) \leq 1$ and $u_0'' - u_0^{-p} \leq 0$. This type of reaction diffusion equation with a singular reaction term arises in connection with the diffusion equation generated by a polarization phenomena in ionic conductors [18]. The problem can also be considered as a limiting case of models in chemical catalyst kinetics (Langmuir-Hinshelwood model) or of models in enzyme kinetics [9], [26].

It is known, see ([29] p.34 Th.3.3.), that the equation (1.1) has a local unique solution in a set $(0, t_\varepsilon) \times (-l, l)$. This solution can be continued until $t < T$, where $T = \inf_\tau \{\tau \geq 0 \quad | \limsup_{t \uparrow \tau, x \in (-l,l)} (u(x,t) + \frac{1}{u(x,t)}) = \infty\}$. It is also known that ([29] p.41 Th.3.8.) $u(x, t)$ is a C^∞-function with respect to x and t in $(x, t) \in (-l, l) \times (0, T)$.

The equation (1.1) has been extensively studied under assumptions implying that the solution $u(x, t)$ approaches zero in finite time. The reaction term then tends to infinity and the smooth solution ceases to exist. This phenomenon is called quenching. We say that a is a quenching point and T is a quenching time for $u(x, t)$, if there exists a sequence $\{(x_n, t_n)\}$ with $x_n \to a$ and $t_n \uparrow T$, such that $u(x_n, t_n) \to 0$ as $n \to \infty$.

For the problem (1.1) it is well-known that for sufficiently large l quenching occurs in finite time [1], [2], [22]. It is also known that the set of quenching points is finite [15]. See also review articles [19], [23].

A primary interest for the problem (1.1) has been the analysis of the local asymptotics of the solution as $t \uparrow T$ in the neighborhood of the quenching point. In particular, it has been shown that the quenching-rate satisfies

$$\lim_{t \uparrow T} u(x, t)(T - t)^{-1/(1+p)} = (1 + p)^{1/(1+p)}, \tag{1.2}$$

uniformly for $|x - a| < C\sqrt{T - t}$. This result was first established by Guo [15] for $p \geq 3$, and subsequently generalized to $p \geq 1$ by Fila and Hulshof [6]. For the weaker singularity $0 < p < 1$, (1.2) has been shown in [17]. The result (1.2) for higher dimensions has been obtained in [7] for cases $p > 0$.

In this paper the quenching problem is studied for the equation

$$\begin{aligned} u_t - u_{xx} = \ln(\alpha u), \qquad x \in (-l, l), \quad t \in (0, T), \\ u(x, 0) = u_0(x), \quad x \in [-l, l], \\ u(\pm l, t) = 1, \quad t \in [0, T), \end{aligned} \tag{1.3}$$

where $\alpha \in (0, 1)$ and $u_0 \in (0, 1]$. The reaction term $f(u) = -u^{-p}$ in the equation (1.1) is thus replaced by a weaker logarithmic singularity $f(u) = \ln(\alpha u)$.

In the equation (1.3), an essential feature is the contest between the linear diffusion term u_{xx} and the nonlinear reaction term $f(u) = \ln(\alpha u)$. If the dissipative diffusion term is dominant, then there is no quenching. Thus the nonlinear reaction term can achcieve quenching. Because the reaction term is now much weaker than in the equation (1.1), it is not obvious that quenching can happen. The first problem is therefore to clear up, whether quenching may at all occur?

It is assumed throughout this paper that the initial function satisfies

$$u_0''(x) + \ln(\alpha u_0(x)) \leq 0. \tag{1.4}$$

This technical assumption guarantees that $u(x,t)$ is decreasing in time. In Section 2 we show that quenching is possible, i.e.

Theorem 1.1. *For l large enough, the solution $u(x,t)$ of* (1.3) *quenches in finite time.*

The proof of this theorem is based on the fact that the stationary problem corresponding to the equation (1.3) has no solution, if l is sufficiently large.

In Section 3 it is studied how the weakening of the singularity affects the set of quenching points? It is known [10] that for a reaction term $f(u) = -u^{-p}$ the x-derivative of the final profile $u(x,T)$ at the quenching point has a singularity when $p \geq 1$ and is smooth ($u_x(a,T) = 0$), when $p \in (0,1)$. Can this regularity be strengthen in the logarithmic case so that $u_x(x,T) = 0$ for all $x \in (c,b) \subset [-l,l]$, in other words can quenching take place on a whole interval?

The following result tells us that the situation does not qualitatively differ from the situation, when we had the power-like singularity.

Theorem 1.2. *Suppose that $u(x,t)$ satisfies* (1.3) *and that* (1.4) *holds. Then the set of quenching points is finite.*

The proof is based on the general method for certain parabolic equations developed by Angenent [3]. It is first deduced, using this method, that u_x cannot oscillate, when the quenching point is approached. Then it is shown that there is a time t^* such that there is a finite number of local minima with respect to x after t^*, and that this number is constant in time. Finally, one shows that quenching cannot occur on the boundary and that the set of quenching points is finite. The proof is essentially the same as Guo's [15] (also adopted from [3]) for the stronger singularities $f(u) = -u^{-p}$.

In Section 4 the local asymptotics of the solution near the quenching point is studied. The main result of this paper is proven:

Theorem 1.3. *Let $u(x,t)$ be the solution of the equation* (1.3)*, where u_0 is even, $u_0'(r) \geq 0$, $u_0(x) \in (0,1]$ and* (1.4) *holds ($r = |x|$). Assume that $u(x,t)$ quenches at $(0,T)$ for some $T < \infty$. Then*

$$\lim_{t \uparrow T}\left(1 + \frac{1}{T-t}\int_0^{u(x,t)} \frac{d\tau}{\ln(\alpha\tau)}\right) = 0, \tag{1.5}$$

uniformly, when $|x| < C\sqrt{T-t}$, for every $C \in (0,\infty)$.

This theorem can also be proved in a somewhat stronger form

Corollary 1.4. *Let $u(x,t)$ quench at (a,T), with an initial function u_0 that satisfies $u_0(x) \in (0,1]$ and (1.4). Then*

$$\lim_{t\uparrow T}(1+\frac{1}{T-t}\int_0^{u(x,t)}\frac{d\tau}{\ln(\alpha\tau)})=0,$$

uniformly, when $|x-a|<C\sqrt{T-t}$, for every $C\in(0,\infty)$.

The content of Theorem 1.3 can be interpreted by comparing the quenching rate to a solution of the corresponding ordinary differential equation $v'=f(v)$ (where $f(v)=\ln(\alpha v)$, with 'initial' condition $v(T)=0$), and concluding that these solutions are asymptotically equal in the region $|x-a|<C\sqrt{T-t}$.

Note that Theorem 1.3 is the same as the result (1.2) for the equation (1.1), if the term $\ln(\alpha\tau)$ in (1.5) is replaced by $-\tau^{-p}$.

Our proof for a logarithmic singularity is not based on earlier results on quenching. The proof here uses similarity variables and energy estimates; in particular observe that our method is different from the earlier versions to prove the corresponding quenching-rate estimate (1.2). (see Giga-Kohn [13], [14], Bebernes-Eberly [4], Guo [15]). Especially note that this is a consequence of the fact that (1.3) does not have the useful scaling property that the equation (1.1) has.

The rather lengthy proof is further commented and given in Section 4.

Consider now the blow-up problem for the equation

$$v_t - v_{xx} = af(v) - bv_x^q \tag{1.6}$$

with Cauchy-Dirichlet data (v is given on boundary and $v(x,0)=v_0(x)$), when $t>0$ and $x\in\Omega$ (bounded). Here q, a and b are positive constants, and $f(v)=v^p$ or $f(v)=e^{rv}$ (p and r are also positive constants). Blow-up means that a solution approaches infinity in finite time. We say that b is a blow-up point and T is a blow-up time for $v(x,t)$, if there exists a sequence $\{(x_n,t_n)\}$ with $x_n\to b$ and $t_n\uparrow T$, such that $v(x_n,t_n)\to\infty$ as $n\to\infty$.

The blow-up problem for the equation (1.6) without the damping term $-bv_x^q$ has been studied extensively [24]. In the problem (1.6), the key question is to find out how this damping term can affect the possible occurrence of blow-up, the set of blow-up points and the asymptotics of blow-up. These questions have been studied in ,e.g., [21], [27], [28].

The equivalence between the quenching problem and the blow-up problem is well-known [21]. Putting $\alpha u = e^{-v}$ in the equation (1.3), we get

$$\begin{aligned} v_t - v_{xx} = \alpha v e^v - v_x^2, \qquad x\in(-l,l),\quad t\in(0,T),\\ v(x,0) = -\ln(\alpha u_0(x)),\quad x\in[-l,l],\\ v(\pm l,t) = -\ln(\alpha),\quad t\in[0,T), \end{aligned} \tag{1.7}$$

Note that quenching for the equation (1.3) corresponds to blow-up in the equation (1.7). Thus the Theorems 1.1, 1.2 and 1.3 yield the following new Corollaries

Corollary 1.5. *For sufficiently large l, the solution $v(x,t)$ of* (1.7) *blows up in finite time.*

Corollary 1.6. *The set of blow-up points for the equation* (1.7) *is finite.*

Corollary 1.7. *Let $(0,T)$ be the blow-up point for the equation* (1.7). *Then*

$$\lim_{t \uparrow T} \frac{1}{T-t} \int_{v(x,t)}^{\infty} \frac{d\tau}{\alpha \tau e^{\tau}} = 1,$$

uniformly, when $|x| \leq C\sqrt{T-t}$.

2 A sufficient condition for quenching

In this Section the possibility of quenching in finite time is studied. There are two reasons why quenching may not happen. In the first case, we may have $u(x,t) \geq c > 0$, for all $t > 0$. This means, that there is a solution to the corresponding stationary equation, which is a subsolution of the equation (1.3). On the other hand, it might happen, that $u(x,t) > 0$ for all t and x, but that $\min u(x,t) \to 0$, as $t \to \infty$. This second case is called quenching in infinite time.

It is known [25] for a general singularity, that $u(x,t)$ quenches in finite time for l sufficiently large, provided that the similar stationary equation does not have a strong solution. We show in Lemma 2.2, that this last fact holds for the logarithmic singularity.

Lemma 2.1. *Let $u(x,t)$ be the solution of* (1.3), *when $x \in (-l,l)$ and $t \in (0,T)$. Suppose that* (1.4) *holds. Then*

(a) $u_t(x,t) < 0$, *when $x \in (-l,l)$ and $t \in (0,T)$.*

(b) $0 < u(x,t) \leq 1$, *when $x \in [-l,l]$ and $t \in [0,T)$.*

Proof. Let $v(x,t) = u_t(x,t)$, when $x \in (-l,l)$ and $t \in (0,T)$. Differentiating the equation (1.3), we get

$$v_t - v_{xx} - \frac{1}{u} v = 0,$$

where $x \in (-l,l)$, $t \in (0,T)$ and $\frac{1}{u}$ is a locally bounded function.

