Helsinki University of Technology Institute of Mathematics Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 1999 A405

RICCATI EQUATIONS FOR H^∞ DISCRETE TIME SYSTEMS: PART I

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Jarmo Malinen: Riccati Equations for H^∞ Discrete Time Systems: Part I; Helsinki University of Technology Institute of Mathematics Research Reports A405 (1999).

Abstract: *This is the first part of a two-part work [26], [27], on the self-adjoint solutions P of the discrete time algebraic Riccati operator equation (DARE), associated to the discrete time linear system (DLS) $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$*

$$\begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P, \\ \Lambda_P = D^*JD + B^*PB, \quad \Lambda_P K_P = -D^*JC - B^*PA, \end{cases}$$

where the indicator operator Λ_P is required to have a bounded inverse. We work under the standing hypothesis that the transfer function $\mathcal{D}_\phi(z) := D + zC(I - zA)^{-1}B$ belongs to $H^\infty(\mathbf{D}; \mathcal{L}(U, Y))$, in which case we call the Riccati equation H^∞DARE. We occasionally require the input operator B to be a compact Hilbert–Schmidt operator, and the DLS ϕ to be approximately controllable. To obtain most complete results, we require that J is nonnegative or the Popov operator $\mathcal{D}^*J\mathcal{D}$ satisfies $\mathcal{D}^*J\mathcal{D} \geq \epsilon\mathcal{I} > 0$.
The solutions of the DARE are classified, and the structure of various solution subsets is considered. To each solution, we associate two additional DLSs, namely the inner DLS ϕ_P and the spectral DLS ϕ^P. In particular, the subset $ric_0(\phi, J)$ of regular H^∞solutions is introduced and described.
To each $P \in ric_0(\phi, J)$, a spectral factorization of the Popov operator $\mathcal{D}^*J\mathcal{D}$ is associated; the spectral factor is the I/O-map of the spectral DLS ϕ_P, and its transfer function belongs to $H^\infty(\mathbf{D}; \mathcal{L}(U))$. Conversely, for certain such spectral factorizations of $\mathcal{D}^*J\mathcal{D}$, a corresponding $P \in ric_0(\phi, J)$ is recovered. The uniqueness of this correspondence is established. Inertia results for the indicator operators Λ_P are considered.

AMS subject classifications: 47A68, 47N70, 49J35, 93B28, 93C55.

Keywords: Spectral factorization, algebraic Riccati equation, discrete time, infinite-dimensional.

ISBN 951-22-4355-5
ISSN 0784-3143
Edita, Espoo, 1999

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1 Introduction

This is the first part of a two-part study on the input-output stable (I/O stable) discrete time linear system (DLS) $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and the associated algebraic Riccati equation (DARE)

$$(1)\qquad \begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P, \\ \Lambda_P = D^*JD + B^*PB, \\ \Lambda_P K_P = -D^*JC - B^*PA, \end{cases}$$

denoted, together with its solution set, by $Ric(\phi, J)$. The input, state and output spaces of the DLS ϕ are separable Hilbert spaces and possibly (but not necessarily) infinite dimensional. In this first part, the necessary technical machinery is developed. Then the (stable) spectral factorizations of the Popov operator $\mathcal{D}^*J\mathcal{D}$ are parameterized by the self-adjoint solutions of the DARE. Here $\mathcal{D}$ denotes the I/O map of the DLS, and J is a cost operator on the outputs of the DLS. Many results given here do not require the cost operator J or the solution of the DARE to be nonnegative.

The second part of this study is [27], where inner-outer type factorizations of the I/O map are considered. These results are valid only for nonnegative cost operators J and nonnegative solutions of the DARE. In fact, this work and [27] provide an order-theoretic characterization of the solution set of the infinite-dimensional operator DARE, in terms of spectral and inner-outer factorizations. We remark that this work and [27] are written in two parts merely for the reasons of page limitations. The section and equation numbers of the latter work [27] start where those of this work end. Also the reference lists of these papers are identical. The conference article [24] is a short presentation of the main lines of [26] and [27].

Let us briefly review some of our relevant previous work on the DLSs and their Riccati equations. In [21], the theory of (well posed) DLSs and their feedbacks is developed; much in the style and notation of the recent continuous time works by O. J. Staffans [36], [37], [38], [39], [40], [41], [42], [43], [44], and [45]. The work [20] contains discrete time minimax control theory, analogous to the early discrete time work [13] by J. W. Helton and paralleling the continuous time works of Staffans and also [54] by M. Weiss and G. Weiss. A considerable part of this work consists of a sharpening the results of [20] under more restrictive assumptions. The conference paper [19] is an abbreviated version [20]. Another direction (related to certain semigroup-invariant subspace problems) on the description of the nonnegative solutions of DARE is outlined in [25] and the associated conference paper [22].

1.1 Outline of the paper

We proceed to give an outline of this paper. In the preliminary Section 2 we extend the discussion of the fundamental notions of discrete time linear systems (DLSs), initiated in [21]. Some additional structure is introduced: transfer functions and their nontangential limit functions (boundary traces) in the Nevanlinna class $N(\mathcal{L}(U;Y))$ of analytic (transfer) functions. Some results from Banach space -valued integration theory and Fourier transform theory are reviewed.

By $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ denote an I/O stable DLS of interest, by $\mathcal{D}_\phi$ its I/O-map, and by $J = J^*$ a cost operator. The corresponding discrete time algebraic Riccati equation (DARE),

$$(2) \qquad \begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P, \\ \Lambda_P = D^*JD + B^*PB, \\ \Lambda_P K_P = -D^*JC - B^*PA, \end{cases}$$

denoted by $Ric(\phi, J)$, is introduced in Section 3. Even though the DARE $Ric(\phi, J)$ can be written for an arbitrary DLS ϕ, our interest mainly lies in the case when the DLS ϕ is I/O stable; i.e., the transfer function $\mathcal{D}_\phi(z)$ of ϕ satisfies $\mathcal{D}_\phi(z) := D + zC(I - zA)^{-1}B \in H^\infty(\mathcal{L}(U;Y))$. In this case, we call equation (2) an H^∞DARE, and write $ric(\phi, J)$ instead of $Ric(\phi, J)$. Furthermore, if $P \in \mathcal{L}(H)$ is a self-adjoint solution of $Ric(\phi, J)$ (or $ric(\phi, J)$), we write $P \in Ric(\phi, J)$. Thus $Ric(\phi, J)$ and $ric(\phi, J)$ represent both the equations itself and their solution sets.

To each $P \in Ric(\phi, J)$, we associate an indicator operator Λ_P and two additional DLSs: the spectral DLS ϕ_P and the inner DLS ϕ^P, centered at $P \in Ric(\phi, J)$ (see Definition 19). These three objects are central in this work. They appear in a natural way in the open and closed loops DLSs when certain state feedbacks (associated to P) are applied to ϕ, as will be seen later in [27, Section 9]. The solutions of the H^∞DARE $ric(\phi, J)$ are classified in Definition 20 according to the stability properties of the spectral DLS ϕ_P, and in Definition 21 according to their residual cost behavior "at infinite time". The smallest subset of solutions for H^∞DARE is denoted by $ric_0(\phi, J)$ — the set of regular H^∞ solutions $P \in ric_0(\phi, J)$. Our strongest results are given in this subset.

In Theorem 27 of Section 4 we prove the equivalence of

- the solvability of a minimax cost optimization problem associated to pair (ϕ, J),
- the solvability of a certain (spectral, inner-outer) factorization problem for the I/O-map $\mathcal{D}_\phi$, and
- the existence of a special (regular critical) solution P_0^{crit} of H^∞DARE $ric(\phi, J)$.

This result appears in a more general form in [19, Theorem 40], and is stated here only in the generality appropriate for this work. For similar results, see also [13], [29], [37], [39], and [54]. We remark that the existence of such a $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ is close to being a standing hypothesis in the present study. Well known sufficient conditions (relying on the nonnegativity of J or $\mathcal{D}_\phi^* J \mathcal{D}_\phi$) for this are given in Proposition 31 and Corollary 32.

In Section 5 we present some auxiliary results from the operator-valued function theory. A result of particular importance to us is Lemma 41, which allows us to deal with an infinite-dimensional input space U in [27], provided that the input operator $B \in \mathcal{L}(U; H)$ is restricted to be a compact Hilbert–Schmidt operator. This result has some application in the following Section 6.

Section 6 contains two spectral factorization results, namely Lemma 45 (the spectral factorization of truncated Toeplitz operators) and Proposition 46 (the spectral factorization of the Popov function $\mathcal{D}_\phi(e^{i\theta})^* J \mathcal{D}_\phi(e^{i\theta})$, constructed from the nontangential boundary trace of the H^2 transfer function $\mathcal{D}_\phi(e^{i\theta})$). Despite of this, our main interest here lies in the output stability and I/O stability question of the spectral DLS ϕ_P, for various solutions $P \in Ric(\phi, J)$. The output stability of ϕ_P is easier, and it is treated in Propoposition 43 by nonnegativity techniques. The I/O stability of ϕ_P is considered in Corollary 47 and the remarks following it. The section in concluded by Lemma 49, which is an inertia result for the indicator operators Λ_P, $P \in Ric(\phi, J)$ in a indefinite metric.

In Section 7, a spectral factorization of the Popov operator

$$\mathcal{D}_\phi^* J \mathcal{D}_\phi = \mathcal{D}_{\phi_P}^* \Lambda_P \mathcal{D}_{\phi_P} \tag{3}$$

is associated to each solution of the Riccati equation $P \in ric_0(\phi, J)$ satisfying a certain residual cost condition. We say that the operator $\mathcal{D}_{\phi_P}$ is a stable spectral factor of the Popov operator $\mathcal{D}_\phi^* J \mathcal{D}_\phi$. Also the converse it true: each such factorization induces a solution of the DARE $ric(\phi, J)$, if $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$ where $\mathcal{B}_\phi$ is the controllability map of ϕ. This is the content of Theorem 50, one of the main results of this paper. We remark that the factorization of the Popov operator does not necessarily require the cost operator J to be nonnegative, if we have an *a priori* knowledge that ϕ_P is output stable and I/O stable. For nonnegative cost, this follows as in previous Section 6, under the indicated technical assumption.

If $P = P_0^{\mathrm{crit}}$ is the regular critical solution in the sense of Theorem 27, then this factorization is the $\Lambda_{P_0^{\mathrm{crit}}}$-spectral factorization $\mathcal{D}_\phi^* J \mathcal{D}_\phi = \mathcal{X}^* \Lambda_{P_0^{\mathrm{crit}}} \mathcal{X}$, where the spectral factor $\mathcal{X}$ is stable and outer, with a bounded causal inverse $\mathcal{X}^{-1}$. This leads to the $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization of the I/O-map $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$ with $\mathcal{X} = \mathcal{D}_{\phi_0^{\mathrm{crit}}}$, see [20, Proposition 20]. We remark that if $P \neq P_0^{\mathrm{crit}}$, then we do not always obtain an analogous factorization of $\mathcal{D}_\phi$, as a composition of two *stable* I/O-maps. The circumstances when we get such stable factors, are considered in the second part of this work [27]. Inertia results, concerning the positivity of the indicator Λ_P for all $P \in ric_0(\phi, J)$, are given in Lemma 53 and Corollary 54. These are variants of Lemma 49, which depends on the restrictive assumption that the input operator $B \in \mathcal{L}(U;H)$ of ϕ is Hilbert–Schmidt.

In Proposition 55, the spectral factor $\mathcal{D}_{\phi_P}$ (i.e. the I/O-map of the spectral ϕ_P) appearing in equation (3) is $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorized as $\mathcal{D}_{\phi_P} = \mathcal{N}_P \mathcal{X}$, under the assumption that the original DARE $ric(\phi, J)$ has a regular critical solution P_0^{crit}. Quite expectedly, the outer part of $\mathcal{D}_{\phi_P}$ does not depend on the choice of the solution $P \in ric_{uw}(\phi, J)$. Realizations for the factors are computed. Section 7 is concluded with Proposition 56, where a realization algebra is considered for the inner factors $\mathcal{N}_P$. This lays foundation to the second part [27] of this work.

1.2 Connections to earlier works

We now briefly consider the appropriate references to earlier works by other authors. The general idea of using the (matrix) Riccati equations for the canonical and spectral factorization of rational transfer functions is quite old. Both the continuous and discrete time case is considered in [15, Chapters 10 and 19] (P. Lancaster and L. Rodman). At the end of both chapters, a short account for the history of such factorizations is given.

The discrete time result [13, Theorem 4.6] (J. W. Helton) is closely related to our Theorem 50 on the spectral factorization, but the information structure of the system and DARE is that of a LQDARE

$$(4) \qquad \begin{cases} A^*PA - P + C^*JC = A^*PB \cdot \Lambda_P^{-1} \cdot B^*PA, \\ \Lambda_P = D^*JD + B^*PB, \end{cases}$$

where the input is penalized by direct cost. The reasons why we discuss the more general DARE (2) instead of DARE (4) will be discussed in [27, Section 8]. In [13, Theorem 4.6], a "nonvanishing residual cost" has been included in the Popov function, whose spectral factor is to be calculated. A similar modification can be done to Theorem 50.

The related results in [10] (P. A. Fuhrmann; continuous time, infinite-dimensional) and [11] (P. A. Fuhrmann and J. Hoffman; discrete time, matrix-valued, a state space factorization of rational inner functions) seem to be most complete. A reference to an earlier work [7] (L. Finesso and G. Picci) is also given there. Unfortunately, there is a considerable overlap between our results and those given in [10] and [11]; we learned about these references at MTNS98 conference (Padova, July 1998), after the present work (in its original form) was completed. In style and basic assumptions these works are quite different from ours, which makes is a hard (but nevertheless a feasible) task to compare the (continuous time) results of [10] to our (discrete time) results. It appears that all the results are in harmony to each other in a beautiful way.

Fuhrmann approaches the general structure from the minimal spectral factorization point of view, rather that from the Riccati equation point of view that we have adopted. In [10], unstable systems and spectral factors are parameterized by solutions of a Riccati equation of a quite special kind. We can roughly say that our generality is in the Riccati equations and classes of stable systems, whereas more general spectral factors and unstable systems are considered in [10]. The work [10] is written under the standing hypothesis of strict noncyclicity of the spectral function (corresponding the Popov function in our work). In [10, Theorem 2.1], this assumption is associated to the existence of Douglas–Shapiro–Shields factorization of the spectral function, see [8] and [9].

We remark that many results such as [10, Theorem 6.1], (analogous to our Lemma 45, Proposition 46 and Theorem 50) are genuinely two-directional where our results are not. By this we mean that in Theorem 50, we do not prove that all spectral factors of $\mathcal{D}_\phi^* J \mathcal{D}_\phi$ can be associated to a solution of DARE. Only those spectral factors are parameterized by the solution in $ric_0(\phi, J)$ that can be realized in a particular way, with the original semigroup generator A and the input operator B of DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$. The full parameterization of spectral factors in [10] comes from the additional minimality assumption of the used realization, and the use of a state space isomorphism result that does not hold in the full generality in our setting. We return to this matters in a later work. It is true that the general lack of a state space isomorphism is quite dissappointing, and it makes the state space idea somewhat "too good to be true" for general infinite-dimensional systems, see [9, Chapter 3].

1.3 Notations

We use the following notations throughout the paper: $\mathbf{Z}$ is the set of integers. $\mathbf{Z}_+ := \{j \in \mathbf{Z} \quad | \quad j \geq 0\}$. $\mathbf{Z}_- := \{j \in \mathbf{Z} \quad | \quad j < 0\}$. $\mathbf{T}$ is the unit circle and $\mathbf{D}$ is the open unit disk of the complex plane $\mathbf{C}$. If H is a Hilbert space, then $\mathcal{L}(H)$ denotes the bounded and $\mathcal{LC}(H)$ the compact linear operators in H. Elements of a Hilbert space are denoted by upper case letters; for example $u \in U$. Sequences in Hilbert spaces are denoted by $\tilde{u} = \{u_i\}_{i \in I} \subset U$, where I is the index set. Usually $I = \mathbf{Z}$ or $I = \mathbf{Z}_+$. Given a Hilbert space Z, we define the sequence spaces

$$
\begin{aligned}
Seq(Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}} \mid z_i \in Z \quad \text{and} \quad \exists I \in \mathbf{Z} \quad \forall i \leq I : z_i = 0\big\},\\
Seq_+(Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}} \mid z_i \in Z \quad \text{and} \qquad \forall i < 0 : z_i = 0\big\},\\
Seq_-(Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}} \in Seq(Z) \mid z_i \in Z \quad \text{and} \qquad \forall i \geq 0 : z_i = 0\big\},\\
\ell^p(\mathbf{Z}; Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}} \subset Z \mid \sum_{i\in\mathbf{Z}} ||z_i||_Z^p < \infty\big\} \quad \text{for} \quad 1 \leq p < \infty,\\
\ell^p(\mathbf{Z}_+; Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}_+} \subset Z \mid \sum_{i\in\mathbf{Z}_+} ||z_i||_Z^p < \infty\big\} \quad \text{for} \quad 1 \leq p < \infty,\\
\ell^\infty(\mathbf{Z}; Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}} \subset Z \mid \sup_{i\in\mathbf{Z}} ||z_i||_Z < \infty\big\}.
\end{aligned}
$$

The following linear operators are defined for $\tilde{z} \in Seq(Z)$:

- the projections for $j, k \in \mathbf{Z} \cup \{\pm\infty\}$

$$
\begin{aligned}
&\pi_{[j,k]}\tilde{z} := \{w_j\}; \quad w_i = z_i \quad \text{for} \quad j \leq i \leq k, \quad w_i = 0 \quad \text{otherwise},\\
&\pi_j := \pi_{[j,j]}, \quad \pi_+ := \pi_{[1,\infty]}, \quad \pi_- := \pi_{[-\infty,-1]},\\
&\bar{\pi}_+ := \pi_0 + \pi_+, \quad \bar{\pi}_- := \pi_0 + \pi_-,
\end{aligned}
$$

- the bilateral forward time shift τ and its inverse, the backward time shift τ^*

$$
\begin{aligned}
\tau\tilde{u} &:= \{w_j\} \quad \text{where} \quad w_j = u_{j-1},\\
\tau^*\tilde{u} &:= \{w_j\} \quad \text{where} \quad w_j = u_{j+1}.
\end{aligned}
$$

Other notations are introduced when they are needed.