On the boundary it holds that: $v(\pm l,t) = 0$, $t \in (0,T)$, and $v(x,0) \leq 0$, by the condition (1.4). The claim (a) follows now from the maximum principle.

The claim (b) follows from (a) and from the fact that $u_0(x) \in (0,1]$.

□

Lemma 2.2. *When l is sufficiently large, then the equation*

$$\frac{d^2 g(x)}{dx^2} + \ln(\alpha g(x)) = 0; \quad g(\pm l) = 1, \tag{2.1}$$

does not have a solution $g \in C^2(-l,l) \cap C[-l,l]$, such that $g(x) \in (0,1]$ for $-l \leq x \leq l$.

Proof. If $g \in C^2(-l,l) \cap C[-l,l]$ is a solution of the equation (2.1) then

$$\frac{g'(x)^2}{2} + \int_1^{g(x)} \ln(\alpha\tau)d\tau = C, \quad x \in (-l,l). \tag{2.2}$$

If $g(x) \in (0,1]$, and because $\alpha \in (0,1)$, then $g''(x) = -\ln(\alpha g(x)) > 0$. So g is strictly convex, and thus it has one minimum. Denote this minimum point by z and the minimum by m: $g(z) = m$ and $g'(z) = 0$. From the formula (2.2), we can now see that

$$\int_1^m \ln(\alpha\tau)d\tau = C. \tag{2.3}$$

From the equations (2.2) and (2.3) it follows that

$$\frac{g'(x)^2}{2} = \int_{g(x)}^m \ln(\alpha\tau)d\tau, \tag{2.4}$$

$$g'(x) = \pm\{2\int_{g(x)}^m \ln(\alpha\tau)d\tau\}^{1/2}. \tag{2.5}$$

On the interval $(-l,z)$, $g(x)$ is decreasing, so an integration of the equation (2.5) yields

$$z + l = -\int_1^m \{2\int_r^m \ln(\alpha\tau)d\tau\}^{-1/2}dr. \tag{2.6}$$

On the other hand, we can conclude from the equation (2.5) that

$$z - l = \int_1^m \{2\int_r^m \ln(\alpha\tau)d\tau\}^{-1/2}dr. \tag{2.7}$$

By the addition of the equations (2.6) and (2.7) we see that $z = 0$.

We have now obtained

$$l = \int_m^1 \{2\int_r^m \ln(\alpha\tau)d\tau\}^{-1/2}dr \overset{def}{=} I(m). \tag{2.8}$$

Because $\int_r^m \ln(\alpha\tau)d\tau {=} -\int_m^r \ln(\alpha\tau)d\tau {>} -\ln(\alpha)(r-m)$ and $r, m \in (0,1)$, then $I(m) \leq M < \infty$. Chosing $l > M$ the lemma follows. □

Theorem 2.3. *Suppose $u(x,t)$ is the solution of the equation* (1.3)*. Then for l sufficienly large, $u(x,t)$ quenches in finite time.*

Proof. Consider the equation (1.3) with an initial function $u_0 = 1$. The solution of this equation is a supersolution of (1.3). By Lemma 2.2 and [25] the claim follows. □

3 There exist at most finitely many quenching points

In this Section we investigate whether the weakening of the reaction term affects the size of the set of quenching points. Does the fact that $f(u) = -u^{-p}$ is replaced by the weaker singularity $f(u) = \ln(\alpha u)$ significantly alter this size? Is it now possible to have quenching on an entire interval?

The answer to these questions is provided by Theorems 3.7 and 3.8 below. In these results, we show that the qualitative properties of the quenching set do not change when a power singularity is replaced by a logarithmic one.

We base our proof on the method developed by Angenent [3] for the analysis of the asymptotics of solutions of certain parabolic equations. More specifically, our proof is essentially that of Guo's concerning the stronger singularity $f(u) = -u^{-p}$ [15]. To make use of that method possible, our quenching problem has to be rewritten as a blow-up problem, i.e., in the form (3.2).

We assume in this Section that quenching happens, and that the assumption (1.4) holds. The quenching time is denoted by T as earlier. By a simple modification, the equation (1.3) can be written in the form

$$\begin{aligned} u_t - u_{xx} &= \epsilon \ln(\alpha u), \qquad x \in (-1,1), \quad t \in (0,T), \\ u(x,0) &= u_0(x), \qquad x \in (-1,1), \\ u(\pm 1, t) &= 1, \qquad t \in (0,T). \end{aligned} \tag{3.1}$$

Let v be defined by $v = -\ln(\alpha u)$. Then v satisfies the equation

$$\begin{aligned} v_t - v_{xx} + v_x^2 - \epsilon \alpha e^v v &= 0, \qquad x \in (-1,1), \quad t \in (0,T), \\ v(x,0) &= -\ln(\alpha u_0(x)), \qquad x \in (-1,1), \\ v(\pm 1, t) &= -\ln(\alpha), \qquad t \in (0,T). \end{aligned} \tag{3.2}$$

By Lemma 2.1, it holds that $v > -\ln(\alpha)$ in the region $A = \{(x,t) | x \in (-1,1), t \in (0,T)\}$. Let also $w = v_t$, then

$$\begin{aligned} w_t - w_{xx} + 2v_x w_x - \epsilon\alpha(1+v)e^v w &= 0, \qquad x \in (-1,1), \quad t \in (0,T), \\ w(x,0) &\geq 0, \qquad x \in (-1,1), \\ w(\pm 1, t) &= 0, \qquad t \in (0,T). \end{aligned} \tag{3.3}$$

It follows from Lemma 2.1, that $w > 0$ in the region A.

The purpose is now to prove the finiteness of blow-up points for the equation (3.2).

Let

$$\operatorname{sgn}(b) = \begin{cases} +1 & \text{when } b > 0, \\ 0 & \text{when } b = 0, \\ -1 & \text{when } b < 0. \end{cases}$$

Lemma 3.1. *For any $a \in [-1,1]$, the limit $\lim_{t \to T} \operatorname{sgn}(v_x(a,t))$ exists.*

Proof. Let $a \in (-1, 0)$ and $T_0 < T$. Consider the function

$$\psi(x,t) = v(x,t) - v(2a - x, t),$$

when $x \in (-1, a)$ and $t \in (0, T_0)$. Then it follows from the equation (3.2) that ψ satisfies

$$\begin{aligned} \psi_t - \psi_{xx} + b_1\psi_x + b_2\psi = 0, \qquad x \in (-1,a), \quad t \in (0,T_0), \\ \psi(-1,t) = -\ln(\alpha) - v(2a+1,t) < 0, \qquad t \in (0,T_0), \\ \psi(a,t) = v(a,t) - v(a,t) = 0, \quad t \in (0,T_0). \end{aligned} \tag{3.4}$$

Here $b_1 = v_x(x,t) - v_x(2a-x,t)$ and $b_2 = b_2(v(x,t), v(2a-x,t))$ are bounded functions. In addition the derivatives of b_1 are bounded. Let also

$$\phi(y,t) = \exp(-\frac{1}{2}\int_{-1}^{y}(b_1(x,t)dx))\psi(y,t).$$

By the equation (3.4) the function ϕ satisfies

$$\begin{aligned} \phi_t - \phi_{yy} + q\phi = 0, \qquad y \in (-1,a), \quad t \in (0,T_0), \\ \phi(-1,t) = \psi(-1,t) < 0, \qquad t \in (0,T_0), \\ \phi(a,t) = 0, \quad t \in (0,T_0). \end{aligned} \tag{3.5}$$

Here $q(y,t)$ is a bounded function. Define

$$U(y,t) = \begin{cases} \phi(y,t), & -1 \le y \le a, \\ -\phi(2a-y,t), & a \le y \le 2a+1, \end{cases}$$

and

$$Q(y,t) = \begin{cases} q(y,t), & -1 \le y \le a, \\ q(2a-y,t), & a \le y \le 2a+1. \end{cases}$$

Then it can be seen by using the equation (3.5), that $U(y,t)$ satisfies the equation

$$\begin{aligned} U_t - U_{yy} + QU = 0, \qquad y \in (-1, 2a+1), \quad t \in (0,T_0), \\ U(-1,t) < 0, \qquad t > 0, \\ U(2a+1,t) > 0, \quad t > 0. \end{aligned} \tag{3.6}$$

Here Q is a bounded function. Denote by $z(t)$ the number of zeros of U on the interval $[-1, 2a+1]$, so

$$z(t) = \#\{y \in [-1, 2a+1]; U(y,t) = 0\}.$$

Concluding now by ref. ([3] Th.D), we have that $z(t)$ is non-increasing and finite, when $0 < t < T_0$. Therefore if there exists a point (y_0, t_0) such that $U(y_0,t_0) = U_y(y_0,t_0) = 0$, then $z(t_1) < z(t_2)$, when $t_2 < t_0 < t_1$. Because $z(t) = 2m(t) - 1$, where

$$m(t) = \#\{x \in [-1, a]; \psi(x,t) = 0\},$$

and because $T_0 < T$, then $m(t)$ is non-increasing and finite, when $0 < t < T$. Furthermore, if there exists a point (x_0, t_0) such that $\psi(x_0, t_0) = \psi_x(x_0, t_0) = 0$, then $m(t_1) < m(t_2)$, when $t_2 < t_0 < t_1$.

We will now deduce the claim by a contradiction. Suppose that $v_x(a,t)$ oscillates (or reaches zero without changing the sign), when $t \uparrow T$. The same is then true for ψ_x, because $\psi_x(a,t) = 2v_x(a,t)$. Thus there exists a sequence $t_n \uparrow T$, when $n \to \infty$ such that $\psi_x(a, t_n) = 0$ for all n, and because $\psi(a,t) = 0$, then $m(t_{n+1}) < m(t_n)$, and

$$m(t_1) = \sum_{i=1}^{N}(m(t_i) - m(t_{i+1})) + m(t_{N+1}) \geq N + m(t_{N+1}),$$

for arbitrary $N \geq 1$. By this, $m(t_1) = \infty$, which contradicts the finiteness of m. The lemma is true, when $a \in (-1, 0)$.

The same proof applies in the case, where $a \in (0, 1)$.