2 A crash course of DLSs

2.1 Notion of causality and shift-invariance

Our basic object is a fixed state space realization of a (well-posed) transfer function analytic in some neighborhood of the origin. We call this realization a *discrete time linear system* (DLS), given by a system of difference equations

$$\text{(5)} \qquad \begin{cases} x_{j+1} &= Ax_j + Bu_j, \\ y_j &= Cx_j + Du_j, \quad j \geq 0, \end{cases}$$

where $u_j \in U$, $x_j \in H$, $y_j \in Y$, and A, B, C and D are bounded linear operators between appropriate Hilbert spaces. We call the ordered quadruple $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ a *DLS in difference equation form.* The operators are as follows: the *semigroup generator* A, *input operator* B, *output operator* C and the *feed-through operator* D of ϕ. The three Hilbert spaces are as follows: U is the input space, H is the state space and Y is the output space of ϕ. It is well known that equations (5) are a state space model for a unique causal shift-invariant operator $\mathcal{D} = \mathcal{D}_\phi : Seq(U) \to Seq(Y)$, called the *I/O-map* of ϕ.

There is also another equivalent form for the same DLS, called *DLS in I/O-form* (see [21, Theorem 11]). It consists of four linear operators in the ordered quadruple

$$\text{(6)} \qquad \Phi := \begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}.$$

Note that ϕ stands for the DLS in difference equation form, and the capital Φ is the same DLS written in I/O-form. The operator $A \in \mathcal{L}(H)$ is called the *semigroup generator*, and the family $\{A^j\}_{j \geq 0}$ is called the *semigroup* of Φ. It is the same operator A that appears in the corresponding DLS ϕ in difference equation form. $\mathcal{B} : Seq_-(U) \to H$ is called the *controllability map* that maps the past input into present state. $\mathcal{C} : H \to Seq_+(Y)$ is called the *observability map* that maps the present state into future outputs. The operator $\mathcal{D} : Seq(U) \to Seq(Y)$ in (6) is the *I/O-map* of Φ that maps the input into output in a causal and shift-invariant way. The DLS Φ is called a *(state space) realization* of its I/O-map $\mathcal{D}$.

By using the bilateral shift operator τ defined on $Seq(U)$, a formula for the I/O-map can be given

$$\text{(7)} \qquad \mathcal{D}_\phi \tilde{u} = D\tilde{u} + \sum_{i \geq 0} CA^i B \tau^{i+1} \tilde{u}.$$

The above converges pointwise: for all $k \in Z$, $\tilde{u} \in Seq(U)$, we have only finitely many nonzero terms in the sum $\pi_k \left(\sum_{i \geq 0} CA^i B \tau^{i+1} \tilde{u} \right)$, by the definition of $Seq(U)$. We remark that the vector spaces $Seq(U)$, $Seq(Y)$ are now given the topology of componentwise (pointwise) convergence. Several DLSs can be realizations for the same I/O-map because formula (7) depends only on the operators $\{CA^iB\}_{i\geq 0}$, and not on the operators A, B and C separately. We remark that each *well-posed* causal shift-invariant operator of form (7) can be written as an I/O-map of a DLS, see [21, Lemma 8].

Consider again the two forms $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ and $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of the same DLS. The operators appearing in ϕ and Φ connected by straightforward algebraic relations (see [21, Lemma 7 and Definition 9]):

- $\mathcal{B} : Seq_-(U) \to H$, $\mathcal{C} : H \to Seq_+(Y)$ and $\mathcal{D} : Seq(U) \to Seq(Y)$.
- $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{C}$ are causal; i.e. they satisfy
$$\pi_-\mathcal{D}\bar{\pi}_+ = 0, \quad \mathcal{B}\bar{\pi}_+ = 0, \quad \pi_-\mathcal{C} = 0.$$
- $\mathcal{B}$ satisfies
$$\begin{aligned} &\mathcal{B}\tau^* = A\,\mathcal{B} + \mathcal{B}\tau^*\pi_0, \\ &\mathcal{B}\tau^{*j}\tilde{u} = A^j\,\mathcal{B}\tilde{u} + \sum_{i=0}^{j-1} A^iBu_{j-i-1}, \\ &B = \mathcal{B}\pi_{-1} \in \mathcal{L}(U, H), \end{aligned}$$
where U is identified with range$\,(\pi_{-1})$ on $Seq(U)$ in the natural way.
- $\mathcal{C}$ satisfies
$$\begin{aligned} &\bar{\pi}_+\tau^*\mathcal{C} = \mathcal{C}A, \\ &C = \pi_0\mathcal{C} \in \mathcal{L}(H, Y), \end{aligned}$$
where Y is identified with range$\,(\pi_0)$ on $Seq(Y)$ in the natural way.
- $\mathcal{D}$ satisfies
$$\begin{aligned} &\bar{\pi}_+\mathcal{D}\pi_- = \mathcal{C}\mathcal{B}, \\ &\mathcal{D}\tau = \tau\mathcal{D}, \quad \mathcal{D}\tau^* = \tau^*\mathcal{D} \\ &D = \pi_0\mathcal{D}\pi_0 \in \mathcal{L}(U, Y), \end{aligned}$$
where U, Y are identified with range$\,(\pi_0)$ in the natural way.

For the input, output and state sequences the following notation is used:

- The state of ϕ at time $j \geq 0$ is denoted by $x_j(x_0, \tilde{u})$, and it is defined by
$$x_j(x_0, \tilde{u}) := A^jx_0 + \sum_{i=0}^{j-1} A^iBu_{j-i} = A^jx_0 + \mathcal{B}_\phi\tau^{*j}\tilde{u}. \tag{8}$$
- The output sequence $\tilde{y}(x_0, \tilde{u}) := \{y_j(x_0, \tilde{u})\}_{j\in\mathbf{Z}_+}$ of ϕ is defined by
$$y_j(x_0, \tilde{u}) := CA^jx_0 + \sum_{i=0}^{j-1} CA^iBu_{j-i} + Du_j = \pi_j(\mathcal{C}_\phi x_0 + \mathcal{D}_\phi\tilde{u}), \tag{9}$$

where $x_0 \in H$ denotes the initial state at time $j = 0$, and $\tilde{u} \in Seq_+(U)$ is an input sequence. We remark that the verification of these relations essentially gives the correspondence between DLSs in difference equation form and in I/O-form.

Definition 1. *Let $\phi_1 = \left(\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right)$, $\phi_2 = \left(\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix}\right)$ be two DLSs. Assume that the input space of ϕ_2 is U, the output space of ϕ_2 and the input space of ϕ_1 is W, and the output space of ϕ_1 is Y.*

(i) If $D_1^{-1} \in \mathcal{L}(Y;U)$ exists, then define

$$\phi_1^{-1} = \begin{pmatrix} A_1 - B_1 D_1^{-1} C_1 & B_1 D_1^{-1} \\ -D_1^{-1} C_1 & D_1^{-1} \end{pmatrix}$$

This DLS is called the inverse DLS of ϕ_1.

(ii) Define

$$\phi_1 \phi_2 = \left(\begin{matrix} \begin{bmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{bmatrix} & \begin{bmatrix} B_1 D_2 \\ B_2 \end{bmatrix} \\ \begin{bmatrix} C_1 & D_1 C_2 \end{bmatrix} & D_1 D_2 \end{matrix} \right),$$

provided ϕ_1 and ϕ_2 as such that all the proposed operator compositions are sensible. This DLS is called the product DLS of ϕ_1 and ϕ_2.

(iii) Define

$$\widetilde{\phi_1} = \begin{pmatrix} A_1^* & C_1^* \\ B_1^* & D_1^* \end{pmatrix}.$$

This DLS is called the adjoint DLS of ϕ_1.

Proposition 2. *Let ϕ_1, ϕ_2 be as in Definition 1.*

(i) $\mathcal{D}_{\phi_1} : Seq(W) \to Seq(Y)$ is invertible and its inverse is a I/O-maps of a DLS if and only if $D_1^{-1} \in \mathcal{L}(Y;W)$ exists. In this case, the inverse $\mathcal{D}_{\phi_1}^{-1} : Seq(Y) \to Seq(W)$ is given by $\mathcal{D}_{\phi_1}^{-1} = \mathcal{D}_{\phi_1^{-1}}$.

(ii) The composition of the I/O-maps $\mathcal{D}_{\phi_1}$ and $\mathcal{D}_{\phi_2}$ satisfies $\mathcal{D}_{\phi_1} \mathcal{D}_{\phi_2} = \mathcal{D}_{\phi_1 \phi_2}$.

(iii) The adjoint DLSs satisfy $\widetilde{(\widetilde{\phi_1})} = \phi_1$, and $\widetilde{(\phi_1^{-1})} = (\widetilde{\phi_1})^{-1}$. Furthermore, $\mathcal{D}_{\widetilde{\phi_1 \phi_2}} = \mathcal{D}_{\widetilde{\phi_2} \widetilde{\phi_1}}$.

Proof. Consider first the "if" part of claim (i). Assume $\tilde{y} \in Seq(Y)$, $\tilde{u} \in Seq(W)$ satisfy $\tilde{y} = \mathcal{D}_{\phi_1}\tilde{u}$, such that D_1^{-1} is bounded. Then

$$
\begin{aligned}
&\begin{cases} x_{j+1} = A_1 x_j + B_1 u_j, \\ y_j = C_1 x_j + D_1 u_j, \end{cases} \qquad \text{for all} \quad j \geq 0, \\
\Leftrightarrow \quad &\begin{cases} x_{j+1} = A_1 x_j + B_1 u_j, \\ u_j = -D_1^{-1} C_1 x_j + D_1^{-1} y_j, \end{cases} \qquad \text{for all} \quad j \geq 0, \\
\Leftrightarrow \quad &\begin{cases} x_{j+1} = (A_1 - B_1 D_1^{-1} C_1) x_j + B_1 D_1^{-1} u_j, \\ u_j = -D_1^{-1} C_1 x_j + D_1^{-1} y_j, \end{cases} \qquad \text{for all} \quad j \geq 0, \\
\Leftrightarrow \quad &\tilde{u} = \mathcal{D}_{\phi_1^{-1}} \tilde{y}
\end{aligned}
$$

where the initial value is $x_J = 0$ for so large J that both $\tilde{u}$, $\tilde{y}$ have no nonzero components with index less than J. This gives $\mathcal{D}_{\phi_1^{-1}}\mathcal{D}_{\phi_1} = I$ on $Seq(U)$. By using $(\phi_1^{-1})^{-1} = \phi_1$, also $\mathcal{D}_{\phi_1}\mathcal{D}_{\phi_1^{-1}} = I$. So $\mathcal{D}_{\phi_1^{-1}}$ is a two-sided inverse of $\mathcal{D}_{\phi_1}$.

To prove the "only if" part of claim (i), assume that $\mathcal{D}_{\phi_1}^{-1}$ is an I/O-map of some DLS ϕ'. Then, because $\mathcal{I} = \mathcal{D}_{\phi_1}^{-1}\mathcal{D}_{\phi_1} = \mathcal{D}_{\phi_1}\mathcal{D}_{\phi_1}^{-1}$, we have $\pi_0 = \pi_0\mathcal{D}_{\phi_1}^{-1}\mathcal{D}_{\phi_1}\pi_0 = \pi_0\mathcal{D}_{\phi_1}^{-1}\pi_0 \cdot \pi_0\mathcal{D}_{\phi_1}\pi_0$, by causality of both $\mathcal{D}_{\phi_1}^{-1}$ and $\mathcal{D}_{\phi_1}$. Now, $\pi_0\mathcal{D}_{\phi_1}\pi_0 = D$, and $I = D'D$, where $D' = \pi_0\mathcal{D}_{\phi_1}^{-1}\pi_0$. Similarly, $I = DD'$. It follows that D is a bounded bijection between Hilbert spaces U, Y. It thus has a bounded inverse $D^{-1} = D'$. This completes the proof of claim (i).

For the second claim (ii), recall formula (7) for the I/O-map of a DLS. Use this to obtain a formula for $\mathcal{D}_{\phi_1}\mathcal{D}_{\phi_2}$

$$
\mathcal{D}_{\phi_1}\mathcal{D}_{\phi_2} = D_1 D_2 + \sum_{k\geq 1} T_k \tau^k, \tag{10}
$$

where

(11)

$$
\begin{aligned}
T_k &:= D_1 C_2 A_2^{k-1} B_2 + C_1 A_1^{k-1} B_1 D_2 + \sum_{j=1}^{k-1} C_1 A_1^{j-1} B_1 C_2 A_2^{k-j-1} B_2, \quad k \geq 2 \\
T_1 &:= D_1 C_2 B_2 + C_1 B_1 D_2
\end{aligned}
$$

and all $T_k \in \mathcal{L}(U;Y)$. Sum (10) converges in the same sense as formula (7). We then calculate the similar formula for the I/O-map of the DLS $\phi_1\phi_2$. For this end, note that the powers of an upper triangular (block) matrix can be calculated by

$$
\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a^k & \sum_{j=0}^{k-1} a^j\, c\, b^{k-j-1} \\ 0 & b^k \end{bmatrix}, \quad k \geq 1.
$$

An application of this gives for $k \geq 1$

$$
\begin{aligned}
C_{\phi_1\phi_2} A^j_{\phi_1\phi_2} B_{\phi_1\phi_2} &= \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix} \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}^k \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix} \\
&= \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix} \begin{bmatrix} A_1^k & \sum_{j=0}^{k-1} A_1^j\, B_1C_2\, A_2^{j-i-1} \\ 0 & A_2^k \end{bmatrix} \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix} \\
&= C_1A_1^kB_1D_2 + D_1C_2A_2^kB_2 + C_1 \left(\sum_{j=0}^{k-1} A_1^j\, B_1C_2\, A_2^{j-i-1} \right) B_1.
\end{aligned}
$$

But this equals T_{k+1} of equation (11). The case $k = 0$ gives

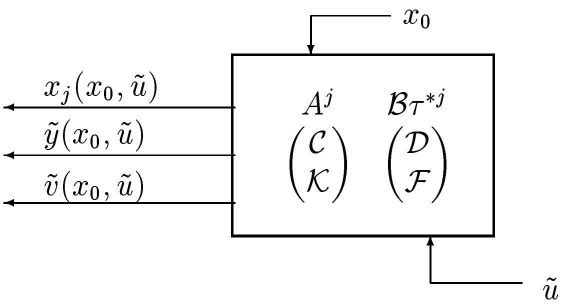
$$
\begin{bmatrix} C_1 & D_1C_2 \end{bmatrix} \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix} = C_1B_1D_2 + D_1C_2B_2 = T_1.
$$

Because also the static parts of $\mathcal{D}_{\phi_1}\mathcal{D}_{\phi_2}$ and $\mathcal{D}_{\phi_1\phi_2}$ are both D_1D_2, equations (10) and (11) give also the I/O-map of $\phi_1\phi_2$. The last claim (iii) is immediate. This completes the proof. ☐

We remark the there is no uniqueness in the realization $\phi_1\phi_2$ for the causal shift-invariant operator $\mathcal{D}_{\phi_1}\mathcal{D}_{\phi_2}$. Furthermore, generally $\widetilde{\phi_1\phi_2} \neq \widetilde{\phi_2\phi_1}$ but the state spaces of these product DLSs are unitarily isomorphic. Given an I/O-map $\mathcal{D}_\phi$, its adjoint I/O-map $\widetilde{\mathcal{D}_\phi}$ is defined by $\widetilde{\mathcal{D}_\phi} := \mathcal{D}_{\tilde{\phi}}$. It is easy to show, by using formula (7), that $\widetilde{\mathcal{D}_\phi}$ is independent of the choise of the realization ϕ.

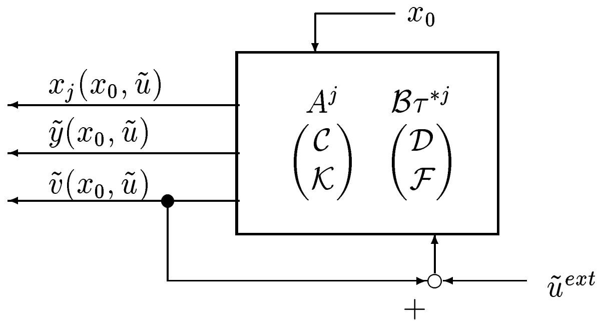
2.2 Notion of feedback

The notion of *state feedback* is central in this work. In difference equation form, we realize the state feedback by first adding still another equation $v_j = Kx_j + Fu_j$ to equations (5), where $K \in \mathcal{L}(U)$. This gives us an *extended DLS* ϕ^{ext}. We get the *closed loop DLS* $\phi^{ext}_\diamond$ in difference equation form by simple manipulation. The same structure written in I/O-form is somewhat more complicated but nevertheless useful. In I/O-form, the new output signal given by K provides a new output $\ell^2(\mathbf{Z}_+; U) \ni \tilde{v} = \mathcal{K}x_0 + \mathcal{F}\tilde{u}$ to Φ, thus giving an *(open loop) extended DLS* $\Phi^{ext} := [\Phi, [\mathcal{K}, \mathcal{F}]]$. This is a Cartesian product of two DLSs with the same input and semigroup structure, as presented in the following picture:



The ordered pair of operators $[\mathcal{K}, \mathcal{F}]$ is called a *feedback pair* of Φ. Here $\mathcal{K}$ is a valid observability map and $\mathcal{F}$ is a valid I/O-map for the system with semigroup generator A and controllability map $\mathcal{B}$

The operator $(\mathcal{I} - \mathcal{F})^{-1} : Seq(U) \to Seq(U)$ is required to be an I/O-map of a DLS (well-posed, causal and shift invariant), to ensure that the closed loop is well defined. From an *I/O stable feedback pair* we require that $\mathrm{dom}\,(\mathcal{C}) \subset \mathrm{dom}\,(\mathcal{K})$, and both $\mathcal{F}$ and $(\mathcal{I} - \mathcal{F})^{-1}$ are bounded in the ℓ^2-topology. If, in addition, $\mathcal{K} : H \to \ell^2(\mathbf{Z}_+; U)$ is bounded, then we say that $[\mathcal{K}, \mathcal{F}]$ is stable. The *closed loop extended DLS* $\Phi_\diamond^{ext}$ is the DLS that we obtain when we close the following state feedback connection:



Here $\tilde{u}^{ext} = \{u_j^{ext}\}$ is an external (disturbance) signal, so that $v_j = K_j + F(v_j + u_j^{ext})$ holds in the closed loop. The formulae for the closed loop system in terms of the open loop operators can be easily calculated (see [21, Definition 18]). Thus we have two different notions of state feedback; one for DLSs in difference equation form, the other for DLSs in I/O-form. It follows that these feedback notions are equivalent in the same way than the two notions of the DLS are equivalent (see [21, Section 5]). The stability properties of the open and closed loop feedback systems are discussed in [21, Section 9].