Consider now the situation, when $a = 0$. Then, by the equation (3.2), the function ψ satisfies

$$\begin{aligned} \psi_t - \psi_{xx} + b_1\psi_x + b_2\psi = 0, \qquad x \in (-1, 0), \quad t \in (0, T_0), \\ \psi(-1, t) = \psi(0, t) = 0, \qquad t > 0. \end{aligned} \tag{3.7}$$

Applying ([3] th. C), we deduce, that $m(t)$ is non-increasing and finite when $0 < t < T$. Furthermore, if there exists a point (x_0, t_0) such that $\psi(x_0, t_0) = \psi_x(x_0, t_0) = 0$, then $m(t_1) < m(t_2)$, when $t_2 < t_0 < t_1$. We can now conclude the claim by a contradiction as in the case $a \in (-1, 0)$.

When $a = \pm 1$, then $v_x(-1, t) > 0$ and $v_x(1, t) < 0$. □

Lemma 3.2. *There exists a $t_\star \in (0, T)$ such that*

$$n(t) = \#\{a \in [-1, 1]; v_x(a, t) = 0\}$$

is a positive constant for all $t \in [t_\star, T)$. There also exist C^1-functions $s_1(t), .., s_m(t)$ from the interval $[t_\star, T)$ to the interval $[-1, 1]$, such that $s_1(t) < ... < s_m(t)$, and

$$\{a \in [-1, 1]; v_x(a, t) = 0\} = \{s_1(t), ..., s_m(t)\},$$

when $t \geq t_\star$. Moreover, the limits $s_i = \lim_{t \uparrow T} s_i(t)$ exist for all $i = 1, ..., m$.

Proof. [15], p.61.

□

Lemma 3.3. *Let s_1 and s_m be defined as in the previous lemma. Then it holds that $s_1 > -1$ and $s_m < 1$.*

Proof. [15], p.62.

□

Lemma 3.4. *The solution v cannot blow-up, when $x = \pm 1$.*

Proof. Let δ be an arbitrary positive constant. Consider the equation

$$\begin{aligned} w_t - w_{xx} = 0, \qquad x \in (-1,1), \quad t \in (\delta,\infty), \\ w(\pm 1,t) = 1, \qquad t > \delta, \\ w(x,t) = u(x,t), \qquad x \in (-1,1), \quad t = \delta. \end{aligned} \tag{3.8}$$

By writing the equation for $w - u$, we can see by the equations (3.1) and (3.8), and by the maximum principle, that $w > u$, when $t \in (\delta, T)$, and $x \in (-1,1)$. It also holds that $w_x(-1,t) > u_x(-1,t)$, when $t \in (\delta,T)$. Because $v_x(-1,t) = -u_x(-1,t)$, and because there exists a constant $c > 0$ such that $w_x(-1,t) \leq -c$ (see. [12] p.156), when $t \in (\delta,T)$, we obtain that

$$v_x(-1,t) > c,$$

for all $t \in (\delta,T)$. Let $a = \frac{-1+s_1}{2}$. By Lemma 3.3, there exists a constant $T_0 \in (\delta,T)$, such that $v_x > 0$ in the set $A = [-1,a] \times [T_0,T)$. Consider now the function

$$J(x,t) = v_x(x,t) - n\epsilon h(x) e^{v(x,t)},$$

on A, where $h(x) = a - x$ and $n > 0$ (a constant, determined later). Then we get from (3.2) for J that

$$J_t - J_{xx} + b_1 J_x + b_2 J = \alpha\epsilon^2 n h(x) e^{2v(x,t)}, \tag{3.9}$$

where $b_1 = 2v_x$ and $b_2 = -\epsilon\alpha e^v(1+v)$. When n is chosen small enough, then $J \geq 0$ on the parabolic boundary of A. By the maximum principle, $J \geq 0$ in A. From this it follows that

$$v_x - n\epsilon h e^v \geq 0,$$

in A. Dividing by e^v and integrating this from $x \in [-1, \frac{1}{2}(a-1)]$ to a, when $t \in (T_0,T)$, we get

$$\int_{v(x,t)}^{v(a,t)} e^{-\tau} d\tau \geq \int_x^a n\epsilon(a-\xi)d\xi = \frac{1}{2}n\epsilon(a-x)^2 \geq \frac{1}{8}n\epsilon(1+a)^2 \geq c > 0,$$

by Lemma 3.3. Thus $e^{-v(x,t)} \geq c > 0$, when $x \in [-1, \frac{1}{2}(a-1)]$. It follows that v is bounded in the set $[-1, \frac{1}{2}(a-1)] \times (T_0,T)$, and v cannot blow-up, when $x = -1$. The case $x = 1$ is concluded similarly. □

Lemma 3.5. *Let $[a,b] \subset [-1,1] \backslash \{s_1,...,s_m\}$. If v has a blow-up point $c \in (a,b)$, then $\lim_{t\uparrow T} v(x,t) = \infty$, either for all $x \in (c,b]$ or for all $x \in [a,c)$.*

Proof. [15], p.64.

□

Lemma 3.6. *The function v does not have blow-up points in $[-1,1]\backslash\{s_1,...,s_m\}$.*

Proof. Let $s_0 = -1$ and $s_{m+1} = 1$. Suppose that there exists a blow-up point $c \in (s_i, s_{i+1})$. Let $a = \frac{1}{2}(s_i + c)$ and $b = \frac{1}{2}(s_{i+1} + c)$. Then there is $T_0 \in (0, T)$, such that v_x does not change the sign in the set $[a, b] \times [T_0, T)$ (denote this set by R_0). We can now suppose that $v_x > 0$ in R_0. Consider the function,

$$J(x, t) = v_x(x, t) - n\epsilon h(x) e^{v(x,t)},$$

when $(x, t) \in R = [d, b] \times [T_0, T)$, and $h(x) = \sin(\frac{(x-d)\pi}{b-d})$ and $d \in (c, b)$, also $n > 0$ (determined later). Then by (3.2), the function J satisfies

$$J_t - J_{xx} + 2v_x J_x + bJ = n\epsilon e^v h\{\epsilon\alpha v e^v - \frac{\pi^2}{(b-d)^2}\},$$

where $b = -\epsilon\alpha e^v(1 + v)$. By Lemma 3.5 there is a $T_1 \geq T_0$ such that, if $t \geq T_1$, then

$$\epsilon\alpha v e^v \geq (\frac{\pi}{b-d})^2,$$

for all $x \in (d, b)$. For this T_1 we can choose $n > 0$ so small that $J \geq 0$, when $t = T_1$ and $x \in (d, b)$. Furthermore, $J \geq 0$, when $x = d$ or $x = b$, $t \geq T_1$. By the maximum principle, we can see that $J > 0$ in R. Hence $v_x e^{-v} \geq n\epsilon h$. Integrating this from d to b, when $t \geq T_1$, we get

$$e^{-v(d,t)} - e^{-v(b,t)} \geq n\epsilon \int_d^b \sin(\frac{x-d}{b-d}\pi) dx.$$

Letting here $t \uparrow T$, we can see that the left-hand side converges to zero, because of Lemma 3.5, but the right-hand side is strictly positive. This is a contradiction. ☐

Theorem 3.7. *Let the initial function u_0 in the equation* (1.3) *satisfy the condition* (1.4)*. Then the set of quenching points is finite.*

Proof. The claim follows directly from Lemma 3.6. ☐

Thus quenching cannot occur on the whole interval. Theorem 3.7 is of crucial importance in the proof of the main result 4.2 about the quenching rate.

If it is also supposed, that u_0 is symmetric ($u_0 = u_0(r)$ and $u_0'(r) \geq 0$), then it follows from the uniquness of the solution and from Theorem 3.7, that the quenching point is $(0, T)$. We formulate this by following Corollary.

Corollary 3.8. *Let the assumptions of Theorem 3.7 hold. Let also the initial function u_0 be even and $u_0'(r) \geq 0$. Then the quenching point is $(0, T)$.*

4 Local asymptotics in the neighborhood of a quenching point

The main result of this paper is Theorem 4.2.

We first prove the preliminary Lemma 4.1 concerning the asymptotics of the solution $u(x,t)$. This Lemma gives a lower bound as a function of t for $u(x,t)$ ($x \in (-\varepsilon, \varepsilon)$), when the quenching point is approached. It also gives an upper bound at a minimum point with respect to the x-variable. Theorem 4.2 improves this Lemma by giving a pointwise asymptotic behavior of $u(x,t)$ in the region $|x| < C\sqrt{T-t}$, when the quenching point is approached.

After this Lemma 4.1, the main Theorem 4.2 is formulated and the proof is commented on.

Lemma 4.1. *Suppose that the initial function* u_0 *satisfies the condition* (1.4), *and that quenching occurs at* $t = T$. *Then there exist positive constants* β,l_1 *and* t_1 *such that*

(a) $u_t - \beta \ln(\alpha u) \leq 0$, *when* $x \in (-l_1, l_1)$ *(the quenching points belong to this interval) and* $t \in [t_1, T)$.

(b) u_t *blows up, when* u *quenches.*

(c) $u_t(\underline{x}, t) - \ln(\alpha u(\underline{x}, t)) \geq 0$, *when* $t \in (0,T)$, *and* $\underline{x}$ *is a local minimum point of* $u(x,t)$ *with respect to* x.

Proof. Because we know that quenching happens (Theorem 2.3) and that the set of quenching points is finite (Theorem 3.7), we can apply ([5], p. 1053-1054) to conclude the claim (a).

The item (b) follows directly from (a).

By the local existence theorem ([29] p.34) $u_{xx}(\underline{x}, t) \geq 0$, where $\underline{x}$ is a local minimum point of $u(x,t)$ with respect to x, and the claim (c) follows. □

Suppose in the following, that u_0 is symmetric in the sense that $u_0 = u_0(r)$ and $u_0'(r) \geq 0$. Then it follows from Corollary 3.8 that the only possible quenching point is $(0,T)$. Suppose that l is sufficiently large, so that we know by Theorem 2.3 that quenching occurs.

The local asymptotics of $u(x,t)$ as the quenching point is approached, will be now studied.

Define new variables:

$$y = \frac{x}{\sqrt{T-t}},$$

$$s = -\ln(T-t),$$

where $x \in [-l, l]$, $t \in [0, T)$, $y \in [-le^{\frac{1}{2}s}, le^{-\frac{1}{2}s}]$ and $s \in [-\ln T, \infty)$. Then the inverse functions $x = x(y,s)$ and $t = t(s)$ are well defined.

The function w is defined in terms of these new variables:

$$w(y,s) \overset{def}{=} 1 + \frac{1}{T - t(s)} \int_0^{u(x(y,s),t(s))} \frac{d\tau}{\ln(\alpha\tau)} = 1 + \frac{1}{T-t} \int_0^{u(x,t)} \frac{d\tau}{\ln(\alpha\tau)}. \quad (4.1)$$

The equation (1.3) can now be written in the form

$$w_s = w_{yy} - \frac{1}{2} y w_y + w + F, \quad (4.2)$$

where $F = \frac{u_x^2}{u(\ln(\alpha u))^2}$, and $(y,s) \in (-le^{\frac{1}{2}s}, le^{\frac{1}{2}s}) \times (-\ln T, \infty)$. Analogously to the formula (4.1), we can write $F = F(x(y,s), t(s)) = F(x,t)$. The boundary conditions are

$$w(\pm le^{\frac{1}{2}s}, s) = 1 + e^s \int_0^1 \frac{d\tau}{\ln(\alpha\tau)}, \tag{4.3}$$

when $s \geq -\ln T$. The initial condition is now

$$w(y, -\ln T) = 1 + \frac{1}{T} \int_0^{u_0(y\sqrt{T})} \frac{d\tau}{\ln(\alpha\tau)}, \tag{4.4}$$

where $y \in [-le^{-\frac{1}{2}s}, le^{-\frac{1}{2}s}]$.

Remark

Note that in the transformed equation (4.2) the nonlinear term F cannot be expressed explicitly as a function of y, s and w. For this reason in the following the variables x and s, or y and t might sometimes appear in a same equation. Another reason for this procedure is that in some cases it simplifies notations.

The goal is to prove:

Theorem 4.2. *Assume that u_0 is even, $u_0'(r) \geq 0$ and u_0 satisfies* (1.4). *Let $u(x,t)$ be the corresponding solution of* (1.3). *Assume that $u(x,t)$ quenches at $(0,T)$ for some $T < \infty$. Then for any constant C,*

(a) $w(y,s) - w(0,s)(1-\frac{1}{2}y^2) \to 0$, as $s \to \infty$ uniformly with respect to $|y| \leq C$, and

(b) $w(0,s) \to 0$, when $s \to \infty$.

Comment on the proof of the Theorem 4.2

The proof of (a) is built on Lemmas 4.3-4.12 and Corollary 4.13. The statement (b) follows from (a) and from Lemmas 4.14-4.18.

The proof of (a) is made more difficult by the fact that in the transformed equation (4.2) the nonlinear term F cannot be expressed explicitly as a function of y, s and w. Furthermore, F depends on both y and s; therefore stationary equation cannot be defined. Finally note that on y-intervals we only get lower estimates on w_{yy}. An upper bound can be obtained only at $y = 0$ (Lemma 4.4).

At the beginning of the proof of (a) it is shown, that $F \to 0$ uniformly on compact y-intervals for the equation (4.2)(Lemma 4.3). Therefore, on compact y-intervals the equation $h'' - \frac{1}{2}yh' + h = 0$ can be considered as the stationary equation for (4.2), when s is large. A particular solution of this equation is $h_2(y) = (1 - \frac{1}{2}y^2)$. Using Lemmas 4.3 and 4.4, we obtain first the limit of the theorem 4.2 (a) in a weak sense (Lemma 4.5(b)). The important Lemmas 4.10 and 4.12 show that $w(y,s) - w(0,s)(1-\frac{1}{2}y^2) \leq \varepsilon$ ($\varepsilon > 0$) for large s and for bounded y. The argument of Theorem 4.2 (a) is based on this, weak convergence (Lemma 4.5(b)) and the estimates of w (Lemma 4.4). Lemmas 4.6, 4.7, 4.8 and 4.9 are needed in the proof of Lemma 4.10, and Lemma 4.11 is needed in the proof of Lemma 4.12.

The idea of the part (2) (the proof of Theorem 4.2 (b)) is to conclude the claim from the nonlinear term F of the equation (4.2), as $s \to \infty$. It is known by the part (1), that for large s the solution is formally $w(y,s) \approx w(0,s)h_2(y)$ (for bounded y), and then $\mathcal{L}w = w_{yy} - \frac{1}{2}w_y + w \approx 0$. Concerning the reaction term F it is known , that it is zero only at the point $y = 0$ and otherwise positive. Thus for a large s^*, the reaction term F hasn't any contribution on $w(0,s)$, but for a large y it has a small increasing contribution on $w(y,s)$. (In fact it is shown, that for large s, the reaction term is formally $F \approx f_1(y,s)f_2(s)$, where $\frac{\partial}{\partial y}f_1(y,s) \geq 0$ (when $y > 0$) and $f_2(s) \to 0$). Therefore, somewhat later ($s = s^* + \delta$), the profile of the solution $w(y,s)$ is formally $w(0,s)[h_2(y) + g(s)\epsilon(y)]$, where the function $\epsilon(y)$ is non-decreasing ($y \geq 0$) and $\epsilon(0) = 0$. Because the solution $w(y,s)$ has to preserve the asymptotical form obtained in the part (1), the only possibility is, that $w(0,s)$ has to be decreasing, and that the limit value has to be zero.

The equation (4.2) is studied as a dynamical system in the space $L^2_\rho(R)$. Then the eigenvalues and eigenfunctions of the operator $\mathcal{L}$ are well-known. The scaled Hermite polynomials form an orthonormal base on that space, and the eigenvalue of the second order polynomial $h_2(y)$ is zero. By the part (1), it is known that this polynomial is dominant, as $s \to \infty$. So we obtain that the multiplier function $a_2(s)$ of h_2 in the Fourier expansion of the function w is asymptotically equal to $w(0,s)$ (Lemma 4.16), and that $a_2(s) \to 0$ (see (4.88)).

The domain of the solution $w(y,s)$ of the equation (4.2) is not (with respect to y) the whole R. Therefore, the above properties of L^2_ρ and $\mathcal{L}$ cannot be applied directly to the equation (4.2). This difficulty can be avoided by first extending the equation (1.3) to all $x \in R$, and observing that the solution of this equation in the region $\{(x,t) \in R^2 | x \in (-l,l), t \in (0,T)\}$ is the same as the solution of the original equation (1.3). Then the transformed solution $\tilde{w}(y,s)$ corresponding to the extended solution $\tilde{u}(x,t)$ is also defined for all $y \in R$.

4.1 Proof of Theorem 4.2 (a)

We begin by

Lemma 4.3.

(a) $u(x,t) \to 0$ *uniformly, when* $t \uparrow T$ *and* $|x| \leq C\sqrt{T-t}$.

(b) F *is uniformly bounded, when* $(x,t) \in [-l,l] \times [0,T)$.

(c) $F \to 0$ *uniformly, when* $t \uparrow T$ *and* $|x| \leq C\sqrt{T-t}$.

Proof. We show first the inequality

$$
\begin{aligned}
P(x,t) \overset{def}{=} \frac{1}{2}u_x(x,t)^2 - u(x,t)(1 - \ln(\alpha u(x,t))) \\
+ u(0,t)(1 - \ln(\alpha u(0,t))) \leq 0,
\end{aligned}
\tag{4.5}
$$

in the region $A = \{(x,t) | x \in (-l,l), t \in (0,T)\}$.

Because the initial function u_0 is symmetric, we can see that $u_x(0,t) = 0$ for all $t \in [0,T)$, and thus $P(0,t) = 0$ for all $t \in [0,T)$. Differentiating the function P with respect to x, we get

$$P_x = u_x u_{xx} + u_x \ln(\alpha u) - u_x + u_x = u_x u_t,$$

by the equation (1.3). It follows from the symmetry and Lemma 2.1 that $P_r \leq 0$ ($r = |x|$). So we have obtained the inequality (4.5).

By (4.5), we have that the function u_x is uniformly bounded. Claim (a) follows now from Corollary 3.8.

Claim (b) can be directly obtained from (4.5) by estimating:

$$0 \leq F \leq \frac{2(1 - \ln(\alpha u))}{(\ln(\alpha u))^2} \leq \frac{C}{|\ln(\alpha u)|} \leq M < \infty, \tag{4.6}$$

because $u \in (0,1]$ and $\alpha \in (0,1)$.

The claim (c) follows from the inequality (4.6), the continuity of the logarithmic function and the item (a). □

Lemma 4.4. *There exist positive constants c_1,c_2 and δ such that for all $s \geq -\ln T$,*

(a) $-c_1 \leq w_{yy}(0,s) \leq 0$.

(b) $-c_2 \leq w_{yy}(y,s)$, *when* $-le^{\frac{1}{2}s} \leq y \leq le^{\frac{1}{2}s}$.

(c) $-c_2 y \leq w_y(y,s) \leq 0$, *when* $0 \leq y \leq le^{\frac{1}{2}s}$. *Furthermore,* $0 \leq w_y(y,s) \leq c_2 y$, *when* $-le^{\frac{1}{2}s} \leq y \leq 0$.

(d) $0 \leq w(0,s) \leq 1 - \delta$.

(e) $-\frac{1}{2}c_2 y^2 \leq w(y,s) \leq 1 - \delta$, *when* $-le^{\frac{1}{2}s} \leq y \leq le^{\frac{1}{2}s}$.

Proof. Differentiating the equation (4.1) with respect to y, we get

$$w_y = \frac{1}{\sqrt{T-t}} \frac{u_x}{\ln(\alpha u)}, \tag{4.7}$$

and

$$w_{yy} = \frac{u_{xx}}{\ln(\alpha u)} - F = \frac{u_t}{\ln(\alpha u)} - 1 - F, \tag{4.8}$$

by the equation (1.3).

Applying Lemma 4.1 (c) and the fact that $u_x(0,t) = 0$ to (4.8), we get $w_{yy}(0,s) \leq 0$. Recalling, in addition, Lemma 4.1 (a), we get $w_{yy}(0,s) \geq -c_1$. This proves (a).

Applying Lemmas 2.1(a,b) and 4.3(b) to (4.8), we obtain the claim (b).

The item (c) follows by integrating (b), and noticing the symmetry of u.

By Lemma 4.1 (a), we get

$$\int_{u(0,t)}^{0} \frac{d\tau}{\ln(\alpha\tau)} \geq \beta \int_t^T d\tau. \tag{4.9}$$

From this it follows that

$$w(0,s) = 1 + \frac{1}{T-t}\int_0^{u(0,t)} \frac{d\tau}{\ln(\alpha\tau)} \leq 1-\beta. \tag{4.10}$$

Similarly by Lemma 4.1 (c), we get

$$\int_{u(0,t)}^0 \frac{d\tau}{\ln(\alpha\tau)} \leq \int_t^T d\tau, \tag{4.11}$$

and so

$$w(0,s) = 1 + \frac{1}{T-t}\int_0^{u(0,t)} \frac{d\tau}{\ln(\alpha\tau)} \geq 0. \tag{4.12}$$

We get the claim (d) from (4.10) and (4.12).