2.3 Notion of energy

As proposed earlier, the sequence spaces $Seq(Z)$, for Z Hilbert space, can be given the topology of componentwise convergence. This is a rather weak topology; so weak that it does not admit a useful energy notion. To fix this, we introduce a smaller vector subspace $\ell^2(\mathbf{Z}_+; Z) \subset Seq(Z)$, which is a Hilbert space of square summable sequences. The norm of $\ell^2(\mathbf{Z}_+; Z)$ is regarded as the energy of the signal. A DLS is called *I/O stable* if the *Toeplitz operator of the I/O-map* $\mathcal{D}\bar{\pi}_+ : \ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; Y)$ is bounded. Then the Toeplitz operator has a unique bounded extension, by continuity and shift invariance, to the whole of $\ell^2(\mathbf{Z}; U)$. This extension is also denoted by $\mathcal{D}$.

For the study of the operators $\mathcal{B}$ and $\mathcal{C}$, a suitable definition is needed for their domains ([21, Definition 24]). We define $\mathrm{dom}\,(\mathcal{B}) := Seq_-(U)$, equipped with the $\ell^2(\mathbf{Z}; U)$-inner product. If $\overline{\mathrm{dom}\,(\mathcal{B})} = H$, we say that Φ is *(infinite* *time) approximately controllable.* The domain of $\mathcal{C}$ is given by

$$\operatorname{dom}(\mathcal{C}) := \{x_0 \in H \mid \mathcal{C}x_0 \in \ell^2(\mathbf{Z}_+;Y)\}, \tag{12}$$

equipped with the inner product topology of H. Neither of the operators $\mathcal{B}, \mathcal{C}$ are assumed to be bounded in their domains. The domains for the operators $\mathcal{D}\bar{\pi}_+$ and $\mathcal{D}\pi_0$ (*the impulse response operator*) are defined similarly (see [21, Definition 24]). This makes all the operators $\mathcal{D}\bar{\pi}_+$, $\mathcal{D}\pi_0$ and $\mathcal{C}$ closed (see [21, Lemma 27]). The case for the observability map is given below:

Lemma 3. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a DLS. Define the domain of the observability map as above. Then* $\mathcal{C} : \operatorname{dom}(\mathcal{C}) \to \ell^2(\mathbf{Z}_+;Y)$ *is closed.*

Proof. Let $\operatorname{dom}(\mathcal{C}) \ni x_j \to x_0 \in H$ be a convergent sequence in H such that

$$\mathcal{C}x_j \to \tilde{y} \in \ell^2(\mathbf{Z}_+;Y)$$

in the norm of $\ell^2(\mathbf{Z}_+;Y)$. We shall show that $x_0 \in \operatorname{dom}(\mathcal{C})$ and $\mathcal{C}x_0 = \tilde{y}$, which proves the closed graph property for $\mathcal{C}$. For each $k \geq 0$ we have

$$(\mathcal{C}x_j)_k = CA^k x_j \to CA^k x_0 \quad \text{as} \quad j \to \infty \tag{13}$$

in the norm of Y, because both A and C are bounded. On the other hand, because $\mathcal{C}x_j \to \tilde{y}$ in the norm of $\ell^2(\mathbf{Z}_+;Y)$, for all $k \geq 0$:

$$(\mathcal{C}x_j)_k \to y_k \quad \text{as} \quad j \to \infty \tag{14}$$

in the norm of Y. Now equations (13) and (14) imply, by the uniqueness of the limit, that $CA^k x_0 = y_k$ for all $k \geq 0$, or equivalently $\tilde{y} = \mathcal{C}x_0$ for the algebraic observability map. But then, because $\tilde{y} \in \ell^2(\mathbf{Z}_+;Y)$, we have $x_0 \in \operatorname{dom}(\mathcal{C})$ and $\tilde{y} = \mathcal{C}x_0$. This completes the proof of this lemma. □

If $\mathcal{B}$ or $\mathcal{C}$ is bounded, we say that Φ is *input stable* or *output stable*, respectively. There is one more important stability condition used in this paper, namely the strong H^2-stability. We say that ϕ is strongly H^2 stable if its impulse response operator $\mathcal{D}_\phi\pi_0$ is bounded from $U = \operatorname{range}(\pi_0)$ to $\ell^2(\mathbf{Z}_+;Y)$. This implies that $\mathcal{D}_\phi\bar{\pi}_+ : \ell^1(\mathbf{Z}_+;U) \to \ell^2(\mathbf{Z}_+;Y)$ boundedly. I/O stability and output stability are sufficient conditions for the strong H^2-stability. We note that the strong H^2-stability is characterized by the following proposition

Proposition 4. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be a DLS. Then the following are equivalent:*

(i) $BU \subset \operatorname{dom}(\mathcal{C})$,

(ii) $\operatorname{range}(\mathcal{B}) \subset \operatorname{dom}(\mathcal{C})$,

(iii) Φ *is strongly* H^2 *stable.*

Proof. The equivalence of (i) and (ii) is [21, Lemma 39]. Assume (i). Because $BU \subset \mathrm{dom}(\mathcal{C})$, then $\mathcal{C}Bu \in \ell^2(\mathbf{Z}_+;Y)$ for all $u \in U$. But then $\mathcal{D}\pi_0\tilde{u} = D\pi_0\tilde{u} + \tau\mathcal{C}Bu_0 \in \ell^2(\mathbf{Z}_+;Y)$ for all $\tilde{u} = \{u_j\}_{j\geq 0}$ such that $u_0 = 0$. Because $u \in U$ is arbitrary, it follows that $\mathrm{dom}(\mathcal{D}\pi_0) = U$. But now $\mathcal{D}\pi_0 : U \to \ell^2(\mathbf{Z}_+;Y)$ maps boundedly by the Closed Graph Theorem, because $\mathcal{D}\pi_0$ is closed with a complete domain. Claim (iii) follows. The implication (iii) $\Rightarrow$ (i) is from [21, Lemma 40]. □

The inclusion $\mathrm{range}(\mathcal{B}) \subset \mathrm{dom}(\mathcal{C})$ is known as the *compatibility condition* in [21, Lemma 39]). Unfortunately, the description of the strong H^2 stability remained incomplete there.

For an I/O stable (and even strongly H^2 stable) DLSs, we can restrict the state space H to $\overline{\mathrm{dom}(\mathcal{C})}$ without essentially changing the I/O-properties, by Proposition 4. The observability map $\mathcal{C}|\overline{\mathrm{dom}(\mathcal{C})}$ of the restricted state space DLS is now densely defined and closed. For this reason, we assume throughout this paper that $\overline{\mathrm{dom}(\mathcal{C})} = H$. By introducing the graph norm of $\mathcal{C}$, we can make the vector space $\mathrm{dom}(\mathcal{C})$ into a Hilbert space. In fact, the strenghtening the topology of the state space in this manner gives us another, output stable DLS whose properties are roughly the same as those of the original DLS, see [21, Theorem 46]. We conclude that any non-output stable, I/O stable DLS ϕ can always be made output stable, and the lack of output stability tells us that the state space of ϕ is "too large" or "inconveniently normed". For this reason, we do not regard it as a grave sin to assume that our I/O stable DLSs are, in addition, output stable — provided that one such fixed topology of the state space H of ϕ is "good" for the full description of the essential structure of ϕ, seen as a realixation of its I/O-map. Fortunately, this is the case, as the present papers [26] and [27] show.

The stability notions associated to the semigroup generator A of the DLS Φ are the following (see [21, Definition 21]): A is power (or exponentially) stable, if $\rho(A) < 1$; A is strongly stable, if $A^j x_0 \to 0$ as $j \to \infty$; A is power bounded, if $\sup_{j\geq 0} ||A^j||_H < \infty$. All these imply that $\rho(A) \leq 1$. The power bounded behaviour is discussed in [31] and the references therein. So as to the strongly stable operators with a ℓ^p growth bound, [48] is an interesting reference.

We say that Φ is *stable* if it is I/O stable, input stable, output stable and its A semigroup generator is power bounded. If Φ is stable and A is strongly stable, then Φ is *strongly stable.* The relations between various stability conditions of open and closed loops systems are discussed in [21, Section 6].

The following notions are from [20, Definition 17]:

Definition 5. *Let $J \in \mathcal{L}(Y)$ be self-adjoint, and let $S \in \mathcal{L}(U)$ self-adjoint and invertible. Let $\mathcal{D}$, $\mathcal{N}$ and $\mathcal{X}$ be I/O-maps of I/O stable DLSs.*

(i) The operator $E \in \mathcal{L}(U)$ is S-unitary, if it is boundedly invertible and $E^*SE = S$.

(ii) $\mathcal{N} \in \mathcal{L}(\ell^2(\mathbf{Z};U), \ell^2(\mathbf{Z};Y))$ is (J,S)-inner, if $\mathcal{N}^*J\mathcal{N} = S$.

(iii) *$\mathcal{X} \in \mathcal{L}(\ell^2(\mathbf{Z}; U))$ is outer, if* $\overline{\text{range}\,(\mathcal{X}\bar{\pi}_+)} = \ell^2(\mathbf{Z}_+; U)$.

(iv) *$\mathcal{X} \in \mathcal{L}(\ell^2(\mathbf{Z}_+; U))$ is S-spectral factor of $\mathcal{D}^* J \mathcal{D}$, if $\mathcal{X}$ has a bounded causal shift invariant inverse $\mathcal{X}^{-1} \in \mathcal{L}(\ell^2(\mathbf{Z}; U))$ and $\mathcal{D}^* J \mathcal{D} = \mathcal{X}^* S \mathcal{X}$.*

Spectral factors of $\mathcal{D}^* J \mathcal{D}$ and *coercive* outer factors of $\mathcal{D}$ have one-to-one correspondence, by [20, Proposition 20].

2.4 Notion of the transfer function

For the I/O-map $\mathcal{D}$ of a DLS $\Phi = \phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, formula (7) is given. The bilateral shift operator can be formally replaced by a complex variable z, and the formal sum is obtained:

$$D + \sum_{i \geq 0} C A^i B z^{i+1}. \tag{15}$$

Because A is bounded by the definition of the DLS, this sum converges for $|z| < ||A^{-1}||^{-1}$, thus defining an analytic $\mathcal{L}(U; Y)$-valued function $\mathcal{D}(z)$ in a neighborhood of the origin. In fact, $\mathcal{D}(z) = D + zC(I - zA)^{-1}$ for $|z| < ||A^{-1}||^{-1}$. The analytic function $\mathcal{D}(z)$ is called the *transfer function* of Φ. Because all I/O-maps of DLSs have transfer functions analytic in a neighborhood of origin, we say that the the DLS is a *well-posed* linear system. The well-posedness makes it possible to add and multiply two transfer functions of appropriate type in a common neighborhood of the origin where both are analytic. We remark that the corresponding continuous time notion of well-posedness is deeper, see [45]. Because the power series coefficient (centered at the origin) of an analytic function are unique, we have one-to-one correspondence between the I/O-maps of DLSs and operator-valued functions, analytic in a neighborhood of the origin of the complex plane, see [21, Lemma 8].

In the following definition, we consider signals instead of systems.

Definition 6. *Let Z be a Hilbert space.*

(i) *The sequence $\tilde{u} = \{u_j\}_{j \in \mathbf{Z}_+} \in Seq_+(Z)$ is well posed, if the power series*

$$\tilde{u}(z) := \sum_{j=0}^{\infty} u_j z^j$$

converge to an analytic function in some neighborhood of the origin.

(ii) *The mapping $\mathcal{F}_z : \tilde{u} \mapsto \tilde{u}(z)$ is called the z-transform.*

The set $WSeq_+(Z)$ of well-posed sequences is a vector subspace of $Seq_+(Z)$. It is a matter of taste whether z-transform should be defined to be analytic in a neighborhood of the origin or of the infinity. It seems that in the function theory the former alternative is used, and in the control theory the latter is preferred.

Proposition 7. *Let $\mathcal{D}$ be an I/O-map of a DLS, and $\mathcal{D}(z)$ its transfer function. Let $\tilde{u} \in WSeq_+(U)$ and $\tilde{u}(z)$ its z-transform. Let $\tilde{y} \in Seq_+(Y)$. Then the following are equivalent:*

(i) $\tilde{y} = \mathcal{D}\tilde{u}$

(ii) $\tilde{y}$ *is well-posed, and* $\tilde{y}(z) = \mathcal{D}(z)\tilde{u}(z)$ *in some neighborhood of the origin.*

Proof. Assume claim (i). Because both $\mathcal{D}(z)$ and $\tilde{u}(z)$ are analytic in a some common neighborhood of the origin, so is the Y-valued function $f(z) := \mathcal{D}(z)\tilde{u}(z)$. Identify the unilateral shift τ by the multiplication by the complex variable z. By comparing the expression of both $\mathcal{D}\tilde{u}$ and $\mathcal{D}(z)\tilde{u}(z)$ it is clear that the power series coefficients f_j of f equal y_j. So $\tilde{y} \in WSeq_+(Y)$ is well posed and (ii) follows. The converse direction is similar. □

Corollary 8. *Let ϕ_1 and ϕ_2 be DLSs with compatible input and output spaces. Then $\mathcal{D}_{\phi_1\phi_2}(z) = (\mathcal{D}_{\phi_1}\mathcal{D}_{\phi_2})(z) = \mathcal{D}_{\phi_1}(z)\mathcal{D}_{\phi_2}(z)$.*

Proof. Let $\tilde{u} \in WSeq(U)$ be arbitrary. Then $\mathcal{D}_{\phi_1\phi_2}\tilde{u}$ and $\mathcal{D}_{\phi_2}\tilde{u}$ are well posed by Proposition 7, and

$$\begin{aligned}&(\mathcal{D}_{\phi_1\phi_2}\tilde{u})(z) = \mathcal{D}_{\phi_1\phi_2}(z)\tilde{u}(z) = (\mathcal{D}_{\phi_1}(\mathcal{D}_{\phi_2}\tilde{u}))(z)\\ &= \mathcal{D}_{\phi_1}(z)(\mathcal{D}_{\phi_2}\tilde{u})(z) = \mathcal{D}_{\phi_1}(z)\mathcal{D}_{\phi_2}(z)\tilde{u}(z),\end{aligned}$$

where all the equalities are by Proposition 7, except the second which is by claim (ii) of Proposition 2. Because $\tilde{u}$ is arbitrary, the claim follows. □

We conclude that the algebraic structure of corresponding I/O-maps and transfer functions is equivalent, when the well-posed inputs are considered. In particular, because I/O-map is known if its action on sequences $\tilde{u} \in WSeq(U)$ satisfying $\pi_j\tilde{u} = 0$ for $j \neq 0$, no uniqueness problems can arise if we restrict to well posed inputs. We have $\ell^2(\mathbf{Z}_+; U) \subset WSeq_+(U)$. This is trivially true because $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ is a bounded sequence, and thus the power series $\sum_{j\geq 0} u_j z^j$ converge for all $z \in \mathbf{D}$ by a simple argument.

At this point, it is necessary to introduce the Hardy spaces $H^p(\mathbf{D}; \mathcal{L}(U; Y))$ (operator-valued) and $H^p(\mathbf{D}; U)$ (Hilbert space -valued) for each $1 \leq p < \infty$. They are defined as the Banach spaces of analytic functions in $\mathbf{D}$ with the norms

$$||\mathcal{D}(z)||^p_{H^p(\mathbf{D};\mathcal{L}(U;Y))} := \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} ||\mathcal{D}(re^{i\theta})||^p_{\mathcal{L}(U;Y)}\, d\theta,$$

$$||\tilde{u}(z)||^p_{H^p(\mathbf{D};U))} := \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} ||\tilde{u}(re^{i\theta})||^p_U\, d\theta.$$

Out of these, the cases $p = 2$ are most important to us. The space $H^2(\mathbf{D}; U)$ is Hilbert, with the inner product

$$\langle \tilde{u}(z), \tilde{v}(z) \rangle_{H^2(\mathbf{D};U)} = \lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} \langle \tilde{u}(re^{i\theta}), \tilde{v}(re^{i\theta}) \rangle_U \, d\theta$$

and the Parseval identity

$$(16) \qquad \langle \tilde{u}(z), \tilde{v}(z) \rangle_{H^2(\mathbf{D};U)} = \langle \tilde{u}, \tilde{v} \rangle_{\ell^2(\mathbf{Z}_+;U)} .$$

The interpretation of equation (16) is that the z-transform $\mathcal{F}_z : \tilde{u} \mapsto \tilde{u}(z)$ is an isometric isomorphism of the Hilbert spaces $\ell^2(\mathbf{Z}_+; U)$ and $H^2(\mathbf{D}; U)$. For further information, see [33, Section 1.15] and [14, Chapter III].