An integration of the item (c) yields the claim (e) by the item (d). □

Lemma 4.5. *Let $\rho(y) = \exp(\frac{-y^2}{4})$, and let $a(s)$ be a bounded function for $s \geq -\ln T$. Let $h_2(y) = (1 - \frac{1}{2}y^2)$ be the second order (Hermite) polynomial. Then*

(a) $\int_0^{le^{\frac{1}{2}s}} w(y,s)\rho(y)dy \to 0$, when $s \to \infty$.

(b) $\int_0^{le^{\frac{1}{2}s}} (w(y,s) - a(s)h_2(y))\rho(y)dy \to 0$, when $s \to \infty$.

Proof. (a) Multiply the equation (4.2) by ρ to obtain

$$(w_s - w)\rho = (w_y\rho)_y + F\rho.$$

Integrating this with respect to y from 0 to $le^{\frac{1}{2}s}$, we get

$$\int_0^{le^{\frac{1}{2}s}} (w_s(y,s) - w(y,s))\rho(y)dy = w_y(le^{\frac{1}{2}s}, s)\rho(le^{\frac{1}{2}s}) + \int_0^{le^{\frac{1}{2}s}} F\rho(y)dy. \tag{4.13}$$

Because $u_x(l,t)$ is bounded for all t, then

$$w_y(le^{\frac{1}{2}s}, s)\rho(le^{\frac{1}{2}s}) = e^{\frac{1}{2}s - \frac{l^2e^s}{4}} \frac{u_x(l,t)}{\ln(\alpha)} \to 0, \tag{4.14}$$

as $s \to \infty$. Writing

$$\int_0^{le^{\frac{1}{2}s}} F\rho dy = \int_0^K F\rho dy + \int_K^{le^{\frac{1}{2}s}} F\rho dy,$$

we can see, by Lemma 4.3(b,c), that

$$\int_0^{le^{\frac{1}{2}s}} F\rho(y)dy \to 0, \tag{4.15}$$

as $s \to \infty$. By the formulas (4.14) and (4.15) the terms on the right-hand side of the equation (4.13) converge to zero, as $s \to \infty$. Furthermore,

$$\frac{\partial}{\partial s}\int_0^{le^{\frac{1}{2}s}} w(y,s)\rho(y)dy = \frac{1}{2}le^{\frac{1}{2}s}w(le^{\frac{1}{2}s},s)\rho(le^{\frac{1}{2}s}) + \int_0^{le^{\frac{1}{2}s}} w_s(y,s)\rho(y)dy. \tag{4.16}$$

Define

$$I(s) = \frac{\partial}{\partial s}\int_0^{le^{\frac{1}{2}s}} w(y,s)\rho(y)dy - \int_0^{le^{\frac{1}{2}s}} w_s(y,s)\rho(y)dy. \tag{4.17}$$

By (4.16), (4.17) and the definition (4.1), we get

$$I(s) = \frac{1}{2}l(\exp(\frac{1}{2}s - \frac{l^2e^s}{4}))(1 + e^s\int_0^1 \frac{d\tau}{\ln(\alpha\tau)}) \to 0. \tag{4.18}$$

Let now $J(s) = \int_0^{le^{\frac{1}{2}s}} w(y,s)\rho(y)dy$. Using the equations (4.13), (4.14), (4.15), (4.17) and (4.18), we can see that

$$J'(s) - J(s) = I(s) + \int_0^{le^{\frac{1}{2}s}} (w_s(y,s) - w(y,s))\rho(y)dy \to 0, \tag{4.19}$$

as $s \to \infty$.

To obtain (a) from (4.19) we argue as follows. Suppose that there exists a sequence $\{s_i\}$ such that $|J(s_i)| \geq \varepsilon$. Then by (4.19) $|J(s_i)| \to \infty$, as $s_i \to \infty$. This contradicts Lemma 4.4.

The claim (b) follows from the item (a) and from the partial integration:

$$\int_0^\infty h_2(y)\rho(y)dy = \int_0^\infty \rho(y)dy + (y(\rho(y)))(\infty) - \int_0^\infty \rho(y)dy = 0.$$

□

Lemma 4.6. *There exist constants $\gamma \in (0,\infty)$, $l_1 \in (0,l)$ and $t_1 \in (0,T)$ such that $u_{rt} - \gamma u_t \geq 0$, when $x \in [-l_1, l_1]$ and $t \in [t_1, T)$.*

Proof. Let $J(r,t) = u_{rt}(r,t) - \gamma u_t(r,t)$. Differentiating this, we get

$$J_t - J_{rr} - \frac{1}{u}J = -\frac{u_r u_t}{u^2}.$$

The right-hand side of this equation is non-negative, by the facts that $u_r \geq 0$ and $u_t < 0$. Because there is only one quenching point, we can choose γ sufficiently big, so that the point $(0,T)$ has a neighborhood, for which $J \geq 0$ on the parabolic boundary of that neighborhood. This follows from Lemma 4.1, and from boundedness of the functions $u_{rt}(r,t)$ and $u_t(r,t)$. We obtain the claim now from the maximum principle. □

Lemma 4.7. *The function $\frac{-\ln(\alpha u(0,t))(T-t)}{u(0,t)}$ is bounded, when $t \in [0,T)$.*

Proof. The problem is the vanishing $u(0,t)$, as $t \uparrow T$. Using the inequality (4.9), we get

$$0 \leq \frac{-\ln(\alpha u(0,t))(T-t)}{u(0,t)} \leq \frac{\ln(\alpha u(0,t)) \int_0^{u(0,t)} \frac{d\tau}{\ln(\alpha\tau)}}{\beta u(0,t)}.$$

Applying L'Hospital's rule, we obtain

$$\frac{\int_0^{u(0,t)} \frac{d\tau}{\ln(\alpha\tau)}}{\frac{u(0,t)}{\ln(\alpha u(0,t))}} \to \frac{\frac{1}{\ln(\alpha u(0,t))}}{\frac{1}{\ln(\alpha u(0,t))} - \frac{1}{(\ln(\alpha u(0,t)))^2}} \to 1, \tag{4.20}$$

when $u(0,t) \to 0$. □

Lemma 4.8. *The function* $\frac{-u_t(x,t)(T-t)}{u(x,t)}$ *is bounded, for* $x \in [-l_1, l_1]$ *and* $t \in [t_1, T)$.

Proof. Because $u(x,t)$ is symmetric, it is sufficient to study the case $x > 0$. By Lemma 4.6,

$$\begin{aligned} u_t(x,t) - u_t(0,t) = \int_0^x u_{tx}(\eta,t)d\eta \geq \gamma \int_0^x u_t(\eta,t)d\eta = \\ \gamma \int_0^x (u_{xx}(\eta,t) + \ln(\alpha u(\eta,t)))d\eta \geq \\ \gamma(u_x(x,t) + x\ln(\alpha u(0,t))). \end{aligned} \tag{4.21}$$

Thus it follows from Lemma 4.1 that

$$-u_t(x,t) \leq -u_t(0,t) - \gamma x \ln(\alpha u(0,t)) \leq -\ln(\alpha u(0,t))(1+\gamma x), \tag{4.22}$$

and so

$$0 \leq \frac{-u_t(x,t)(T-t)}{u(x,t)} \leq \frac{-\ln(\alpha(u(0,t)))(T-t)}{u(0,t)}(1+\gamma x).$$

The claim now follows from Lemma 4.7. □

Lemma 4.9. *For* $|y| = |\frac{x}{\sqrt{T-t}}| \leq C < \infty$, *we have*

$$\limsup_{t\uparrow T} \sqrt{T-t}\frac{u_{rt}}{\ln(\alpha u)} \leq 0,$$

uniformly.

Proof. By Lemma 4.6,

$$\sqrt{T-t}\frac{u_{xt}}{\ln(\alpha u)} \leq \gamma\sqrt{T-t}\frac{u_t}{\ln(\alpha u)},$$

when $x > 0$. So it sufficies to show that $(T-t)\frac{u_t^2}{(\ln(\alpha u))^2} \to 0$, as $t \uparrow T$.

Lemma 4.8 implies

$$u_t^2 \frac{T-t}{(\ln(\alpha u))^2} \leq Mu \frac{-u_t}{(\ln(\alpha u))^2}.$$

Therefore it is sufficient, by Lemma 4.3(a), to obtain that $\frac{-u_t}{(\ln(\alpha u))^2}$ is bounded. The Taylor expansions give:

$$\begin{aligned} \ln(\alpha u(x,t)) = \ln(\alpha u(0,t)) + \frac{1}{\eta}(u(x,t) - u(0,t)) \leq \\ \ln(\alpha u(0,t)) + \frac{1}{u(0,t)}(u(x,t) - u(0,t)) \leq \ln(\alpha u(0,t)) + \frac{u(x,t)}{u(0,t)}, \end{aligned} \tag{4.23}$$

where $\eta = \eta(x,t) \in (u(0,t), u(x,t))$, and

$$\begin{aligned} u(x,t) = u(0,t) + \frac{1}{2}u_{xx}(\xi,t)x^2 = \\ u(0,t) + \frac{1}{2}(u_t - \ln(\alpha u))(\xi,t)x^2, \end{aligned} \tag{4.24}$$

where $\xi \in (0,x)$. Using Lemma 2.1, and the fact that $u(0,t) \leq u(\xi,t)$, in the expansion (4.24), we get

$$u(x,t) \leq u(0,t) - \frac{1}{2}\ln(\alpha u(0,t))x^2. \tag{4.25}$$

By Lemma 2.1, and by (4.22) and (4.23), we have

$$\begin{aligned} 0 \leq \frac{-u_t(x,t)}{(\ln(\alpha u(x,t)))^2} \leq \frac{-C\ln(\alpha u(0,t))}{(\ln(\alpha u(x,t)))^2} \leq \\ \frac{-C}{\ln(\alpha u(x,t))} + \frac{Cu(x,t)}{(\ln(\alpha u(x,t)))^2 u(0,t)}. \end{aligned}$$

Making use of (4.25) and applying the definition of y yield

$$\begin{aligned} 0 \leq \frac{-u_t(x,t)}{(\ln(\alpha u(x,t)))^2} \leq \frac{-C}{\ln(\alpha u(x,t))} + \frac{C}{(\ln(\alpha u(x,t)))^2} - \\ \frac{1}{2}Cy^2 \frac{(T-t)\ln(\alpha u(0,t))}{u(0,t)\ln(\alpha u(x,t))^2}. \end{aligned}$$

The claim now follows from Lemmas 4.3(a) and 4.7. □

Lemma 4.10. *For $0 < y < C$, we have*

$$\limsup_{s \to \infty} w_{yyy}(y,s) \leq 0$$

uniformly.

Proof. Differentiating the equation (4.8) with respect to x, we get

$$\frac{1}{\sqrt{T-t}} w_{yyy} = \sum_{i=1}^{4} G_i, \tag{4.26}$$

where

$$G_1(x,t) = \sqrt{T-t}\frac{u_{xt}}{\ln(\alpha u)} \tag{4.27}$$

$$G_2(x,t) = -\sqrt{T-t}\frac{u_x u_t}{u(\ln(\alpha u))^2} \tag{4.28}$$

$$G_3(x,t) = \sqrt{T-t}\frac{u_x}{u^2}(\frac{u_x}{\ln(\alpha u)})^2 \tag{4.29}$$

$$G_4(x,t) = -\sqrt{T-t}\frac{2u_x}{u(\ln(\alpha u))}(\frac{u_{xx}}{\ln(\alpha u)} - \frac{u_x^2}{u(\ln(\alpha u))^2}). \tag{4.30}$$

By Lemma 4.9 it is sufficient to prove that $G_i \to 0$ uniformly for bounded y, as $s \to \infty$ and $i = 2, 3, 4$.

Using the symmetry of u, Lemmas 2.1 and 4.8, we obtain

$$0 \leq G_2 = -\frac{u_x}{\sqrt{T-t}\ln(\alpha u)} \frac{(T-t)(-u_t)}{u} \frac{1}{-\ln(\alpha u)} \leq M w_y \frac{1}{\ln(\alpha u)}.$$

Applying Lemmas 4.4 and 4.3, we get the claim $\lim_{t \uparrow T} G_2 = 0$.

From the inequality (4.5) and the symmetry of u, it follows that

$$0 \leq G_3 = \frac{T-t}{u} \frac{-u_x}{(\sqrt{T-t}\ln(\alpha u))} \frac{u_x^2}{(-u\ln(\alpha u))} \leq 2\frac{T-t}{u}(-w_y)\frac{1-\ln(\alpha u)}{-\ln(\alpha u)}.$$

By Lemmas 4.4 and 2.1, the two last term on the right-hand side are bounded. Using also the fact $u(0,t) \leq u(x,t)$, Lemma 4.1 (b) and applying L'Hospital's rule to the term $\frac{T-t}{u(x,t)}$, we obtain the claim $\lim_{t \uparrow T} G_3 = 0$.

The function G_4 can be written in the form

$$G_4 = 2G_2 + 2\frac{T-t}{u}(w_y)(1+F).$$

Applying Lemmas 4.4, 4.3 and 4.1, we conclude that the last term at the right-hand side converges to zero. Because $\lim_{t \uparrow T} G_2 = 0$, then also $\lim_{t \uparrow T} G_4 = 0$. □

Lemma 4.11. *There exists a positive constant M such that $(T-t)u_{tt} \leq M$ in some neighborhood $N = (-a,a) \times (T-\delta, T)$ of $(0,T)$.*

Proof. Let $H = (T-t)u_{tt} - M$, where the constant $M > 0$ will be determined later. Then

$$H_t = -u_{tt} + (T-t)u_{ttt},$$

$$H_{xx} = (T-t)u_{ttxx}.$$

Therefore we get from the equation (1.3)

$$H_t - H_{xx} = (T-t)(\ln(\alpha u))_{tt} - u_{tt} = (T-t)\{\frac{u_{tt}}{u} - (\frac{u_t}{u})^2\} - \frac{1}{T-t}(H+M),$$

and further

$$H_t - H_{xx} + bH = -(T-t)(\frac{u_t}{u})^2 + \frac{M}{u}(1 - \frac{u}{T-t}),$$

where $b = \frac{1}{T-t} - \frac{1}{u}$ is locally bounded function. We can see that $\frac{u(x,t)}{T-t} \geq \frac{u(0,t)}{T-t} \to \infty$, by Lemma 4.1 (b). Hence there is a neighborhood $N = (-a,a) \times (T-\delta, T)$ of $(0,T)$, which is independent of M, such that

$$H_t - H_{xx} + bH \leq 0,$$

on N. When M is chosen big enough, then $H \leq 0$ on the parabolic boundary of N by Theorem 3.7. The claim now follows from the maximum principle. □

Lemma 4.12. *Let $w(y,s)$ be the solution of (4.2). Then*

$$\lim_{s\to\infty} w_s(0,s) = 0.$$

Proof. Using the formulas (4.7) and (4.8), the equation (4.2) can be written in the form

$$w_s - w = w_{yy} - \frac{1}{2}yw_y + F = \frac{u_t}{\ln(\alpha u)} - 1 - \frac{xu_x}{2(T-t)\ln(\alpha u)}.$$

Differentiating this with respect to t, we get

$$\begin{aligned}\frac{1}{T-t}\{w_{ss} + \frac{1}{2}yw_{ys} - w_s - \frac{1}{2}yw_y\} = &-\frac{1}{2}\frac{x}{T-t}\{\frac{u_x}{(T-t)\ln(\alpha u)} + \\ &+ \frac{u_{tx}}{\ln(\alpha u)} - \frac{u_x u_t}{u(\ln(\alpha u))^2}\} + \\ &+ \frac{u_{tt}}{\ln(\alpha u)} - \frac{u_t^2}{u(\ln(\alpha u))^2}.\end{aligned} \tag{4.31}$$

In (4.31), take $y = 0$ to obtain

$$w_{ss}(0,s) - w_s(0,s) = (T-t)\{\frac{u_{tt}(0,t)}{\ln(\alpha u(0,t))} - \frac{(u_t(0,t))^2}{u(0,t)(\ln(\alpha u(0,t)))^2}\}.$$

Here the last term of the right-hand side converges to zero by Lemma 4.1(b,c). Thus, applying Lemma 4.11, we have

$$\liminf_{s\to\infty}\{w_{ss}(0,s) - w_s(0,s)\} \geq 0. \tag{4.32}$$

We prove next that $\liminf_{s\to\infty} w_s(0,s) \geq 0$.

By Lemma 4.10, for every $\varepsilon > 0$ and $C > 0$, there exists $s_1 > -\ln T$ such that $w_{yyy} < \varepsilon$, when $s \geq s_1$ and $0 \leq y \leq C < \infty$. Integrating this inequality three times with respect to y, we get

$$w_{yy}(y,s) - w_{yy}(0,s) < \varepsilon y, \tag{4.33}$$

$$w_y(y,s) - y w_{yy}(0,s) < \frac{1}{2}\varepsilon y^2, \tag{4.34}$$

$$w(y,s) - w(0,s) - \frac{1}{2}y^2 w_{yy}(0,s) < \frac{1}{6}\varepsilon y^3, \tag{4.35}$$

when $y \in [0, C]$. Because $w_s(0,s) = w_{yy}(0,s) + w(0,s)$, it follows from the inequality (4.35) that

$$\limsup_{s\to\infty}\{w(y,s) + w_{yy}(0,s)(1 - \frac{1}{2}y^2) - w_s(0,s)\} \leq 0, \tag{4.36}$$

uniformly, when $0 \leq y \leq C < \infty$. Let $g(y,s) = w(y,s) + w_{yy}(0,s)(1 - \frac{1}{2}y^2)$, and consider the function

$$G(y,s) = \int_0^y (g(\eta,s) - w_s(0,s))\rho(\eta)d\eta.$$

An application of (4.36) to this definition gives that for every $\varepsilon > 0$ and $K \in (0,\infty)$ there exists $s_2 \geq -\ln T$ such that

$$G(y,s) < \varepsilon, \tag{4.37}$$

when $s \geq s_2$ and $0 < y \leq K$.

Choosing y big enough, we can see, by Lemmas 4.4 and 4.5 (b), that

$$G(y,s) + w_s(0,s)\int_0^y \rho(\eta)d\eta \to 0, \tag{4.38}$$

as $s \to \infty$. Using the formulas (4.37) and (4.38), we get

$$\liminf_{s\to\infty} w_s(0,s) \geq 0. \tag{4.39}$$

Suppose now that there exists a sequence $s_i \to \infty$ such that $w_s(0,s_i) \geq \varepsilon$. From this it follows by (4.32) and (4.39), that $w_s(0,s_i) \to \infty$. This contradicts Lemma 4.4. □

After these preliminary Lemmas we turn to the proof of Theorem 4.2(a).

Let g be as in the proof of Lemma 4.12. By this Lemma 4.12 and by (4.36),

$$\limsup_{s\to\infty} g(y,s) \leq 0 \tag{4.40}$$

uniformly, when $0 \leq y \leq C < \infty$. Furthermore, by Lemma 4.5(b) (where $a(s) = -w_{yy}(0,s)$)

$$\lim_{s\to\infty}\int_0^{le^{\frac{1}{2}s}} g(\eta,s)\rho(\eta)d\eta = 0. \tag{4.41}$$

By Lemma 4.4 it holds that $|g_y| \leq Cy$. Therefore it follows from (4.40), (4.41) and the symmetry of the solution that

$$\lim_{s\to\infty} g(y,s) = 0, \tag{4.42}$$

uniformly, when $|y| \leq C < \infty$. By Lemma 4.12, $w_{yy}(0,s) + w(0,s) \to 0$, as $s \to \infty$, and so Theorem 4.2 (a) follows from the equation (4.42).

Corollary 4.13. *For $|y| \leq C < \infty$, we have $\lim_{s\to\infty} w_s(y,s) = 0$ uniformly.*

Proof. By Lemma 4.12, $w_s(0,s) = w_{yy}(0,s) + w(0,s) \to 0$. Combining this with inequality (4.34), we have

$$\limsup_{s\to\infty}\{w_y(y,s) + yw(0,s)\} \leq 0, \tag{4.43}$$

uniformly, when $0 \leq y \leq C < \infty$. Writing

$$w(y,s) - w(0,s)(1 - \frac{1}{2}y^2) = \int_0^y (w_\eta(\eta,s) + \eta w(0,s))d\eta,$$

we obtain, by Theorem 4.2(a), (4.43) and Lemma 4.4, that

$$\lim_{s\to\infty}(w_y(y,s) + yw(0,s)) = 0, \tag{4.44}$$

uniformly for bounded y. Correspondingly, to conclude (4.46), we first write

$$w_y(y,s) + yw(0,s) = \int_0^y (w_{\eta\eta}(\eta,s) + w(0,s))d\eta. \tag{4.45}$$

Using the inequality (4.33) and Lemma 4.12, we get

$$\lim_{s\to\infty}(w_{yy}(y,s) + w(0,s)) \leq 0,$$

uniformly for bounded y. Applying this together with (4.44) and Lemma 4.4 to the equation (4.45), we can see that

$$\lim_{s\to\infty}(w_{yy} + w(0,s)) = 0, \tag{4.46}$$

uniformly for bounded y. Writing

$$w_s = (w_{yy} + w(0,s)) - \frac{1}{2}y(w_y + yw(0,s)) + (w - w(0,s)(1 - \frac{1}{2}y^2)) + F,$$

we obtain the claim from (4.46), (4.44), Theorem 4.2(a), and Lemma 4.3(c). ∎

4.2 Proof of Theorem 4.2 (b)

We will now replace the equation (1.3) by an extended one, defined on the whole real line with respect to x. This equation of course admits the same solutions as (1.3) on the original interval $(-l, l)$. The technical construction is done similarly as in [30] or in [10]. Without loss of generality, we may assume that $l = 1$ in the equation (1.3). So let $x \geq 1$, and define the kernels:

$$V(x,t) = \frac{1}{\sqrt{\pi t}} \exp(-\frac{x^2}{4t}),$$

$$W(x,t) = \frac{x}{2\sqrt{\pi t^3}} \exp(-\frac{x^2}{4t}),$$

when $|x| < \infty$ and $0 < t < \infty$.