Now that we have identified the z-transforms of finite energy signals, we identify the transfer functions of I/O stable DLSs. For this end, we meet one more Hardy space, namely the celebrated $H^\infty(\mathbf{D}; \mathcal{L}(U; Y))$. We say that $\mathcal{D}(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U; Y))$ if it is $\mathcal{L}(U; Y)$-valued analytic function in the whole of $\mathbf{D}$, and

$$||\mathcal{D}(z)||_{H^\infty(\mathbf{D};\mathcal{L}(U;Y))} := \sup_{z \in D} ||\mathcal{D}(z)||_{\mathcal{L}(U;Y)} < \infty.$$

Proposition 9. *Let $\mathcal{D}$ be a I/O-map of a DLS, such that all the Hilbert spaces U, H and Y are separable. Then $\mathcal{D}$ is I/O stable if and only if $\mathcal{D}(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U; Y))$. Furthermore, $||\mathcal{D}(z)||_{H^\infty(\mathbf{D};\mathcal{L}(U;Y))} = ||\mathcal{D}||_{\ell^2(\mathbf{Z}_+;U) \mapsto \ell^2(\mathbf{Z}_+;Y)}$.*

Proof. This is the contents of [33, Theorem 1.15B]), or [8, Theorem 1.1, Section IX, p. 235], to mention few possible references. In [33], the input and output spaces are written to be the same space. However, by using the Cartesian product Hilbert space $W = U \times Y$ as both input and output space, and extending the operators $T \in \mathcal{L}(U; Y)$ to $T' = \left(\begin{smallmatrix} 0 & 0 \\ T & 0 \end{smallmatrix}\right)$, the notational inconvenience is resolved. □

For the representation of bounded causal shift-invariant operators by H^∞functions, see also [47] and [49]. Related to the operator-valued $H^2(\mathbf{D}; \mathcal{L}(U, Y))$-space, another less known variant, called the strong $H^2(\mathbf{D}; \mathcal{L}(U, Y))$ is defined as follows:

Definition 10. *The strong $H^2(\mathbf{D}; \mathcal{L}(U, Y))$ (briefly: $\mathrm{sH}^2(\mathbf{D}; \mathcal{L}(U; Y))$) is the set of $\mathcal{L}(U; Y)$-valued analytic functions $\mathcal{D}(z)$ in $\mathbf{D}$, such that $\mathcal{D}(z)u_0 \in H^2(\mathbf{D}; Y)$, for all $u_0 \in U$.*

Clearly $\mathrm{sH}^2(\mathbf{D}; \mathcal{L}(U; Y))$ is a vector space. The following proposition gives a hint why $\mathrm{sH}^2(\mathbf{D}; \mathcal{L}(U; Y))$ is important to us.

Proposition 11. *If the DLS $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is output stable or, more generally, strongly H^2 stable, then the transfer function $\mathcal{D}_\phi(z) \in \mathrm{sH}^2(\mathbf{D}; \mathcal{L}(U; Y))$.*

Proof. We first show that the transfer function $\mathcal{D}_\phi(z)$ is analytic in the whole of $\mathbf{D}$. If ϕ is output stable, then

$$||\mathcal{C}_\phi x_0||^2_{\ell^2(\mathbf{Z}_+;Y)} = ||\{CA^j x_0\}_{j\geq 0}||^2_{\ell^2(\mathbf{Z}_+;Y)} = \sum_{j\geq 0} ||CA^j x_0||^2_Y < \infty$$

for all $x_0 \in H$. Then, in particular $\sup_{j\geq 0} ||CA^j x_0||_Y < \infty$ for all $x_0 \in H$. Now Banach–Steinhaus Theorem implies that the family $\{CA^j\}_{j\geq 0}$ is uniformly bounded, and then easily the power series $\sum_{j=0}^\infty CA^j B z^j$ converges for all $z \in \mathbf{D}$. The power series expansion of transfer function $\mathcal{D}_\phi(z)$ is given by

$$\mathcal{D}_\phi(z) = D + \sum_{j\geq 1} CA^{j-1} B z^j$$

By output stability, $\{CA^j B u_0\}_{j\geq 0} \subset \ell^2(\mathbf{Z}_+;Y)$ for any $u_0 \in U$. The Parseval identity implies now that $\mathcal{D}_\phi(z) u_0 \in H^2(\mathbf{D};Y)$ for each $u_0 \in U$. So $\mathcal{D}_\phi(z) \in \mathrm{sH}^2(\mathbf{D};\mathcal{L}(U;Y))$. The case of the strong H^2-stability is similar. □

2.5 Nontangential limits of transfer functions

We have seen that the I/O-maps of DLSs and well-posed signals have a one-to-one correspondence to their transfer functions and z-transforms, respectively. Furthermore, the I/O stability and finite signal energy notions are well behaved under the z-transform. The following question arises: what essentially new does the replacement of the bilateral shift τ by the complex variable z bring us? A (partial) answer is: point evaluations of the transfer function $\mathcal{D}(z)$ at all points of analyticity z. This gives us the notion of *zeroes* and *poles* of the transfer function, at least in the case when all the Hilbert spaces U, H and Y are finite dimensional.

The notions of zeroes and poles are not central in our work, and if it was only for this reason, we would not need to define the transfer functions in the first place. However, there is another reason to introduce transfer functions that is important to us. Namely, there are classes of (transfer) functions $\mathcal{D}(z)$ and (signals) $\tilde{u}(z)$, analytic for $z \in \mathbf{D}$, that can be evaluated in a useful sense at the boundary points $e^{i\theta} \in \mathbf{T} = \partial\mathbf{D}$, too. In these classes, the notion of the *nontangential limit functions* or, equivalently, *boundary traces* $\mathcal{D}(e^{i\theta})$ and $\tilde{u}(e^{i\theta})$ can be defined by

$$\begin{aligned}
\mathcal{D}(e^{i\theta}) u_0 &= \lim_{z_j \to e^{i\theta}} \mathcal{D}(z_j) u_0 \quad \text{for all} \quad u_0 \in U, \\
\tilde{u}(e^{i\theta}) &= \lim_{z_j \to e^{i\theta}} \tilde{u}(z_j),
\end{aligned}$$

for all such $e^{i\theta} \in \mathbf{T}$, where the limit exists for all $u_0 \in U$ and all sequences $\mathbf{D} \ni z_j \to e^{i\theta} \in \mathbf{T}$ lying inside some nontangential approach region, as defined in [6, p. 6], [34, Theorem 11.18], or any other book of basic function theory. We remark that the operator limit $\mathcal{D}(e^{i\theta})$ is taken pointwise, in the strong operator topology. If $\mathcal{D}(z)$ is matrix-valued, then the strong nontangential limit is actually a nontangential norm limit, because in a finite dimensional space pointwise convergence implies norm convergence. We proceed to define the classes where boundary traces $\tilde{u}(e^{i\theta})$ and $\mathcal{D}(e^{i\theta})$ are available in a practical sense.

Suppose now that $\tilde{u}(z) \in H^p(\mathbf{D}; U)$ for $1 \leq p < \infty$, and $\mathcal{D}(z) \in H^p(\mathbf{D}; \mathcal{L}(U; Y))$ for $1 \leq p \leq \infty$. By [33, Theorem 4.6A], if U, Y are separable, the nontangential limit functions, denoted by $\tilde{u}(e^{i\theta})$ and $\mathcal{D}(e^{i\theta})$, exist a.e. $e^{i\theta} \in \mathbf{T}$ modulo the Lebesgue measure of the unit circle $\mathbf{T}$. Actually this is true in much larger classes $N(\mathbf{D}; U)$, $N(\mathbf{D}; \mathcal{L}(U; Y))$, $N_+(\mathbf{D}; U)$, $N_+(\mathbf{D}; \mathcal{L}(U; Y))$, defined in the following.

Definition 12. *Let X be U or $\mathcal{L}(U; Y)$.*

(i) Then $N(\mathbf{D}; X)$ is the set of analytic X-valued functions $f(z)$, such that

$$\sup_{0<r<1} \int_0^{2\pi} log_+ ||f(re^{i\theta})||_X \, d\theta < \infty.$$

The set $N(\mathbf{D}; X)$ is called the Nevanlinna class, and its elements are called the functions of bounded type.

(ii) $\mathcal{H}_g(D; X)$ is the set of analytic X-valued functions $f(z)$, such that

$$\sup_{0<r<1} \int_0^{2\pi} g(log_+ ||f(re^{i\theta})||_X) \, d\theta < \infty,$$

where g is a strongly convex function. The space $\mathcal{H}_g(\mathbf{D}; X)$ is called the Hardy-Orlicz class.

(iii) $N_+(\mathbf{D}; X) := \cup \mathcal{H}_g(\mathbf{D}; X)$, where the union is taken over all strongly convex functions g.

A function $g : \mathbf{R} \to \mathbf{R}_+$ is strongly convex (in the sense of [33, p. 135]) if it is convex, nondecreasing, satisfies $\lim_{t \to \infty} g(t)/t = \infty$, and for some $c > 0$ there exists $M \geq 0$ and $a \in \mathbf{R}$ such that $g(t + c) \leq M g(t)$ for all $t \geq a$. All the sets $\mathcal{H}_g(\mathbf{D}; X), N_+(\mathbf{D}; X), N(\mathbf{D}; X)$ are vector spaces, and $\mathcal{H}_g(\mathbf{D}; X) \subset N_+(\mathbf{D}; X) \subset N(\mathbf{D}; X)$. For additional information, see [33, Chapter 4]. In particular, choosing $g(t) = e^{pt}$ gives the $H^p(\mathbf{D}; X)$ space, for $0 < p < \infty$. Because $H^\infty(\mathbf{D}; X) \subset H^2(\mathbf{D}; X)$, also the bounded analytic functions are of bounded type.

These spaces are introduced because for $f(z) \in N(\mathbf{D}; X)$, the boundary trace function $f(e^{i\theta})$ exists almost everywhere on $\mathbf{T}$. The set of the corresponding boundary traces is denoted, quite naturally, by $N(\mathbf{T}; X)$. The mapping $N(\mathbf{D}; X) \ni f(z) \mapsto f(e^{i\theta}) \in N(\mathbf{T}; X)$ is one-to-one and linear. Furthermore, the operator products of such functions are well behaved: If $F(e^{i\theta}) \in N(\mathbf{T}; \mathcal{L}(U; Y))$ and $G(e^{i\theta}) \in N(\mathbf{T}; \mathcal{L}(U))$, then $F(e^{i\theta})G(e^{i\theta}) \in$

$N(\mathbf{T};\mathcal{L}(U;Y))$. If $f(e^{i\theta}) \in N(\mathbf{T};U)$, then $F(e^{i\theta})f(e^{i\theta}) \in N(\mathbf{T};\mathcal{L}(Y))$. Not only the sensible products of bounded type functions are of bounded type, but also the boundary trace of the product is always the product of the boundary traces. In the infinite-dimensional cases these are nontrivial facts because the operator multiplication is not continuous in the strong operator topology; or in the poetic words of [33, p. 88]: "there is more here than meets the eye". The proofs of these results are based on the powerful representation for the Nevanlinna class functions as a fraction of two H^∞ functions, with a scalar zero-free denominator. The H^∞ case can then be handled more easily. For further information, see [33, Theorem 4.2D and Theorem 4.5A].

Let us return to discuss the special case of $H^p(\mathbf{D};X)$-spaces and the corresponding boundary trace spaces $H^p(\mathbf{T};X)$. Ultimately, the spaces $H^p(\mathbf{T};X)$ are identified with subspaces of certain Lebesque spaces $L^p(\mathbf{T};\mathcal{L}(U;Y))$ (operator-valued) and $L^p(\mathbf{T};U)$ (Hilbert space -valued), for each $1 \leq p \leq \infty$. In order to introduce the operator and vector Lebesque spaces, it is necessary to remind some notions of measure theory.

Definition 13. *Let U, Y be separable Hilbert spaces. Let the measure space $(\mathbf{T},\mathcal{B},d\theta)$ be the usual (Lebesque completion of the) Borel σ-algebra of the unit circle $\mathbf{T}$, where $d\theta$ denotes the Lebesque measure of $\mathbf{T}$.*

(i) The U-valued function $f(e^{i\theta})$, defined $d\theta$-almost everywhere on $e^{i\theta} \in \mathbf{T}$, is weakly (Lebesque) measurable, if for all $u \in U$, the $\mathbf{C}$-valued function $f_u(e^{i\theta}) := \langle f(e^{i\theta}), u\rangle_U$ is $(\mathbf{T},\mathcal{B},d\theta)$-measurable.

(ii) The $\mathcal{L}(U;Y)$-valued function $F(e^{i\theta})$, defined $d\theta$-almost everywhere on $e^{i\theta} \in \mathbf{T}$, is weakly (Lebesque) measurable, if for all $u \in U$, $y \in Y$, the $\mathbf{C}$-valued function $F_{u,y}(e^{i\theta}) := \langle F(e^{i\theta})u, y\rangle_Y$ is $(\mathbf{T},\mathcal{B},d\theta)$-measurable.

If $f(e^{i\theta}), g(e^{i\theta}), F(e^{i\theta}), G(e^{i\theta})$ are weakly measurable, then so are $F(e^{i\theta})f(e^{i\theta})$ and $F(e^{i\theta})G(e^{i\theta})$, if the products make sense. Furthermore, the following scalar functions are measurable: $\langle f(e^{i\theta}), g(e^{i\theta})\rangle_U$, $||f(e^{i\theta})||_U$ and $||F(e^{i\theta})||_{\mathcal{L}(U;Y)}$. If $r(e^{i\theta})$ is a measurable scalar function and $u \in U$, $A \in \mathcal{L}(U;Y)$, then $r(e^{i\theta})u$ and $r(e^{i\theta})A$ are weakly measurable, see [5, Part I, Chapter III]), [14, Chapter III, p. 74], [33, comment on p. 81], and [47].

Definition 14. *Let $1 \leq p < \infty$. The Lebesque spaces are defined as follows:*

(i) $L^p(\mathbf{T};U)$ is the vector space of weakly measurable U-valued functions $f(e^{i\theta})$, defined a.e. $e^{i\theta} \in \mathbf{T}$, such that

$$||f(e^{i\theta})||^p_{L^p(\mathbf{T};U)} := \frac{1}{2\pi}\int_0^{2\pi} ||f(e^{i\theta})||^p_U \, d\theta < \infty.$$

(ii) $L^p(\mathbf{T};\mathcal{L}(U;Y))$ is the vector space of weakly measurable $\mathcal{L}(U;Y)$-valued functions $F(e^{i\theta})$, defined a.e. $e^{i\theta} \in \mathbf{T}$, such that

$$||F(e^{i\theta})||^p_{L^p(\mathbf{T};\mathcal{L}(U;Y))} := \frac{1}{2\pi}\int_0^{2\pi} ||F(e^{i\theta})||^p_{\mathbf{T};\mathcal{L}(U;Y)} \, d\theta < \infty.$$

(iii) $L^\infty(\mathbf{T}; \mathcal{L}(U;Y))$ *is the vector space of weakly measurable* $\mathcal{L}(U;Y)$*-valued functions* $F(e^{i\theta})$*, such that*

$$||F(e^{i\theta})||_{L^\infty(\mathbf{T};\mathcal{L}(U;Y))} := ess\ sup_{e^{i\theta}\in\mathbf{T}}||F(e^{i\theta}||_{\mathcal{L}(U;Y)} < \infty.$$

Note that the scalar integrals appearing in Definition 14 are well defined, by the assumed weak measurability. All the Lebesque spaces are Banach spaces. $L^2(\mathbf{T}; U)$ is a Hilbert space with the inner product

$$\left\langle f(e^{i\theta}), g(e^{i\theta})\right\rangle_{L^2(\mathbf{T};U)} := \frac{1}{2\pi}\int_0^{2\pi} \left\langle f(e^{i\theta}), g(e^{i\theta})\right\rangle_U \, d\theta.$$

Because of the nice properties of the weak measurability, much of the scalar Lebesque space theory can be carried over to the corresponding vector-valued theory, by quite straightforward arguments. For example, because $\mathbf{T}$ is of the finite Lebesque measure, the Hölder inequality implies that if $1 \le p_1 \le p_2 \le \infty$, then $L^{p_2}(\mathbf{T}; X) \subset L^{p_1}(\mathbf{T}; X)$.

For $1 \le p \le \infty$, $H^p(\mathbf{T}; X)$ can be regarded as a closed subspace of $L^2(\mathbf{T}; X)$, such that the Fourier coefficients of $f(e^{i\theta})$ (to be introduced in next Subsection 2.6) satisfy $f_j = 0$ for all $j < 0$, see [33, Theorem 4.7C]. Furthermore, $f(z)$ can be recovered from $f(e^{i\theta})$ by both Poisson and Cauchy integrals. Finally, the $H^p(\mathbf{D}; X)$-functions $f(z)$ and their boundary traces $f(e^{i\theta}) \in H^p(\mathbf{T}; X)$ can be and usually are identified by an isometry, see [33, Theorem 4.7D].

2.6 Vector-valued integration and Fourier transform

Let U and Y be separable Hilbert spaces. In order to define the Fourier transform in the Lebesque spaces $L^p(\mathbf{T}; \mathcal{L}(U;Y))$ and $L^p(\mathbf{T}; U)$ for $p \ge 1$, we must have an integration theory for these Banach space -valued functions. Note that in Subsection 2.5, only a scalar Lebesque integration theory, together with a characterization of weakly measurable Banach space -valued functions, was required to define the spaces $L^p(\mathbf{T}; \mathcal{L}(U;Y))$ and $L^p(\mathbf{T}; U)$. Also recall that if $1 \le p_1 \le p_2 \le \infty$, then $L^{p_2}(\mathbf{T}; \mathcal{L}(U;Y)) \subset L^{p_1}(\mathbf{T}; \mathcal{L}(U;Y))$ and $L^{p_2}(\mathbf{T}; U) \subset L^{p_1}(\mathbf{T}; U)$. It is well known that in the largest classes $L^1(\mathbf{T}; \mathcal{L}(U;Y))$ and $L^1(\mathbf{T}; U)$, a vector-valued integration theory (and in fact many of those) can be developed:

Proposition 15. *Let* U *and* Y *be separable Hilbert spaces. Let* $f(e^{i\theta}) \in L^1(\mathbf{T}; U)$ *and* $F(e^{i\theta}) \in L^1(\mathbf{T}; \mathcal{L}(U;Y))$*.*

(i) There is a unique $c \in U$ *such that for all* $u \in U$

$$\langle c, u\rangle_U = \int_0^{2\pi} \left\langle f(e^{i\theta}), u\right\rangle_U \, d\theta.$$

We call c *the weak Lebesque (Pettis) integral of* $f(e^{i\theta})$ *and write* $\int_0^{2\pi} f(e^{i\theta})\, d\theta := c$*.*

(ii) There is a unique $C \in \mathcal{L}(U;Y)$ such that for all $u \in U$, $y \in Y$

$$\langle Cu, y\rangle_Y = \int_0^{2\pi} \langle F(e^{i\theta})u, y\rangle_Y \, d\theta.$$

We call C the weak Lebesque (Pettis) integral of $F(e^{i\theta})$ and write $\int_0^{2\pi} F(e^{i\theta})\, d\theta := C$.

Proof. For claim (i), see [14, Definition 3.7.1 and Theorem 3.7.1], and note that U, as a Hilbert space, is reflexive. We outline the proof how claim (ii) follows from claim (i). Let $u \in U$. Then $F(e^{i\theta})u$ is a Y-valued weakly measurable function, and by claim (i) there is a unique $c_u \in Y$ such that

$$\langle c_u, y\rangle_Y = \int_0^{2\pi} \langle F(e^{i\theta})u, y\rangle_Y \, d\theta$$

for all $y \in Y$. It is easy to show that the mapping $U \ni u \mapsto c_u \in Y$ is linear, and we write $C : U \to Y$ by $Cu := c_u$. It remains to be shown that C is bounded. Let now $u \in U$ and $y \in Y$ be arbitrary. Then

$$|\langle Cu, y\rangle_Y| \le \int_0^{2\pi} |\langle F(e^{i\theta})u, y\rangle_Y| \, d\theta \le ||u||_U \cdot ||y||_Y \cdot \int_0^{2\pi} ||F(e^{i\theta})|| \, d\theta,$$

where the first estimate holds by the property of scalar Lebesque integral, and second by the Schwartz inequality. Because $F(e^{i\theta}) \in L^1(\mathbf{T}; \mathcal{L}(U;Y))$, the integral of its norm is finite, and it the follows that

$$||T||_{\mathcal{L}(U;Y)} := \sup_{||u||_U = ||y||_Y = 1} |\langle Cu, y\rangle_Y| \le ||F(e^{i\theta})||_{L^1(\mathbf{T};\mathcal{L}(U;Y))} < \infty.$$

We regard this proposition as proved. □

Now that we can integrate, we are prepared to consider the Fourier transforms. Let $f(e^{i\theta}) \in L^1(\mathbf{T}; U)$ and $F(e^{i\theta}) \in L^1(\mathbf{T}; \mathcal{L}(U;Y))$. Trivially, the functions $e^{i\theta} \mapsto e^{ij\theta} f(e^{i\theta}) \in L^1(\mathbf{T}; U)$ and $e^{i\theta} \mapsto e^{ij\theta} F(e^{i\theta}) \in L^1(\mathbf{T}; \mathcal{L}(U;Y))$ for all $j \in \mathbf{Z}$, and we can uniquely define the weak integrals

$$f_j := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ij\theta} \, d\theta \in U, \quad F_j := \frac{1}{2\pi} \int_0^{2\pi} F(e^{i\theta}) e^{-ij\theta} \, d\theta \in \mathcal{L}(U;Y).$$

These integrals are called the *Fourier coefficients* of the respective functions. We call the formal series

$$f(e^{i\theta}) \sim \sum f_j e^{ij\theta}, \quad F(e^{i\theta}) \sim \sum F_j e^{ij\theta}$$

the *Fourier series* of the respective functions. Two Fourier series are identical if all their respective coefficients f_j or F_j are identical. The mappings

$$f(e^{i\theta}) \mapsto \{f_j\}_{j\in\mathbf{Z}} \subset U, \quad F(e^{i\theta}) \mapsto \{F_j\}_{j\in\mathbf{Z}} \subset \mathcal{L}(U;Y)$$

are called the *Fourier transforms* of the respective spaces. It is easy to show that the Fourier transform is a linear mapping, and the Fourier coefficient are uniformly bounded: $||f_j|| \leq ||f(e^{i\theta})||_{L^1(\mathbf{T};U)} \leq \sqrt{2\pi}||f(e^{i\theta})||_{L^2(\mathbf{T};U)}$ and $||F_j|| \leq ||F(e^{i\theta})||_{L^1(\mathbf{T};\mathcal{L}(U;Y))} \leq \sqrt{2\pi}||F(e^{i\theta})||_{L^2(\mathbf{T};\mathcal{L}(U;Y))}$. The questions of convergence of the Fourier series (in various topologies) are generally highly nontrivial. In this paper, the classes $L^2(\mathbf{T};U)$ and $L^2(\mathbf{T};\mathcal{L}(U;Y))$ are of particular interest. The case of the Hilbert space is well known:

Proposition 16. *The Fourier transform* $f(e^{i\theta}) \mapsto \{f_j\}_{j\in\mathbf{Z}}$ *is an isometric isomorphism of the Hilbert space* $L^2(\mathbf{T};U)$ *onto the Hilbert space* $\ell^2(\mathbf{Z};U)$*. The Fourier series* $\sum f_j e^{i\theta}$ *converges to* $f(e^{i\theta})$ *in* $L^2(\mathbf{T};U)$*. The Parseval identity holds*

$$\left\langle f(e^{i\theta}), g(e^{i\theta})\right\rangle_{L^2(\mathbf{T};U)} = \langle\{f_j\},\{g_j\}\rangle_{\ell^2(\mathbf{Z};U)}\,.$$

The closed subspace $H^2(U) \subset L^2(\mathbf{T};U)$ *is mapped onto the closed subspace* $\ell^2(\mathbf{Z}_+;U) \subset \ell^2(\mathbf{Z};U)$*.*

However, we need the following result on the operator-valued $L^2(\mathbf{T};\mathcal{L}(U;Y))$.

Proposition 17. *Let* U *and* Y *be separable Hilbert spaces, and* $u \in U$ *arbitrary. Let* $F(e^{i\theta}) \in L^1(\mathbf{T};\mathcal{L}(U;Y))$*. Define the* Y*-valued function* $F_u(e^{i\theta}) := F(e^{i\theta})u$*. Then*

(i) $F_u(e^{i\theta}) \in L^1(\mathbf{T};Y)$,

(ii) *the Fourier coefficients* $\{F_j\}_{j\in\mathbf{Z}}$ *of* $F(e^{i\theta})$ *and* $\{(F_u)_j\}_{j\in\mathbf{Z}}$ *of* $F_u(e^{i\theta})$ *satisfy*

$$F_j u = (F_u)_j \quad \text{for all} \quad j \in \mathbf{Z},$$

(iii) *the Fourier series* $\sum_{j\in Z}(F_j u)\, e^{ij\theta}$ *converges in* $L^2(\mathbf{T};Y)$ *to* $F(e^{i\theta})u$*.*

Proof. Claim (i) is trivial. To prove claim (ii), fix $u \in U$, $j \in \mathbf{Z}$, and let $y \in Y$ be arbitrary. By the definition of weak integral, the Fourier coefficient $F_j \in \mathcal{L}(U;Y)$ is an operator such that

(17)

$$\langle F_j u, y\rangle_Y = \frac{1}{2\pi}\int_0^{2\pi}\left\langle F(e^{i\theta})\, e^{-ij\theta}u, y\right\rangle_Y d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left\langle F(e^{i\theta})u, y\right\rangle_Y e^{-ij\theta}\, d\theta.$$

for all $y \in Y$. By the definition of the weak Hilbert space -valued integral, the Fourier coefficient $(F_u)_j \in Y$ is an element such that

(18)

$$\langle (F_u)_j, y \rangle_Y = \frac{1}{2\pi} \int_0^{2\pi} \left\langle F_u(e^{i\theta})\, e^{-ij\theta}, y \right\rangle_Y \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left\langle F(e^{i\theta}) u, y \right\rangle_Y \, e^{-ij\theta} \, d\theta.$$

for all $y \in Y$. Comparing the right hand sides of equations (17) and (18) implies that $\langle F_j u, y \rangle_Y = \langle (F_u)_j, y \rangle_Y$ for all y, or equivalently $F_j u = (F_u)_j$. Because u and j are arbitrary, this proves claim (ii). The last claim (iii) follows from the previous claim and Proposition 16. □

2.7 Discussion

We conclude this section with a general discussion. In this section we have introduced three (essentially) equivalent formalisms (DLS in difference equation form, DLS in I/O-form and transfer function formalism) to realize and describe the same class of objects (well posed, causal shift-invariant linear operators in discrete time). At first sight, this might seem a little superfluous, and we try to defend ourselves in the following.

We note that all the operators A, B, C, D, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ appearing in quadruples $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ and $\left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right]$ are separate functional blocks, present in any linear state space model. From the control theoretic point of view, the interaction between controllability-, observability- and I/O-maps can be conveniently described because these operators constitute the DLS in I/O form in our formalism. What we have actually done, it to collect the operator of the same kind into two different structures: DLS in I/O-form and in difference equation form. In this notational framework, also nonlinear generalizations are admitted.

Notationally our DLS-formalism is very similar to the formalism used in [37], [39], [40], and [45] for continuous time *stable well-posed linear systems*. In continuous time, however, the notions corresponding to our "difference equation form" and "I/O-form" are not equivalent because generally the feedthrough operator D cannot be separated from the I/O-map $\mathcal{D}$ without an extra regularity assumption, see [45], [50], and [51].

So as to the transfer function representation, we also remark that the operator theoretic study of these linear systems becomes notationally clumsy, if the basic operators are always stated as multiplications by transfer functions. In monographs [33] and [46], the basic objects are the unilateral shift operators and the Toeplitz operators — the complex analysis results are presented more or less as an important application. The only reason for us to introduce the transfer functions is to get the additional structure associated to the boundary trace algebra of the functions in the Nevanlinna class $N(\mathbf{D}; \mathcal{L}(U))$. We remark that this is only possible under stronger assumptions, requiring e.g. the separability of all the Hilbert spaces involved.

3 H^∞ Riccati equation

In this section we give basic definitions of the discrete time algebraic Riccati equation, associated to an output stable and I/O stable DLS Φ and a possibly indefinite cost operator $J \in \mathcal{L}(U)$. The solutions P of such equation are classified according to stability properties of an associated DLS ϕ_P, see Definitions 19 and 20. An additional classification is done according to the residual cost properties, as introduced in Definition 21. After that, inclusions of the various solution sets are considered.

Definition 18. *Let $J \in \mathcal{L}(Y)$ be self-adjoint, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS. Then the following system of operator equations*

$$(19) \qquad \begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P \\ \Lambda_P = D^*JD + B^*PB \\ \Lambda_P K_P = -D^*JC - B^*PA \end{cases}$$

is called the discrete time algebraic Riccati equation (DARE) and denoted by $Ric(\Phi, J)$. The linear operators are required to satisfy $\Lambda_P, \Lambda_P^{-1} \in \mathcal{L}(U)$ and $K_P \in \mathcal{L}(H;U)$. Here P is a unknown self-adjoint operator to be solved. If $P \in \mathcal{L}(H)$ satisfies (19), we write $P \in Ric(\Phi, J)$.

We use the same symbol $Ric(\Phi, J)$ both for the solution set of a DARE, and the DARE itself. This should not cause confusion. Clearly the equations (19) can be put into form

$$(20) \qquad \begin{aligned} &A^*PA - P + C^*JC \\ &= (D^*JC + B^*PA)^* (D^*JD + B^*PB)^{-1} (D^*JC + B^*PA). \end{aligned}$$

This is the usual form of the DARE in the literature. Because Λ_P and K_P are quite fundamental objects in our treatment, the system (19) is used instead. For a given $P \in Ric(\Phi, J)$, the operator Λ_P is called the indicator of P, and the operator K_P is called the (state) feedback operator of solution P. The operators $A_P := A + BK_P$ and $C_P = C + DK_P$ are the closed loop semigroup generator and the closed loop output operator, respectively. Sometimes DARE (20) has a trivial solution; if we can write $(D^*JD)^{-1} = D^{-1}J^{-1}(D^{-1})^*$, then clearly $0 \in Ric(\Phi, J)$.

To each solution $P \in Ric(\Phi, J)$, two additional DLSs are associated:

Definition 19. *Let $J \in \mathcal{L}(Y)$ be self-adjoint, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS. Let K_P, A_P and C_P be as above.*

(i) For $P \in Ric(\Phi, J)$, the DLS

$$\phi_P := \begin{pmatrix} A & B \\ -K_P & I \end{pmatrix}$$

is the spectral DLS, associated to the pair (Φ, J) and centered at P.

(ii) *For* $P \in Ric(\Phi, J)$, *the DLS*

$$\phi^P := \begin{pmatrix} A_P & B \\ C_P & D \end{pmatrix}$$

is called the inner DLS, associated to the pair (Φ, J) *and centered at* P.

In this work, we consider DAREs $Ric(\Phi, J)$, such that Φ is output stable and I/O stable. These are called H^∞DAREs, and defined as follows:

Definition 20. *Let the objects* Φ, J, $Ric(\Phi, J)$, $P \in Ric(\Phi, J)$, *and* ϕ_P *be as in Definitions 18 and 19. Assume that* Φ *is, in addition, I/O stable and output stable.*

(i) *We denote the DARE (19) by* $ric(\Phi, J)$ *instead of* $Ric(\Phi, J)$. *The DARE* $ric(\Phi, J)$ *is called* H^∞*DARE.*

(ii) *If* $P \in Ric(\Phi, J)$ *is such that the spectral DLS* ϕ_P *is I/O stable and output stable, then we say that* $P \in ric(\Phi, J)$. *We say that such* P *is an* H^∞ *solution of a* H^∞*DARE.*

When we write inclusions and equalities like $Ric(\Phi, J) \subset Ric(\Phi', J')$, $Ric(\Phi, J) = Ric(\Phi', J')$, then these symbols refer to the solution sets of the respective DAREs. We remark that a H^∞DARE $ric(\Phi, J)$ could have a non-H^∞ solution P. This this case we write $P \in Ric(\Phi, J)$ instead of $P \in ric(\Phi, J)$.

A number of residual cost conditions are required in our work.

Definition 21. *Let the objects* Φ, J, $Ric(\Phi, J)$, $P \in Ric(\Phi, J)$, *and* ϕ_P *be as in Definitions 18 and 19.*

(i) *If the residual cost operator*

$$L_{A,P} := \mathrm{s} - \lim_{j \to \infty} A^{*j} P A^j$$

exists as a bounded operator in $\mathcal{L}(H)$, *we write* $P \in Ric_{00}(\Phi, J)$.

(ii) *If* $L_{A,P} = 0$, *we write* $P \in Ric_0(\Phi, J)$. *Such* P *satisfies the strong residual cost condition.*

(iii) *If* $\langle PA^j x_0, A^j x_0 \rangle \to 0$ *for all* $x_0 \in H$, *we write* $P \in Ric_{000}(\Phi, J)$. *Such* P *satisfies the weak residual cost condition.*

(iv) *If* $\langle PA^j x_0, A^j x_0 \rangle \to 0$ *for all* $x_0 \in \mathrm{range}\,(\mathcal{B})$, *we write* $P \in Ric_{uw}(\Phi, J)$. *Such* P *satisfies the ultra weak residual cost condition.*

We also define the solution sets $ric_0(\Phi, J) := Ric_0(\Phi, J) \cap ric(\Phi, J)$, $ric_{00}(\Phi, J) := Ric_{00}(\Phi, J) \cap ric(\Phi, J)$, $ric_{000}(\Phi, J) := Ric_{000}(\Phi, J) \cap ric(\Phi, J)$ *and* $ric_{uw}(\Phi, J) := Ric_{uw}(\Phi, J) \cap ric(\Phi, J)$. *The elements of* $ric_0(\Phi, J)$ *are called regular* H^∞ *solutions.*

We remark that the residual cost conditions (i), (ii), and (iii) depend on the structure of the solution P in the whole state space H. The ultra weak residual cost condition (iii) imposes only requirements on P restricted to the (possibly nonclosed) controllable vector subspace range$(\mathcal{B})$. Recall that range$(\mathcal{B}) = \mathcal{B}(\mathrm{dom}(\mathcal{B}))$ where $\mathrm{dom}(\mathcal{B}) := Seq_-(U)$ consists of sequences in $\ell^2(\mathbf{Z}_-; U)$ with only finitely many nonzero components. Equivalently, $P \in Ric_{uw}(\Phi, J)$ if and only if $\lim_{j\to\infty} \langle P\mathcal{B}\tau^{*j}\tilde{u}, \mathcal{B}\tau^{*j}\tilde{u}\rangle = 0$ for all $\{u_j\}_{j\geq 0} = \tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ having only finitely many nonzero components u_j. Solutions $P \in Ric_{uw}(\Phi, J)$ are of particular interest in the factorization theory of Theorem 27 and Theorem 50. The residual cost conditions (i) and (ii) of Definition 21 are convenient for the Liapunov equation techniques. The following inclusions are basic:

Proposition 22. *Let the objects Φ, J, $Ric(\Phi, J)$, $P \in Ric(\Phi, J)$, and ϕ_P be as in Definitions 18 and 21. Then the following holds*

(i) If A is strongly stable, then $Ric(\Phi, J) = Ric_0(\Phi, J)$.