Differentiating these, we can see that $V_x = -W$, $V_t = V_{xx}$ and $W_t = W_{xx}$.

Define the extension $\overline{u}$ of $u(x,t)$, when $x \geq 1$ and $t > 0$ by

$$\overline{u}(x,t) = (x-1)\int_0^t W(x-1, t-\tau)u_x(1,\tau)d\tau + 1. \tag{4.47}$$

Here $u_x(1,t)$ is obtained from the equation (3.1) ($u_x(1,t) = \lim_{z\uparrow 1} u_x(z,t)$).

Lemma 4.14. *The function $\overline{u}$ satisfies:*

$$\overline{u}_t - \overline{u}_{xx} = 2u_x(1,0)V(x-1,t) + 2\int_0^t V(x-1,t-\tau)u_{x\tau}(1,\tau)d\tau,$$

when $1 < x < \infty$.

Proof. Differentiating (4.47), we get

$$\overline{u}_t(x,t) = (x-1)\int_0^t W_t(x-1,t-\tau)u_x(1,\tau)d\tau, \tag{4.48}$$

$$\overline{u}_x(x,t) = \int_0^t W(x-1,t-\tau)u_x(1,\tau)d\tau + (x-1)\int_0^t W_x(x-1,t-\tau)u_x(1,\tau)d\tau, \tag{4.49}$$

and

$$\begin{aligned}\overline{u}_{xx}(x,t) = 2\int_0^t W_x(x-1,t-\tau)u_x(1,\tau)d\tau + \\ (x-1)\int_0^t W_{xx}(x-1,t-\tau)u_x(1,\tau)d\tau.\end{aligned} \tag{4.50}$$

Subtract (4.50) from (4.48), yields

$$\overline{u}_t - \overline{u}_{xx} = -2\int_0^t V_\tau(x-1,t-\tau)u_x(1,\tau)d\tau,$$

by the formulas $W_x = -V_{xx} = -V_t$ and $W_t = W_{xx}$. An integration by parts of the right-hand side gives the claim. $\square$

Correspondingly in the extension of u to the left of $x = -1$ the term $u_x(1,t)$ in the equation (4.47) is replaced by the term $u_x(-1,t)$.

An extended equation is now defined

$$\tilde{u}_t - \tilde{u}_{xx} = f(\tilde{u}(x,t)); \qquad x \in R\backslash\{\pm 1\}, \quad 0 < t < T, \tag{4.51}$$

where

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & \text{when } |x| \le 1 \\ \overline{u}(x,t), & \text{when } |x| > 1, \end{cases}$$

$$f(\tilde{u}) = \begin{cases} \ln(\alpha u), & \text{when } |x| \le 1 \\ \overline{g}(x,t), & \text{when } |x| > 1, \end{cases} \tag{4.52}$$

and

$$\overline{g}(x,t) = 2u_x(1,0)V(x-1,t) + 2\int_0^t V(x-1,t-\tau)u_{x\tau}(1,\tau)d\tau. \tag{4.53}$$

We can see that $\tilde{u} \in C^1(R)$ (fixed t), but f is not continuous at $x = \pm 1$, and therefore $\tilde{u}$ is not twice continuously differentiable.

Because $u(x,t)$ cannot quench at $x = \pm 1$, then the functions $u_x(1,t)$ and $u_{xt}(1,t)$ are uniformly bounded.

Lemma 4.15. *The functions $\overline{u}(x,t)$ and $\overline{g}(x,t)$ satisfy*

$$1 \le \overline{u}(x,t) < c_1 < \infty,$$

and

$$0 \le \overline{g}(x,t) < c_2 < \infty,$$

when $|x| \ge 1$ and $0 \le t < T$.

Proof. Using the inequality (4.5), we get

$$u_x \le \sqrt{2u(1-\ln(\alpha u))} \to \sqrt{2(1-\ln(\alpha))}, \tag{4.54}$$

when $x \uparrow 1$. Writing the equation (4.47) in the form

$$\overline{u} = \int_0^t \frac{(x-1)^2}{2\sqrt{\pi(t-\tau)^3}} \exp(-\frac{(x-1)^2}{4(t-\tau)})u_x(1,\tau)d\tau + 1,$$

and substituting (4.54), we can see that

$$\overline{u} \le C\sqrt{2(1-\ln(\alpha))}\int_R \xi e^{-\frac{\xi^2}{4}}d\xi + 1 < \infty. \tag{4.55}$$

Also

$$\sup \overline{g} \le c|V(x-1,t)| + 2c_3\int_0^t V(x-1,t-\tau)d\tau < \infty.$$

□

From (4.55) we get the inequality

$$\alpha\overline{u} \leq C\alpha\sqrt{1-\ln(\alpha)}+\alpha, \tag{4.56}$$

and from (4.49) there follows

$$|\overline{u}_x| \leq c_4 < \infty, \tag{4.57}$$

when $|x| \geq 1$ and $0 \leq t < T$.

Define, when $y \in R$ ($y = \frac{x}{\sqrt{T-t}}$) and $s \geq -\ln T$:

$$\tilde{w}(y,s) = 1 + \frac{1}{T-t}\int_0^{\tilde{u}(x,t)} \frac{d\tau}{f(\tau)}, \tag{4.58}$$

where

$$f(\tau) = \begin{cases} \ln(\alpha\tau) & , \quad 0 < \tau \leq 1, \\ \ln(\alpha\tau) + g(\tau) & , \quad \tau > 1. \end{cases} \tag{4.59}$$

The function g is defined such that $f(\tau)$ is smooth, increasing and negative, when $\tau > 0$. Differentiating the definition (4.58), we get

$$\tilde{w}_s - \tilde{w}_{yy} + \frac{1}{2}y\tilde{w}_y - \tilde{w} = \tilde{F}, \tag{4.60}$$

where $\tilde{F} = F$, when $y \in (-e^{\frac{1}{2}s}, e^{\frac{1}{2}s})$, and at intervals $(-\infty, -e^{\frac{1}{2}s})$ and $(e^{\frac{1}{2}s}, \infty)$:

$$\tilde{F} = \frac{\overline{g}(x,t)}{f(\overline{u})} - 1 + \frac{\overline{u}_x^2}{f(\overline{u})^2}f'(\overline{u}), \tag{4.61}$$

where $\overline{g}$ is defined by the equation (4.53) and f by the equation (4.59). Using (4.56)- (4.59), (4.61) and Lemma 4.15, we obtain, when $|y| > le^{\frac{1}{2}s}$ and $s \geq -\ln T$,

$$|\tilde{w}(y,s)| \leq Cy^2, \tag{4.62}$$

$$|\tilde{w}_y(y,s)| \leq Cy, \tag{4.63}$$

and

$$|\tilde{F}| \leq M < \infty. \tag{4.64}$$

Consider now the extended equation (4.60) as a dynamical system in the space

$$L^2_\rho(R) = \{g(y) | \int_R g(y)^2\rho(y)dy < \infty\}. \tag{4.65}$$

We use from now on the notation $w = \tilde{w}$. Then

$$w_s - \mathcal{L}w = F, \tag{4.66}$$

where

$$\mathcal{L}w = w_{yy} - \frac{1}{2}yw_y + w, \tag{4.67}$$

on the set $R \times [-\ln T, \infty)$.

The space L^2_ρ is a Hilbert space with an inner product

$$\langle f, g\rangle_{L^2_\rho} = \int_R f(y)g(y)\rho(y)dy.$$

Concerning the linear operator $\mathcal{L}$ it is known that (see [11]), it is self-adjoint i.e, that

$$\langle \mathcal{L}f, g\rangle_{L^2_\rho} = \langle f, \mathcal{L}g\rangle_{L^2_\rho}, \tag{4.68}$$

with spectrum

$$\lambda_k = 1 - \frac{1}{2}k; \quad k = 0, 1, 2, ... \tag{4.69}$$

The corresponding eigenfunctions are

$$\tilde{h}_k(y) = \alpha_k H_k(\frac{1}{2}y), \tag{4.70}$$

where H_k are the (standard) Hermite polynomial and $\alpha_k = (\pi^{\frac{1}{2}} 2^{k+1} k!)^{-\frac{1}{2}}$. The first three eigenfunctions are

$$\tilde{h}_0 = \frac{1}{\sqrt{2}}\pi^{\frac{-1}{4}}, \quad \tilde{h}_1 = \frac{1}{2}\pi^{\frac{-1}{4}}y, \quad \tilde{h}_2 = \frac{1}{2}\pi^{\frac{-1}{4}}(\frac{1}{2}y^2 - 1). \tag{4.71}$$

The Fourier-expansion of w with respect to this base is:

$$w(y, s) = \sum_{k=0}^{\infty} \tilde{a}_k(s)\tilde{h}_k(y). \tag{4.72}$$

Then one has

Lemma 4.16. *Let* $a_2(s) = -\frac{1}{2}\pi^{\frac{-1}{4}}\tilde{a}_2(s)$. *Then* $a_2(s) - w(0, s) \to 0$, *as* $s \to \infty$.

Proof. Let $\phi(y, s) = w(y, s) - w(0, s)h_2(y)$, where h_2 is defined as in the Lemma 4.5. Projecting the function ϕ to the subspace generated by the function $\tilde{h}_2$, we get

$$\langle \phi, \tilde{h}_2\rangle_{L^2_\rho} = \sum_k \tilde{a}_k(s)\langle \tilde{h}_k, \tilde{h}_2\rangle_{L^2_\rho} - w(0, s)\langle h_2, \tilde{h}_2\rangle_{L^2_\rho} = -2\pi^{\frac{-1}{4}}(a_2(s) - w(0, s)).$$

Applying Hölder's inequality to this, it follows that

$$|a_2(s) - w(0, s)| \le C(\int_R (w(y, s) - w(0, s)h_2)^2\rho)^{\frac{1}{2}} \|h_2\|^{\frac{1}{2}}_{L^2_\rho} \to 0,$$

by Theorem 4.2(a) and the inequality (4.62). □

Lemma 4.17. *The inequalities*

$$0 \le \frac{(T-t)(-\ln(T-t))}{u(x, t)} \le M < \infty$$

hold on the set $[-l, l] \times [0, T)$.