(ii) $\{P \in Ric_{000}(\Phi, J) \mid P \geq 0\} \subset Ric_0(\Phi, J) \subset Ric_{000}(\Phi, J)$.

(iii) $Ric_0(\Phi, J) \cup Ric_{000}(\Phi, J) \subset Ric_{uw}(\Phi, J)$.

(iv) $Ric_{00}(\Phi, J) \cap Ric_{000}(\Phi, J) \subset Ric_0(\Phi, J)$. If $\overline{\mathrm{range}(\mathcal{B})} = H$, then $Ric_{00}(\Phi, J) \cap Ric_{uw}(\Phi, J) \subset Ric_0(\Phi, J)$.

(v) If $\overline{\mathrm{range}(\mathcal{B})} = H$ and A is power bounded, then $Ric_{uw}(\Phi, J) \subset Ric_{000}(\Phi, J)$ and $\{P \in Ric_{uw}(\Phi, J) \mid P \geq 0\} \subset Ric_0(\Phi, J)$.

(vi) We have the inclusion:

$$\{P \in Ric(\Phi, J) \mid \lim_{j\to\infty} \langle P\mathcal{B}\tau^{*j}\tilde{u}, \mathcal{B}\tau^{*j}\tilde{u}\rangle = 0 \text{ for all } \tilde{u} \in \ell^2(\mathbf{Z}_+; U)\}$$
$$\subset Ric_{uw}(\Phi, J).$$

If Φ is, in addition, input stable, then the inclusion is equality.

Proof. If A is strongly stable, then for all $x_0 \in H$ we have

$$||A^{*j}PA^jx_0|| \leq ||A^{*j}|| \cdot ||P|| \cdot ||A^jx_0|| \quad \text{for all} \quad j \geq 1.$$

By the strong stability of A, $||A^jx_0|| \to 0$ as $j \to \infty$. Furthermore, by Banach–Steinhaus Theorem, $\sup_{j\geq 1} ||A^j|| < \infty$ and thus also $\sup_{j\geq 1} ||A^{*j}|| < \infty$. Thus $||A^{*j}PA^jx_0|| \to 0$ for all x_0 and $L_{A,P} := \mathrm{s} - \lim_{j\to\infty} A^{*j}PA^j = 0$. This verifies claim (i).

Assume that $P \in Ric_{000}(\Phi, J)$ is nonnegative. Then it follows that $\langle PA^jx_0, A^jx_0\rangle = ||P^{\frac{1}{2}}A^jx_0||^2 \to 0$ for all $x_0 \in H$. Again, by Banach–Steinhaus Theorem, $C := \sup_{j\geq 1} ||A^{*j}P^{\frac{1}{2}}|| < \infty$. It now follows that $||A^{*j}PA^jx_0|| \leq C \cdot ||P^{\frac{1}{2}}A^jx_0|| \to 0$, and thus $P \in Ric_0(\Phi, J)$. Now claim (ii) follows. Claim (iii) is trivial.

Let $P \in Ric_{00}(\Phi, J) \cap Ric_{000}(\Phi, J)$. Thus $L_{A,P}$ exists, and for all $x_0 \in H$ we have

$$0 = \lim_{j\to\infty} \left\langle PA^j x_0, A^j x_0 \right\rangle = \lim_{j\to\infty} \left\langle A^{*j} PA^j x_0, x_0 \right\rangle = \left\langle L_{A,P} x_0, x_0 \right\rangle .$$

Now, [35, Theorem 12.7] implies that $L_{A,P} = 0$, and the first part of claim (iv) follows. Because $\overline{\text{range}\,(\mathcal{B})} = H$ and $P \in Ric_{uw}(\Phi, J)$, it follows that $\lim_{j\to\infty} \left\langle PA^j x_0, A^j x_0 \right\rangle = 0$ for all x_0 in a dense set. Thus $L_{A,P} x_0 = 0$ in a dense set, and vanishes, by continuity. Now claim (iv) follows.

To prove claim (v), assume that $\overline{\text{range}\,(\mathcal{B})} = H$ and $\sup_{j\geq 0} ||A^j|| < \infty$. Because $P \in Ric_{uw}(\Phi, J)$, we have $\left\langle PA^j x, A^j x \right\rangle \to 0$ for all $x \in \text{range}\,(\mathcal{B})$. Let $x_0 \in H$ be arbitrary, and let $\text{range}\,(\mathcal{B}) \ni x_k \to x_0$ in the norm of H, as $k \to \infty$. Then

$$\begin{aligned}
&| \left\langle PA^j x_0, A^j x_0 \right\rangle | \\
&\leq | \left\langle A^{*j} PA^j x_k, x_k \right\rangle | + | \left\langle A^{*j} PA^j x_k, (x_0 - x_k) \right\rangle | + | \left\langle A^{*j} PA^j (x_0 - x_k), x_0 \right\rangle | \\
&\leq | \left\langle A^{*j} PA^j x_k, x_k \right\rangle | + \sup_{j\geq 0} ||A^{*j} PA^j|| \cdot ||x_0 - x_k|| \cdot (||x_k|| + ||x_0||)
\end{aligned}$$

Because $\{x_k\}$ is a convergent sequence, it is a bounded set. Because A is power bounded, $\sup_{j\geq 0} ||A^{*j} PA^j|| < \infty$. Then, by first increasing k sufficiently the latter term get arbitrarily small, and the former term gets small as j is increased. Now $\left\langle PA^j x_0, A^j x_0 \right\rangle \to 0$ for all $x_0 \in H$, not just $x_0 \in \text{range}\,(\mathcal{B})$; or $P \in Ric_{000}(\Phi, J)$. The additional claim for $P \geq 0$ follows from claim (ii) of this Proposition.

The inclusion part of claim (vi) is trivial. For the rest, let $\epsilon > 0$, $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ and $P \in Ric_{uw}(\Phi, J)$ be arbitrary. Let $K \geq 0$ so large that $||\pi_{[k,\infty]} \tilde{u}|| \leq \epsilon / ||\mathcal{B}||$ for all $k \geq K$, where the input stability is used. Then for $j > k \geq K$,

$$\begin{aligned}
(21)\quad &| \left\langle P\mathcal{B}\tau^{*j} \tilde{u}, \mathcal{B}\tau^{*j} \tilde{u} \right\rangle | \\
&\leq | \left\langle P\mathcal{B}\tau^{*j} \pi_{[0,k-1]} \tilde{u}, \mathcal{B}\tau^{*j} \pi_{[0,k-1]} \tilde{u} \right\rangle | + | \left\langle P\mathcal{B}\tau^{*j} \pi_{[0,k-1]} \tilde{u}, \mathcal{B}\tau^{*j} \pi_{[k,\infty]} \tilde{u} \right\rangle | \\
&+ | \left\langle P\mathcal{B}\tau^{*j} \pi_{[k,\infty]} \tilde{u}, \mathcal{B}\tau^{*j} \pi_{[0,k-1]} \tilde{u} \right\rangle | + | \left\langle P\mathcal{B}\tau^{*j} \pi_{[k,\infty]} \tilde{u}, \mathcal{B}\tau^{*j} \pi_{[k,\infty]} \tilde{u} \right\rangle | \\
&\leq 2\, ||P|| \cdot ||\mathcal{B}|| \cdot ||\tilde{u}|| \cdot \epsilon + ||P|| \cdot \epsilon^2 + | \left\langle P\mathcal{B}\tau^{*j} \pi_{[0,k-1]} \tilde{u}, \mathcal{B}\tau^{*j} \pi_{[0,k-1]} \tilde{u} \right\rangle | .
\end{aligned}$$

Now we estimate the latter term. Because $j \geq k$, $\mathcal{B}\tau^{*j} \pi_{[0,k-1]} \tilde{u} = A^{j-k} x_0$, where $x_0 = \mathcal{B}\tau^{*k} \pi_{[0,k-1]} \tilde{u} \in \text{range}\,(\mathcal{B})$. But because $P \in Ric_{uw}(\Phi, J)$ by assumption,
$\left\langle PA^{j-k} x_0, A^{j-k} x_0 \right\rangle \to 0$ as $j \to \infty$. So there is $J \geq K$ such that the latter term satisfies $| \left\langle P\mathcal{B}\tau^{*j} \pi_{[0,k-1]} \tilde{u}, \mathcal{B}\tau^{*j} \pi_{[0,k-1]} \tilde{u} \right\rangle | < \epsilon$ for all $j > J$. Now the claim follows from estimate (21). $\square$

The residual cost condition $\lim_{j\to\infty} \left\langle P\mathcal{B}\tau^{*j} \tilde{u}, \mathcal{B}\tau^{*j} \tilde{u} \right\rangle = 0$ for all $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ was used in [20]. For Riccati equation theory of I/O stable DLSs, this residual cost condition is "too strong", and it can be replaced by requiring $P \in Ric_{uw}(\Phi, J)$.

We proceed to consider the H^∞ solutions. The symbols $ric(\phi, J)$ and $ric_{00}(\phi, J)$ can be used synonymously, as far as they refer to the solution sets.

Proposition 23. *Let Φ be an output stable and I/O stable DLS. Let J be a cost operator. Then $ric(\Phi, J) = ric_{00}(\Phi, J)$, and we have*

$$P - L_{A,P} = \mathcal{C}^* J \mathcal{C} - \mathcal{C}^*_{\phi_P} J \mathcal{C}_{\phi_P}.$$

Proof. By iterating on DARE (19), we obtain for all $j \geq 0$:

$$\begin{aligned}(22)\quad P - (A^*)^{j+1} P A^{j+1} &= \left(\pi_{[0,j]}\mathcal{C}\right)^* J \left(\pi_{[0,j]}\mathcal{C}\right) - \left(\pi_{[0,j]}\mathcal{C}_{\phi_P}\right)^* \Lambda_P \left(\pi_{[0,j]}\mathcal{C}_{\phi_P}\right) \\ &= \mathcal{C}^* J \cdot \pi_{[0,j]}\mathcal{C} - \mathcal{C}^*_{\phi_P} J \cdot \pi_{[0,j]}\mathcal{C}_{\phi_P},\end{aligned}$$

where we have written the adjoints by the assumed output stabilities. Clearly $\mathcal{C}^* J \mathcal{C} = \mathcal{C}^* J \pi_{[0,j]}\mathcal{C} + \mathcal{C}^* J \pi_{[j+1,\infty]}\mathcal{C}$. Now $\mathrm{s} - \lim_{j\to\infty} \pi_{[j+1,\infty]}\mathcal{C} = 0$ because $\mathcal{C} : H \to \ell^2(\mathbf{Z}_+; Y)$. Because $\mathcal{C}^* J$ is bounded, $\mathrm{s} - \lim_{j\to\infty} \mathcal{C}^* J \pi_{[j+1,\infty]}\mathcal{C} = 0$ and thus $\mathrm{s} - \lim_{j\to\infty} \mathcal{C}^* J \pi_{[0,j]}\mathcal{C} = \mathcal{C}^* J \mathcal{C}$. Similarly $\mathrm{s} - \lim_{j\to\infty} \mathcal{C}^*_{\phi_P} \Lambda_P \pi_{[0,j]}\mathcal{C}_{\phi_P} = \mathcal{C}^*_{\phi_P} \Lambda_P \mathcal{C}_{\phi_P}$. Now we see from (22) that the strong limit $L_{A,P} := \mathrm{s} - \lim_{j\to\infty} (A^*)^{j+1} P A^{j+1}$ on the left hand side exists, and the claim follows. Also the identity immediately follows. □

Note that the I/O stability of ϕ and ϕ_P played no part in the proof of previous proposition.

The question to what extent the operators Λ_P, K_P (or, equivalently the indicator Λ_P and the spectral DLS ϕ_P in case $\mathrm{range}\,(\mathcal{B}_\phi) = H$) uniquely define a solution $P \in Ric(\phi, J)$, is discussed in the following.

Proposition 24. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable output stable DLS. Let J be a self-adjoint operator. Let $P_1, P_2 \in Ric(\phi, J)$ be such that $\Lambda_{P_1} = \Lambda_{P_2}$ and $K_{P_1} = K_{P_2}$.*

(i) If either P_1 or $P_2 \in Ric_{00}(\phi, J)$, then they both are in $Ric_{00}(\phi, J)$. In this case, $P_1 - P_2 = L_{A,P_1} - L_{A,P_2}$.

(ii) If, in addition, $P_1, P_2 \in Ric_0(\phi, J)$, then $P_1 = P_2$. This is, in particular, always the case when A is strongly stable.

Proof. It follows from equation (19) that $A^* P_1 A - P_1 = A^* P_2 A - P_2$, and immediately $P_1 - P_2 = A^{*j}(P_1 - P_2)A^j = A^{*j} P_1 A^j - A^{*j} P_2 A^j$ for all $j \geq 1$. Now, if $A^{*j} P_2 A^j \to L_{A,P_2} \in \mathcal{L}(H)$ in the strong operator topology, $A^{*j} P_1 A^j$ converges in the strong operator topology, too. Now $L_{A,P_1} - L_{A,P_2} = P_1 - P_2$ and claim (i) follows. The other claim is trivial. □

Proposition 25. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS and $J \in \mathcal{L}(Y)$ a cost operator. Let $P \in Ric_{00}(\phi)$ be arbitrary. If $B^* L_{A,P} B = 0$ and $B^* L_{A,P} A = 0$ then $P' = P - L_{A,P} \in Ric(\phi, J)$, and $\Lambda_P = \Lambda_{P'}$, $K_P = K_{P'}$.*

Proof. The claim immediately follows, by noting that $A^* L_{A,P} A - L_{A,P} = 0$. □

Under stronger assumptions, it in fact follows that $L_{A,P} = 0$ and then $P' = P$, see Lemma 52. In this case, the indicator Λ_P and the spectral DLS ϕ_P uniquely determine $P \in ric(\phi, J)$.

4 Critical solutions of the Riccati equation

There are fundamental connections between a feedback solution of a certain minimax problem, the existence of a certain factorization of the I/O-map, and the existence of a *critical* solution of the DARE, as defined in Definition 28. This connection is the equivalence of Theorem 27, and it has been discussed in [20] under more general assumptions. We remark that it is practically a standing hypothesis of this work that the equivalence of Theorem 27 holds. Before going to this theorem, one more basic notion must be introduced:

Definition 26. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O stable DLS, and* $J \in \mathcal{L}(Y)$ *a self-adjoint operator. The self-adjoint and shift invariant linear operator* $\mathcal{D}^* J \mathcal{D} \in \mathcal{L}(\ell^2(\mathbf{Z}; U))$ *is called the Popov operator (of* Φ *and* J*).*

The Popov operator is clearly bounded, self-adjoint and shift invariant. Its causal Toeplitz operator $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+$ is also called Popov operator, and the Fourier transform is called the Popov function. A fair amount of control theory has recently been written around the Popov operator, see [52], [53] and the references therein. In this respect, our approach is not different.

Theorem 27. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O stable and output stable DLS, and* $J \in \mathcal{L}(Y)$ *be self-adjoint. Then the following conditions (i), (ii) and (iii) are equivalent:*

(i) *a)* Φ *is J-coercive; i.e. the Popov operator* $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+$ *has a bounded inverse.*

b) *There is an I/O stable feedback pair* $[\mathcal{K}, \mathcal{F}]$ *for* Φ *such that the critical control of* Φ *is of feedback form with the critical feedback pair* $[\mathcal{K}, \mathcal{F}]$*.*

(ii) *There is a boundedly invertible operator* $S \in \mathcal{L}(U)$ *such that* $\mathcal{D}$ *has a* (J, S)*-inner-outer factorization* $\mathcal{D} = \mathcal{N}\mathcal{X}$ *where the outer part* $\mathcal{X}$ *has a bounded inverse.*

(iii) *There is a (critical) solution* $P^{\text{crit}} \in Ric_{uw}(\Phi, J)$ *of DARE (19), such that the spectral DLS* $\phi_{P^{\text{crit}}}$ *is I/O stable, and its I/O-map* $\mathcal{D}_{P^{\text{crit}}}$ *is outer with a bounded inverse.*

Proof. The equivalence of claims (i) and (ii) is a particular case of [20, Theorem 27], applied to an output stable DLS Φ. Note that the assumed output stability trivializes the condition $\pi_0 \mathcal{N}^* J \mathcal{C} \in \mathcal{L}(H; U)$ present in [20, Theorem 27].

To study condition (iii), assume that the equivalent conditions (i) and (ii) of this theorem hold. We first note that the critical (closed loop) observability map

$$\mathcal{C}^{\text{crit}} := \left(\mathcal{I} - \bar{\pi}_+ \mathcal{D}(\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}^* J\right) \mathcal{C}$$

is bounded, because all its operators are bounded. The Popov operator $(\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+)^{-1}$ is bounded because Φ is J-coercive, by condition (i). The

Corollary 30. *Let $J \in \mathcal{L}(Y)$ be self-adjoint. Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS, such that $\overline{\mathrm{range}\,(\mathcal{B})} = H$. Assume that a critical solution $P^{\mathrm{crit}} \in Ric_{uw}(\Phi, J)$ exists.*

(i) Then P_0^{crit} is the unique critical solution in the set $Ric_{00}(\Phi, J)$. If A is strongly stable, then P_0^{crit} is the unique critical solution.

(ii) Assume, in addition, that $P^{\mathrm{crit}} \geq 0$. If $P^{\mathrm{crit}} \notin Ric_0(\Phi, J)$, then $\sup_{j\geq 0} ||(P^{\mathrm{crit}})^{\frac{1}{2}} A^j|| = \infty$.

Proof. Let P_0^{crit} be as in Proposition 29. Because both P_0^{crit}, P^{crit} are critical, the I/O-maps $\mathcal{D}_{\phi_{P_0^{\mathrm{crit}}}}$, $\mathcal{D}_{\phi_{P^{\mathrm{crit}}}}$ are outer factors in the $(J, \Lambda_{P_0^{\mathrm{crit}}})$, $(J, \Lambda_{P^{\mathrm{crit}}})$-inner-outer factorizations that they induce, respectively. Using [19, Proposition 21], we conclude that there is $E \in \mathcal{L}(U)$ having a bounded inverse, such that

$$\mathcal{D}_{\phi_{P_0^{\mathrm{crit}}}} = E^{-1}\mathcal{D}_{\phi_{P^{\mathrm{crit}}}}, \quad \text{and} \quad \Lambda_{P_0^{\mathrm{crit}}} = E^*\Lambda_{P^{\mathrm{crit}}}E.$$

Because the feed-through operators of both $\mathcal{D}_{\phi_{P_0^{\mathrm{crit}}}}$ and $\mathcal{D}_{\phi_{P^{\mathrm{crit}}}}$ are identity operators, it follows that $E = I$ and $\Lambda_{P_0^{\mathrm{crit}}} = \Lambda_{P^{\mathrm{crit}}}$. Also the restrictions $K_{P_0^{\mathrm{crit}}}|\mathrm{range}\,(\mathcal{B}) = K_{P^{\mathrm{crit}}}|\mathrm{range}\,(\mathcal{B})$ because the controllability map $\mathcal{B}$ is same for both spectral DLSs in question. Because $\overline{\mathrm{range}\,(\mathcal{B})} = H$, it follows $K_{P_0^{\mathrm{crit}}} = K_{P^{\mathrm{crit}}}$.

By the definition of a critical solution, $P^{\mathrm{crit}} \in Ric_{uw}(\Phi, J)$, see claim (vi) of Proposition 22. Now, if $P^{\mathrm{crit}} \in Ric_{00}(\Phi, J)$, then $P^{\mathrm{crit}} \in Ric_0(\Phi, J)$, by claim (iv) of Proposition 22 and the approximate controllability assumption $\overline{\mathrm{range}\,(\mathcal{B})} = H$. Proposition 24 implies that $P^{\mathrm{crit}} \in Ric_{00}(\Phi, J)$ satisfies

$$P^{\mathrm{crit}} = P_0^{\mathrm{crit}} + L_{A,P^{\mathrm{crit}}} - L_{A,P_0^{\mathrm{crit}}} = P_0^{\mathrm{crit}},$$

because $P_0^{\mathrm{crit}} \in ric_0(\Phi, J)$, by Proposition 29. Now claim (i) follows.

By the definition of a critical solution, $P^{\mathrm{crit}} \in Ric_{uw}(\Phi, J)$. Because $P^{\mathrm{crit}} \geq 0$, it follows that $||(P^{\mathrm{crit}})^{\frac{1}{2}} A^j x|| \to 0$ for all $x \in \mathrm{range}\,(\mathcal{B})$. Assume that $\sup_{j\geq 0} ||(P^{\mathrm{crit}})^{\frac{1}{2}} A^j|| < \infty$. Let $\mathrm{range}\,(\mathcal{B}) \ni x_k \to x \in H \setminus \mathrm{range}\,(\mathcal{B})$. Then,

$$||(P^{\mathrm{crit}})^{\frac{1}{2}} A^j x|| \leq \sup_{j\geq 0} ||(P^{\mathrm{crit}})^{\frac{1}{2}} A^j|| \cdot ||x - x_k|| + ||(P^{\mathrm{crit}})^{\frac{1}{2}} A^j x_k||.$$

The first term on the right can be made small by increasing k, and the latter by increasing j. It follows that $\lim_{j\to\infty} ||(P^{\mathrm{crit}})^{\frac{1}{2}} A^j x|| = 0$ and then $P^{\mathrm{crit}} \in Ric_0(\Phi, J)$, by the Banach–Steinhaus theorem. This completes the proof. □

For a fixed J, the I/O-map $\mathcal{D}$ may have (J, S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ for several different $S = S^* \in \mathcal{L}(U)$. All these are parameterized by [19, Proposition 21]. Given a critical $P^{\mathrm{crit}} \in Ric_{uw}(\Phi, J)$, $\mathcal{D} = \mathcal{N}\mathcal{D}_{\phi_{P^{\mathrm{crit}}}}$ is a $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization, where $\mathcal{N} := \mathcal{D}\mathcal{D}_{\phi_{P^{\mathrm{crit}}}}^{-1}$ by definition. In this case, the feed-through operator of the outer factor $\mathcal{X} = \mathcal{D}_{\phi_{P\text{crit}}}$ is the identity operator in U. This normalization is used throughout this paper.

The rest of this section is devoted to the study of sufficient conditions that quarantee that (one and hence all of) the equivalent conditions of Theorem 27 hold. We remark that this is practically a standing hypothesis in this work.

Proposition 31. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable DLS whose input space U is separable, and $J \in \mathcal{L}(Y)$ be self-adjoint. If $\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+ > 0$ for some $\epsilon > 0$, then the equivalent conditions of 27 hold. In particular, this is true if Φ is J-coercive and $J \geq 0$, or there is $P \in ric_{uw}(\Phi, J)$ such that $\Lambda_P > 0$.*

Proof. $\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+$ is a nonnegative self-adjoint Toeplitz operator with a bounded inverse. By [33, Theorem 3.7], there is an I/O stable I/O-map $\mathcal{G} \in \mathcal{L}(\ell^2(\mathbf{Z}; U))$ such that $\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+ = \bar{\pi}_+\mathcal{G}^*\mathcal{G}\bar{\pi}_+$. By this trick we get rid of the output space Y.

By [33, Theorem 3.4], $\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+ = \bar{\pi}_+\mathcal{G}^*\mathcal{G}\bar{\pi}_+ = \bar{\pi}_+\mathcal{H}^*\mathcal{H}\bar{\pi}_+$, where $\mathcal{H}$ is outer having possibly a nonclosed range. Two problems are present. Firstly, range $(\mathcal{H}\bar{\pi}_+)$ must be closed, so that the outer factor has a bounded inverse. Secondly, [33, Definition 1.6] of outer operator does not require that range $(\mathcal{H}\bar{\pi}_+)$ should be even dense in $\ell^2(\mathbf{Z}_+; U)$, only that its closure reduces the shift and is thus of form $\ell^2(\mathbf{Z}_+; U')$ for some Hilbert *subspace* $U' \subset U$. The first of these problems is easy to resolve. The coercivity $\bar{\pi}_+\mathcal{H}^*\mathcal{H}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+ > 0$ implies that the Toeplitz operator $\mathcal{H}\bar{\pi}_+$ has a closed range, and thus a bounded (pseudo) inverse.

To attack the second problem, note that $U' \subset U$ implies $\dim U' \leq \dim U$. Also $\dim U' \geq \dim U$ holds because for all $z \in \mathbf{D}$, $\ker(\mathcal{H}(z)) = \{0\}$, by a lengthy calculation omitted here. Because $\dim U' = \dim U$, there is a unitary $E \in \mathcal{L}(U'; U)$ such that $E^*E = I$. Define $\mathcal{X} = E\mathcal{H}$. This is the I-spectral factor $\mathcal{X}$ of the Popov operator $\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+$ such that $(\mathcal{X}\bar{\pi}_+)^{-1} \in \mathcal{L}(\ell^2(\mathbf{Z}_+; U))$, or equivalently $\mathcal{X}^{-1}$ is bounded . This is also the outer factor (with a bounded inverse) in the (J, I)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, see [19, Proposition 20]. Now condition (ii) of Theorem 27 holds.

If there is a solution in $P \in ric_{uw}(\Phi, J)$ such that $\Lambda_P > 0$, then we obtain the factorization of the Popov operator $\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+ = \bar{\pi}_+\mathcal{D}_{\phi_P}^* J\mathcal{D}_{\phi_P}\bar{\pi}_+$, by [19, Lemma 37] or claim (i) of Theorem 50 of this paper. By definition, $\mathcal{D}_\phi(0) = I$ has a bounded inverse. We can now proceed as above, with $\mathcal{D}$ replaced by $\mathcal{D}_{\phi_P}$. □

For a further comment on the condition $\Lambda_P > 0$, see Lemma 53. The following equivalence is now an immediate corollary:

Corollary 32. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS, such that the input space U is separable. Let the self-adjoint cost operator $J \in \mathcal{L}(Y)$. Then the following are equivalent:*

(i) $\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ *for some* $\epsilon > 0$.

(ii) *The Popov operator $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+$ is nonnegative, and the equivalent conditions of Theorem 27 hold.*

Proof. The implication (i) $\Rightarrow$ (ii) is in Proposition 31. The converse direction is claim (i) of Theorem 27. □

The case when the Popov operator $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+ \geq \epsilon \mathcal{I} > 0$ occurs in applications. Such Popov operators arise e.g. in the study of linear quadratic optimal control problems and in the factorization versions of Bounded and Positive Real Lemmas, see [41, Section 8].

5 Function theoretic definitions and tools

In this section, we present some relevant results from the operator-valued function theory. We work in terms of the nontangential boundary limits (boundary traces) $\mathcal{D}_\phi(e^{i\theta})$ of transfer functions $\mathcal{D}_\phi(e^{i\theta})$ that must now be of bounded type $\mathcal{D}_\phi(z) \in N(\mathbf{D}; \mathcal{L}(U))$. This requires the separability of the Hilbert spaces U and Y, and a compactness assumption of an input operator B of $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, as we shall later see. Then various factorization problems, initially formulated in terms of the I/O-maps (or equivalently: transfer functions) of DLSs, are stated in the language of the boundary trace vector spaces and algebras. These function spaces contain additional structure that gives us stronger results.

Inner and outer transfer functions are defined and investigated. In Proposition 34, we give a sufficient condition for an inner from the left analytic function to be inner (from both sides). In Proposition 36, the inner functions are characterized in the set $H^2(\mathbf{D}; \mathcal{L}(U; Y))$. Transfer functions and boundary traces of outer I/O-maps (having a bounded inverse) are considered in Proposition 37. The I/O-map $\mathcal{D}$ of an I/O stable and J-coercive DLS is the subject of Proposition 38; it is remarkable that we need the boundary traces and separability of the Hilbert spaces to get a bounded, generally noncausal inverse for such $\mathcal{D}$. The Hilbert–Schmidt class of compact operators is introduced in Definition 39. In Lemma 41 and Corollary 42, we use the Hilbert-Schmidt property of the input operator B to make the transfer function of an output stable DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ to be of bounded type. This makes it possible to extend our main results to DLSs whose input space U can be infinite dimensional, at the prize of a compactness assumption. We remark that much of this section can be replaced by trivial arguments, if the input space U of the DLS is finite dimensional.

We start with giving basic definitions. Let $\Theta(z)$ be an analytic $\mathcal{L}(U; Y))$-valued function in $\mathbf{D}$. The adjoint function $\tilde{\Theta}(z)$ is defined by

$$\tilde{\Theta}(z) := \Theta(\bar{z})^* \quad \text{for all} \quad z \in \mathbf{D}.$$

If $\Theta(z) = \sum_{j \geq 0} c_j z^j$ for $\{c_j\} \subset \mathcal{L}(U, Y)$, then $\tilde{\Theta}(z) = \sum_{j \geq 0} c_j^* z^j$. It follows that $\tilde{\Theta}(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U; Y))$ if and only if $\Theta(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U; Y))$. The nontangential boundary limits behave expectedly $\tilde{\Theta}(e^{i\theta}) = \Theta(e^{-i\theta})^*$ a.e. $e^{i\theta} \in \mathbf{T}$.

Definition 33. *Let $\Theta(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U; Y))$, where U and Y are separable. Then*

(i) $\Theta(z)$ is inner from the left if $\Theta(e^{i\theta}) \in \mathcal{L}(U, Y)$ is an isometry a.e. $e^{i\theta} \in \mathbf{T}$,

(ii) $\Theta(z)$ is inner from the right if the adjoint function $\tilde{\Theta}(z)$ is inner from the left,

(iii) $\Theta(z)$ is inner if the nontangential limit $\Theta(e^{it})$ is unitary a.e. $e^{i\theta} \in \mathbf{T}$.

Clearly $\Theta(z)$ is inner from the left if and only if $\tilde{\Theta}(z)$ is inner from the right. The nontangential limit $\Theta(e^{i\theta})$ of the inner from the right function is is co-isometric a.e. $e^{i\theta} \in \mathbf{T}$. Also $\Theta(z)$ is inner if and only if it is inner from the left and right. In this case we can say, for clarity, that $\Theta(z)$ is inner from the both sides or two-sided inner. In [8, p. 234 and 242], $\Theta(z)$ is inner ($*$-inner) if $\Theta(e^{i\theta})$ is isometric (co-isometric, respectively) a.e. $e^{i\theta} \in \mathbf{T}$. The same notation is used in [46, p. 190]. In [33], inner function is an element of $H^\infty(\mathbf{D}; \mathcal{L}(U))$ such that the nontangential boundary values are partial isometries. In several occasions, it will be necessary to conclude that an inner from the left function is in fact inner. If the spaces U and Y are finite dimensional with the same dimension, it is easy to show that inner from the left implies inner from the both sides. This is because all isometries in a finite dimensional space are unitary, by a basic dimension counting argument. If the involved Hilbert spaces are infinite dimensional, much less it true. Fortunately, the inner from the left factors arising from the solutions of DARE (as studied in this paper) have this special property. It is related to the requirement that the indicator Λ_P for all $P \in Ric(\Phi, J)$ must have a bounded inverse.

Proposition 34. *Assume that* $\Theta(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U; Y))$ *is inner from the left. Then* $\Theta(z)$ *is inner if and only if*

$$U = \overline{\operatorname{span}_{z\in\Omega}\{\operatorname{range}(\Theta(z))\}}, \tag{23}$$

where $\Omega \subset \mathbf{D}$ *is any subset that has an accumulation point in* $\mathbf{D}$*. In particular, if* $\Theta(z_0)$ *has a dense range for some* $z_0 \in \mathbf{D}$*, then* $\Theta(z)$ *is inner.*

Proof. We have to show that $\Theta(z)$ is inner from the right if and only if (23) holds. Equivalently, we have to show that $\tilde{\Theta}(z)$ is inner from the left if and only if (23) holds. By [33, Theorems 5.3A, 5.3B and 5.3C], $\tilde{\Theta}(z)$ is inner from the left if and only if

$$\begin{aligned} U = M_{in}(\tilde{\Theta}(z)) &= \overline{\operatorname{span}_{z\in\Omega}\{\operatorname{range}\left(\tilde{\Theta}(z)^*\right)\}} \\ &= \overline{\operatorname{span}_{z\in\Omega}\{\operatorname{range}(\Theta(\bar{z}))\}} = \overline{\operatorname{span}_{z\in\bar{\Omega}}\{\operatorname{range}(\Theta(z))\}} \end{aligned}$$

where $\Omega \subset \mathbf{D}$ (and hence its complex conjugate set $\bar{\Omega}$) is any set having an accumulation point inside $\mathbf{D}$. □

Clearly, if $\Theta(z_0)$ has a bounded inverse for some $z_0 \in \mathbf{D}$, then $\Theta(z)$ is inner from the both sides. However, such z_0 does not necessary exists. For a counter example, consider the "snake function" in $H^\infty(\mathbf{D}; \mathcal{L}(U))$

$$s(z) = \sum_{j\geq 0} P_j z^j$$

where $\{P_j\}_{j\geq 0}$ are one-dimensional, mutually orthogonal projections on a separable infinite-dimensional separable Hilbert space such that $\sum_{j\geq 0} P_j = I$. It follows that $s(z)$ is injective and compact with dense range for all

$z \in \mathbf{D} \setminus \{0\}$, and thus not boundedly invertible. Also $s(0) = P_0$ is rank one. However, the boundary trace $s(e^{i\theta})$ exists on $\mathbf{T}$, and it is unitary — otherwise the boundary behavior of $s(z)$ is very wild. Definitely, the convergence on the nontangential sequences happens only in the strong operator topology, because the ideal of compact operators is closed. We remark that $s(z)$ is an example of an operator-valued bounded analytic function which is "as bad as it gets", in many respects.

The transfer functions of the isometric and unitary Toeplitz operators of I/O-maps $\mathcal{N}\bar{\pi}_+ : \ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; Y)$ are of particular interest.

Proposition 35. *Let $\mathcal{N}$ be an I/O-map of an I/O stable DLS, with U and Y separable. Then $\mathcal{N}\bar{\pi}_+$ is an isometry on $\ell^2(\mathbf{Z}_+; Y)$ (i.e. $\bar{\pi}_+\mathcal{N}^*\mathcal{N}\bar{\pi}_+ = \bar{\pi}_+$, or $\mathcal{N}$ is (I, I)-inner) if and only if the transfer function $\mathcal{N}(z)$ is inner from the left. Furthermore, $\mathcal{N}\bar{\pi}_+$ is unitary if and only if $\mathcal{N}(z)$ is a unitary constant function.*

Proof. This is [8, part (c) of Theorem 1.1 and Corollary 1.2, Chapter IX]. □

In Definition 33, we have required (as usual) that the inner function $\Theta(z)$ is *a priori* in $H^\infty(\mathbf{D}; \mathcal{L}(U; Y))$. This makes it possible to speak about nontangential limits, defined a.e. on $\mathbf{T}$. Actually, it would have been sufficient to require that $\Theta(z)$ lies in $H^2(\mathbf{D}; \mathcal{L}(U; Y))$ or even in $N_+(\mathbf{D}; \mathcal{L}(U; Y))$:

Proposition 36. *Let $T(z) \in H^2(\mathbf{D}; \mathcal{L}(U; Y))$, with U and Y separable. Assume that the nontangential limit satisfies $\operatorname{ess\,sup}_{e^{i\theta} \in \mathbf{T}} ||T(e^{i\theta})|| \leq \infty$. Then $T(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U; Y))$. In particular, if $T(z) \in H^2(\mathbf{D}; \mathcal{L}(U; Y))$, with isometric nontangential limits $T(e^{i\theta})$ a. e. $e^{i\theta} \in \mathbf{T}$, then $T(z)$ is inner from the left.*

Proof. By the same comment that is present in the proof of Proposition 9, we need to consider only the case $Y = U$. In this case, [33, Theorem 4.7A] proves the claim because $H^2(\mathbf{D}; \mathcal{L}(U; Y)) \subset N_+(\mathbf{D}; \mathcal{L}(U; Y))$. □

Now that we have dealt with the matters concerning the boundary behavior of the inner functions, we proceed to study the outer functions. Recall that an I/O stable I/O-map $\mathcal{X}$ is outer with a bounded inverse, if the Toeplitz operator $\mathcal{X}\bar{\pi}_+$ has a bounded inverse in $\ell^2(\mathbf{Z}_+; U)$, see [19, Definition 17].