Proof. We first prove that

$$\lim_{u\downarrow 0}\frac{(\int_0^u \frac{d\tau}{-\ln(\alpha\tau)})(-\ln(k\int_0^u \frac{d\tau}{-\ln(\alpha\tau)}))}{u}=1, \tag{4.73}$$

holds for a positive constant k. To show this, we use L'Hospital's rule twice.

We first obtain

$$\frac{\ln(k\int_0^u \frac{d\tau}{-\ln(\alpha\tau)})}{\ln(\alpha u)}\to 1. \tag{4.74}$$

The claim (4.74) is true, because

$$\frac{\ln(k\int_0^u \frac{d\tau}{-\ln(\alpha\tau)})}{\ln(\alpha u)}\to\frac{\frac{1}{-ln(\alpha u)\int_0^u\frac{d\tau}{-\ln(\alpha\tau)}}}{\frac{1}{u}}=\frac{u}{\ln(\alpha u)\int_0^u\frac{d\tau}{\ln(\alpha\tau)}}\to 1, \tag{4.75}$$

by (4.20).

The claim (4.73),

$$\frac{(\int_0^u \frac{d\tau}{-\ln(\alpha\tau)})(-\ln(k\int_0^u \frac{d\tau}{-\ln(\alpha\tau)}))}{u}\to\frac{\frac{1}{\ln(\alpha u)}\ln(k\int_0^u\frac{d\tau}{-\ln(\alpha\tau)})+\frac{1}{\ln(\alpha u)}}{1}\to 1,$$

now follows from the equation (4.75).

By the inequalities (4.9) and (4.11) we can see that

$$\int_0^{u(0,t)}\frac{d\tau}{-\ln(\alpha\tau)}\le T-t\le\frac{1}{\beta}\int_0^{u(0,t)}\frac{d\tau}{-\ln(\alpha\tau)},$$

when $t\in[0,T)$. Because $u(0,t)\le u(x,t)$, then

$$0\le\frac{(T-t)(-\ln(T-t))}{u(x,t)}\le\frac{1}{\beta}\frac{(\int_0^{u(0,t)}\frac{d\tau}{-\ln(\alpha\tau)})(-\ln(\int_0^{u(0,t)}\frac{d\tau}{-\ln(\alpha\tau)}))}{u(0,t)}\to\frac{1}{\beta},$$

as $u(0,t)\to 0$, by (4.73). □

Lemma 4.18. *For the solution $u(x,t)$ one has*

$$\frac{(T-t)(-\ln(T-t))}{u(x,t)}-\frac{1}{1-w(0,s)h_2(y)}\to 0$$

uniformly for bounded y, as $t\uparrow T$ ($s\to\infty$).

Proof. By Theorem 4.2(a),

$$1+\frac{1}{T-t}\int_0^u\frac{d\tau}{\ln(\alpha\tau)}-w(0,s)h_2(y)\to 0,$$

uniformly for bounded y, as $t\uparrow T$. Dividing this by the function $1-w(0,s)h_2(y)$ ($\neq 0$ by Lemma (4.4)), we get

$$\frac{\int_0^u\frac{d\tau}{-\ln(\alpha\tau)}}{(T-t)(1-w(0,s)h_2(y))}\to 1, \tag{4.76}$$

uniformly for bounded y, as $t \uparrow T$. From the properties of the logarithmic function it follows that

$$\ln(\frac{1}{1-w(0,s)h_2(y)}\int_0^u \frac{d\tau}{-\ln(\alpha\tau)}) - \ln(T-t) \to 0,$$

and

$$\frac{\ln(\frac{1}{1-w(0,s)h_2(y)}\int_0^u \frac{d\tau}{-\ln(\alpha\tau)})}{\ln(T-t)} \to 1, \tag{4.77}$$

uniformly for bounded y, as $t \uparrow T$.

Let

$$H = \frac{(T-t)(-\ln(T-t))}{u} - \frac{\int_0^u \frac{d\tau}{-\ln(\alpha\tau)}}{1-w(0,s)h_2(y)} \frac{-\ln(\frac{1}{1-w(0,s)h_2(y)}\int_0^u \frac{d\tau}{-\ln(\alpha\tau)})}{u}. \tag{4.78}$$

This can be written in the form

$$H = (\frac{(T-t)(-\ln(T-t))}{u}) \Big(1 - \frac{\int_0^u \frac{d\tau}{-\ln(\alpha\tau)}}{(T-t)(1-w(0,s)h_2(y))} \frac{\ln(\frac{1}{1-w(0,s)h_2(y)}\int_0^u \frac{d\tau}{-\ln(\alpha\tau)})}{\ln(T-t)}\Big) \tag{4.79}$$

By Lemma 4.17, and by the formulas (4.76) and (4.77),

$$H \to 0, \tag{4.80}$$

uniformly for bounded y, as $t \uparrow T$.

Writing the equation (4.78) in the form,

$$H = \frac{(T-t)(-\ln(T-t))}{u} - \frac{1}{1-w(0,s)h_2(y)} + \frac{1}{1-w(0,s)h_2(y)}(1 - \frac{(\int_0^u \frac{d\tau}{-\ln(\alpha\tau)})(-\ln(\frac{1}{1-w(0,s)h_2}\int_0^u \frac{d\tau}{-\ln(\alpha\tau)}))}{u}), \tag{4.81}$$

we can see that the claim follows from (4.80), and 4.73, provided we recall that $(1-w(0,s)h_2(y)) \in [\delta, C]$ for bounded y, by Lemma 4.4. □

Proof of Theorem 4.2(b). Projecting the equation $w_s = \mathcal{L}w + F$ to the subspace generated by the function h_2, we obtain

$$\sum_k \tilde{a}_k'(s)\langle \tilde{h}_k, h_2\rangle_{L^2_\rho} = \langle \mathcal{L}w, h_2\rangle_{L^2_\rho} + \langle F, h_2\rangle_{L^2_\rho}.$$

Note that $\langle F, h_2\rangle_{L^2_\rho} = 2\int_0^{le^{\frac{1}{2}s}} + \int_{le^{\frac{1}{2}s}}^\infty Fh_2\rho$, where the latter integral is less than $C\exp(-\epsilon e^s)$ by (4.64), and so only the first integral is essential in the equation (4.82), as $s \to \infty$. The factor 2 is included in the integrals below, because the solution is symmetric. We can conclude by (4.68) and the orthogonality of the base $\{\tilde{h}_k\}_{k=0}^\infty$, that ($C = 4\sqrt{\pi}$)

$$Ca_2'(s) = 2\int_0^\infty \frac{u_x^2}{u(\ln(\alpha u))^2} h_2(y)\rho(y)dy = 2\int_0^\infty \frac{T-t}{u} w_y^2 h_2(y)\rho(y)dy,$$

and

$$Csa_2'(s) = 2\int_0^\infty \frac{(T-t)(-\ln(T-t))}{u} w_y^2 h_2(y)\rho(y)dy.$$

Write this in the form

$$\begin{aligned} Csa_2' = 2\int_0^\infty & \frac{(T-t)(-\ln(T-t))}{u}(w_y^2 - w(0,s)^2y^2)h_2(y)\rho(y)dy + \\ 2\int_0^\infty & \left(\frac{(T-t)(-\ln(T-t))}{u} - \frac{1}{1-w(0,s)h_2(y)}\right)w(0,s)^2y^2h_2(y)\rho(y)dy + \\ 2\int_0^\infty & \frac{1}{1-w(0,s)h_2(y)}(w(0,s)^2 - a_2(s)^2)y^2h_2(y)\rho(y)dy + \\ 2\int_0^\infty & \frac{a_2(s)^2y^2}{1-w(0,s)h_2(y)}h_2(y)\rho(y)dy = \sum_{j=1}^4 I_j(s). \end{aligned} \tag{4.82}$$

Next we show that $I_j(s) \to 0$, as $s \to \infty$ and $j = 1, 2, 3$.

When $j = 1$, then by Lemmas 4.4, 4.15, 4.17 and the inequality (4.63), we can apply the Lebesgue Dominated Convergence Theorem. Writing $w_y^2 - w(0,s)^2y^2 = (w_y + w(0,s)y)(w_y - w(0,s)y)$, we can conclude by Lemmas 4.4, 4.15,4.17, the inequality (4.63) and the formula (4.44), that

$$\lim_{s\to\infty} I_1(s) \to 0. \tag{4.83}$$

Correspondingly, in the case $j = 2$, we obtain from Lemmas 4.4, 4.18 and 4.15 that

$$\lim_{s\to\infty} I_2(s) \to 0. \tag{4.84}$$

When $j = 3$, we obtain, by Lemmas 4.4 and 4.16,

$$\lim_{s\to\infty} I_3(s) \to 0. \tag{4.85}$$

Finally, we have that there exist positive constants c_1 and c_2 such that

$$-c_1a_2(s)^2 \le I_4(s) \le -c_2a_2(s)^2, \tag{4.86}$$

for all $s \ge -\ln T$.

By the relations (4.82)- (4.86) it follows that

$$\limsup_{s\to\infty}(sa_2'(s) + c_1a_2(s)^2) \le 0. \tag{4.87}$$

Finally, we conclude that (4.87) imply

$$\lim_{s\to\infty} a_2(s) = 0. \tag{4.88}$$

(1): If a_2 has a non-zero limit a^* ($a^* > 0$, because of Lemmas 4.4 and 4.16), then by (4.87) it holds that for every $\varepsilon > 0$ there exists a $s_0 \ge -\ln T$ and $C > 0$ such that $sa_2' \le -C$, as $s \ge s_0$. Integrating this, we obtain

$$a_2(s) - a_2(s_0) \le -C\ln(\frac{s}{s_0}) \to -\infty,$$

which is a contradiction to Lemmas 4.4 and 4.16 .

(2): If a_2 does not have a limit, then it it follows, by Lemmas 4.4 and 4.16, that there exists a sequence $s_j \to \infty$ such that $a_2'(s_j) \geq 0$, and $a_2(s_j) \geq \delta > 0$, which is a contradiction to (4.87).

Theorem 4.2(b) follows from (4.88) and Lemma 4.16. □

From Theorem 4.2 we can deduce Corollary 1.4: In the proof of Corollary 1.4 we can, by Lemma 3.2, replace the minimum point $x = 0$ of u by function $s_i(t)$, and note that the analysis in a neighborhood of quenching point is qualitatively similar as in the proof of Theorem 4.2.

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