Proposition 37. *Let $\mathcal{X} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ be an I/O-map of an I/O stable DLS, which is outer with a bounded inverse. Then the following holds:*

(i) *$\mathcal{X}^{-1} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ exists boundedly, and is an I/O-map of an I/O stable DLS.*

(ii) *$\mathcal{X}(z)^{-1} \in \mathcal{L}(U)$ exists for all $z \in \mathbf{D}$, and $\mathcal{X}(z)^{-1} = \mathcal{X}^{-1}(z)$ where $\mathcal{X}^{-1}$ is the I/O-map of the inverse DLS of a realization of $\mathcal{X}$. Furthermore, $\sup_{z \in \mathbf{D}} ||\mathcal{X}(z)^{-1}||_{\mathcal{L}(U)} < \infty$ and thus $\mathcal{X}(z)^{-1} \in H^\infty(\mathbf{D}; \mathcal{L}(U))$.*

(iii) If, in addition, U is separable, then the nontangential boundary limit $\mathcal{X}(e^{i\theta})$ exists and is boundedly invertible a.e. $e^{i\theta} \in \mathbf{T}$. We have $\mathcal{X}(e^{i\theta})^{-1} = \mathcal{X}^{-1}(e^{i\theta})$ a.e. $e^{i\theta} \in \mathbf{T}$ and, in particular, $\mathcal{X}(e^{i\theta})^{-1} \in H^\infty(\mathbf{T}; \mathcal{L}(U))$.

Proof. The proof of claim (i) is the matter of [19, Chapter 4], with slight additions. To prove claim (ii), we show that $\mathcal{X}(z)^{-1} = \mathcal{X}^{-1}(z)$ for all $z \in \mathbf{D}$. Let ϕ' be a realization: $\mathcal{X} = \mathcal{D}_{\phi'}$. Then $\mathcal{X}^{-1} = \mathcal{D}_{\phi'}^{-1} = \mathcal{D}_{(\phi')^{-1}}$, by Proposition 2, and $\mathcal{I} = \mathcal{D}_{(\phi')^{-1}}\mathcal{D}_{\phi'}$. By Corollary 8, $I = \mathcal{D}_{(\phi')^{-1}}(z)\mathcal{D}_{\phi'}(z)$ and $I = \mathcal{D}_{\phi'}(z)\mathcal{D}_{(\phi')^{-1}}(z)$ for all $z \in \mathbf{D}$. It follows that $\mathcal{D}_{\phi'}(z) = \mathcal{X}(z) : U \to U$ is a bounded bijection and has a bounded inverse $\mathcal{X}(z)^{-1}$, for all $z \in \mathbf{D}$. Also $\mathcal{X}(z)^{-1} = \mathcal{D}_{\phi'}^{-1}(z) = \mathcal{X}^{-1}(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U))$, by claim (i). The last claim (iii) follows now from the theory of nontangential boundary limits of H^∞-functions, see the discussion following Definition 12 or [33, p. 88]. □

An an important application, we consider the noncausal shift-invariant inverse of the I/O-map. This result is used in Lemma 53.

Proposition 38. *Let $J \in \mathcal{L}(Y)$ be self-adjoint. Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and J-coercive DLS, with input space U and output space Y. Then*

(i) both the Toeplitz operator $\mathcal{D}\bar{\pi}_+ : \ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; Y)$ and the I/O-map $\mathcal{D} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$ are coercive.

(ii) Assume, in addition, that U and Y are separable, and the feed-through operator $D \in \mathcal{L}(U; Y)$ of Φ is injective with a dense range. Then $\overline{\mathrm{range}\,(\mathcal{D})} = \ell^2(\mathbf{Z}; Y)$. In this case $\mathcal{D}^{-1} : \ell^2(\mathbf{Z}; Y) \to \ell^2(\mathbf{Z}; U)$ exists, is bounded and shift-invariant. ($\mathcal{D}^{-1}$ is not causal, unless $\mathcal{D}$ is outer with a bounded inverse.)

Proof. The claim about Toeplitz operator $\mathcal{D}\bar{\pi}_+$ is [19, Proposition 6]. It is easy to see, by a density argument and shift invariance of $\mathcal{D}$, that $\mathcal{D}\bar{\pi}_+$ and $\mathcal{D}$ are simultaneously coercive in the indicated spaces.

Consider now claim (ii). Because of the separability of the spaces U and Y, we can study the problem in terms of multiplication operators on the nontangential boundary limits. Because $\mathcal{D}\bar{\pi}_+$ is coercive, it follows that the Popov operator $\bar{\pi}_+\mathcal{D}^*\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+ > 0$. Now Corollary 32 implies that we have the factorization $\mathcal{D} = \mathcal{N}'\mathcal{X}'$, where $\mathcal{N}'$ is (I, I)-inner and $\mathcal{X}'$ is outer with a bounded inverse. On the boundary, this means

$$\mathcal{D}(e^{i\theta}) = \mathcal{N}'(e^{i\theta})\mathcal{X}'(e^{i\theta}) \tag{24}$$

a.e. $e^{i\theta} \in \mathbf{T}$. The boundary trace of the inner (from the left) factor $\mathcal{N}'(e^{i\theta}) \in \mathcal{L}(U; Y)$ is isometric almost everywhere: $\mathcal{N}'(e^{i\theta})^*\mathcal{N}'(e^{i\theta}) = I_U$ a.e. $e^{i\theta} \in \mathbf{T}$. The outer factor $\mathcal{X}'(e^{i\theta}) \in \mathcal{L}(U)$ has a a bounded inverse for almost all $e^{i\theta} \in \mathbf{T}$. By Proposition 37, $\mathcal{X}'(e^{i\theta})^{-1} \in H^\infty(\mathbf{T}; \mathcal{L}(U))$.

We now consider the static part $N := \mathcal{N}(0) \in \mathcal{L}(U; Y)$ of the inner factor. By causality, $D = \pi_0\mathcal{N}\mathcal{X}\pi_0 = \pi_0\mathcal{N}\pi_0 \cdot \pi_0\mathcal{X}\pi_0 = NX$, where $X = \mathcal{X}(0) \in \mathcal{L}(U)$ is the feed-through operator of the outer factor $\mathcal{X}$. By Proposition

37, $X^{-1} \in \mathcal{L}(U)$ and $N = DX^{-1}$. It now follows that range (N) is dense, and $\mathcal{N}(e^{i\theta})$ is inner from both sides, by Proposition 34. This means that $\mathcal{N}'(e^{i\theta})\mathcal{N}'(e^{i\theta})^* = I_Y$ a.e. $e^{i\theta} \in \mathbf{T}$. In particular, $\mathcal{N}'(e^{i\theta})^* \in L^\infty(\mathbf{T};\mathcal{L}(Y;U))$.

Now we can attack the claim about the density of range $(\mathcal{D})$. Let $\tilde{y}(e^{i\theta}) \in L^2(\mathbf{T};Y)$ be arbitrary. Define $\tilde{w}(e^{i\theta}) := \mathcal{N}'(e^{i\theta})^*\tilde{y}(e^{i\theta})$ away from a set of measure zero. Because $\mathcal{N}'(e^{i\theta})^* \in L^\infty(\mathbf{T};\mathcal{L}(Y;U))$ and $\tilde{y}(e^{i\theta}) \in L^2(\mathbf{T};Y)$, [8, part (a) of Theorem 1.1, Chapter IX] implies that $\tilde{w}(e^{i\theta}) \in L^2(\mathbf{T};U)$. Similarly, $\tilde{u}(e^{i\theta}) := \mathcal{X}'(e^{i\theta})^{-1}\tilde{w}(e^{i\theta}) \in L^2(\mathbf{T};U)$. But now, $\mathcal{D}(e^{i\theta})\tilde{u}(e^{i\theta}) = \tilde{y}(e^{i\theta})$ almost everywhere. Because $\tilde{y}(e^{i\theta})$ is arbitrary, this means in the time domain that range $(\mathcal{D}) = \ell^2(\mathbf{Z};Y)$ because the Fourier transform is an isometric isomorphism. The shift invariance of $\mathcal{D}^{-1}$ follows from [8, part (a) of Theorem 1.1, Chapter IX], too. □

As we have stated earlier, functions in the Nevanlinna class $N(\mathbf{D};X)$ can be adequately described by their nontangential boundary limit functions for $X = U$, $X = \mathcal{L}(U)$ or $X = \mathcal{L}(U;Y)$, when U and Y are separable Hilbert spaces. Unfortunately, a general $\mathrm{sH}^2(\mathbf{D};\mathcal{L}(U;Y))$ function need not be in $N(\mathbf{D};\mathcal{L}(U;Y))$ if $\dim U = \infty$. It is even more unfortunate that the strong H^2-stability of the transfer function is an important notion because output stability of its realization implies it. From the state space representation of a transfer function, output stability of the realization is often best we can achieve by Liapunov type methods. The I/O stability is not "built" into the state space model as conveniently as the output stability.

So, in order to work with the nontangential limit function $\mathcal{D}_\phi(e^{i\theta})$, we have to make a extra assumption on the output stable DLS ϕ, as will be done in Lemma 41. The question is about a compactness assumption of the input operator B which, in a sense, forbids the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ to be "too" infinite-dimensional. With this restriction, we can conclude that $\mathcal{D}_\phi(z) \in H^2(\mathbf{D};\mathcal{L}(U;Y)) \subset N_+(\mathbf{D};\mathcal{L}(U;Y))$, by Lemma 41.

Definition 39. *Let H_1, H_2 be separable Hilbert spaces, and $T \in \mathcal{L}(H_1, H_2)$. Let $\{e_j\}_{j\geq 0}$ be an orthonormal base for H_1. T is a Hilbert–Schmidt operator if*

$$||T||_{HS}^2 := \sum_{j\geq 0} ||Te_j||_{H_2}^2$$

is finite. In this case we write $T \in HS = HS(H_1;H_2)$. The number $||T||_{HS}$ is called the Hilbert–Schmidt -norm of T.

It can be shown that the class HS is well defined, and the norm $||\cdot||_{HS}$ is independent of the choice of the basis $\{e_j\}_{j\geq 0}$. All Hilbert–Schmidt operators are compact, and each finite dimensional operator is trivially Hilbert–Schmidt. In the matrix case, HS-norm is the familiar Frobenius matrix norm. The set HS is a vector space, and the norm $||\cdot||_{HS}$ makes it a Banach algebra where the involution $*$ satisfies $||T||_{HS} = ||T^*||_{HS}$, provided $H_1 = H_2$. HS is also a Hilbert space under the inner product

$$[T_1, T_2]_{HS} := \sum_{j\geq 0} \langle T_1 e_j, T_2^* e_j\rangle.$$

The Hilbert–Schmidt operators are exactly those compact operators T whose singular values satisfy $\sum_{j\geq 0} \sigma_j(T)^2 < \infty$. A good general reference here is [5, Chapter XI.6]. However, the following fact is important enough to be stated separately:

Proposition 40. *Let $T \in HS(H_1; H_2)$ and $S \in \mathcal{L}(H_2; H_3)$. Then $ST \in HS(H_1; H_2)$ and $||ST||_{HS(H_2;H_3)} \leq ||S||_{\mathcal{L}(H_2;H_3)} \, ||T||_{HS(H_1;H_2)}$.*

Proof. The following calculation proves the claim:

$$\begin{aligned} ||ST||_{HS(H_2;H_3)} &= \sum_{j\geq 0} ||STe_j||_{H_3}^2 \\ &\leq \sum_{j\geq 0} ||S||_{\mathcal{L}(H_2;H_3)} \, ||Te_j||_{H_2}^2 = ||S||_{\mathcal{L}(H_2;H_3)} \sum_{j\geq 0} ||Te_j||_{H_2}^2. \end{aligned}$$

□

Lemma 41. *Let $\Theta(z) \in \mathrm{sH}^2(\mathbf{D}; \mathcal{L}(U; Y))$, with U and Y separable. Assume that the linear mapping*

$$U \ni u \mapsto \Theta(z)u \in H^2(\mathbf{D}; Y) \tag{25}$$

is a Hilbert–Schmidt operator. Then $\Theta(z) \in H^2(\mathbf{D}; \mathcal{L}(U; Y))$.

Proof. Let $\{e_j\}_{j\geq 0}$ be an countable orthonormal basis for the separable U. Define the analytic functions $\Theta_j(z) := \Theta(z)e_j$. Each $\Theta_j(z)$ belongs to $H^2(\mathbf{D}; Y)$ because $\Theta(z) \in \mathrm{sH}^2(\mathbf{D}; \mathcal{L}(U; Y))$. The Hilbert–Schmidt assumption means that

$$\sum_{j\geq 0} ||\Theta_j(z)||_{H^2(\mathbf{D};Y)}^2 < \infty, \tag{26}$$

where

$$||\Theta_j(z)||_{H^2(\mathbf{D};Y)}^2 := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} ||\Theta_j(re^{i\theta})||_Y^2 \, d\theta.$$

For all $z \in D$, $\Theta(z) \in \mathcal{L}(U; Y)$. Let $u = \sum_{j\geq 0} c_j e_j \in U$ be arbitrary, such that only a finite number of c_j's are nonzero. Then for all $z \in \mathbf{D}$ we have

$$||\Theta(z)u||_Y^2 = ||\sum_{j\geq 0} c_j \, \Theta_j(z)||_Y^2 \leq \sum_{j\geq 0} |c_j|^2 \sum_{j\geq 0} ||\Theta_j(z)||_Y^2 = ||u||_U^2 \cdot \sum_{j\geq 0} ||\Theta_j(z)||_Y^2$$

Because above the set of u's is dense in U, it follows

$$||\Theta(z)||_{\mathcal{L}(U;Y)}^2 \leq \sum_{j\geq 0} ||\Theta_j(z)||_Y^2 \tag{27}$$

for all $z \in \mathbf{D}$.

Now, let $0 < r < 1$ be arbitrary. Then each function $e^{i\theta} \mapsto ||\Theta_j(re^{i\theta})||_Y^2$ is a smooth (and thus a measurable) function, by the analyticity of $\Theta_j(z)$ in D. The function $e^{i\theta} \mapsto \sum_{j\geq 0} ||\Theta_j(re^{i\theta})||_Y^2$ is measurable because the partial sums are increasing, and the supremum of a countable collection of measurable functions is measurable, by [34, Theorem 1.14]. Similarly, because $\Theta(z)$ is analytic inside $\mathbf{D}$, the function $e^{i\theta} \mapsto ||\Theta(re^{i\theta})||_Y^2$ is measurable, too. Now equation (27) gives for all $0 < r < 1$

$$\frac{1}{2\pi}\int_0^{2\pi} ||\Theta(re^{i\theta})||_{\mathcal{L}(U;Y)}^2 \, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{j\geq 0} ||\Theta_j(re^{i\theta})||_Y^2 \right) d\theta \tag{28}$$

$$= \sum_{j\geq 0} \left(\frac{1}{2\pi}\int_0^{2\pi} ||\Theta_j(re^{i\theta})||_Y^2 \, d\theta \right),$$

where the latter equality is by the Lebesque's Monotone Convergence theorem [34, Theorem 1.26] implies (or its immediate corollary [34, Theorem 1.27]), because the partial sums are nondecreasing. Taking supremum over r, gives

$$||\Theta(z)||_{H^2(\mathbf{D};\mathcal{L}(U;Y))}^2 \leq \sum_{j\geq 0} \left(\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} ||\Theta_j(re^{i\theta})||_Y^2 \, d\theta \right)$$

$$= \sum_{j\geq 0} ||\Theta_j(z)||_{H^2(\mathbf{D};Y)}^2.$$

Using the Hilbert–Schmidt assumption in the form of equation (26) shows that $\Theta(z) \in H^2(\mathbf{D};\mathcal{L}(U;Y))$. The proof is now complete. □

Corollary 42. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable DLS, such that the spaces U and Y are separable. Assume that the input operator $B \in \mathcal{L}(U;H)$ is Hilbert–Schmidt. Then $\mathcal{D}_\phi(z) \in H^2(\mathbf{D};\mathcal{L}(U;Y))$.*

Proof. Because ϕ is output stable, $\mathcal{D}_\phi(z) - D \in \mathrm{sH}^2(\mathbf{D};\mathcal{L}(U;Y))$, by Proposition 11. We also have $(\mathcal{D}_\phi(z) - D)u_0 = \sum_{j\geq 1} CA^{j-1}Bu_0z^j = z\cdot(\mathcal{F}_z\mathcal{C}_\phi Bu_0)(z)$, where $\mathcal{F}_z$ denotes the unitary z-transform from $\ell^2(\mathbf{Z}_+;Y)$ onto $H^2(\mathbf{D};Y)$. By output stability, the composition $\mathcal{F}_z\mathcal{C}_\phi : H \to H^2(\mathbf{D};Y)$ is well defined and bounded. It follows from Proposition 40 that the mapping

$$U \ni u_0 \mapsto (\mathcal{F}_z\mathcal{C}_\phi Bu_0)(z) \in H^2(Y)$$

is Hilbert–Schmidt because the input operator B is. Because the multiplication of the variable z in $H^2(\mathbf{D};Y)$ is isometric, the mapping

$$U \ni u_0 \mapsto (\mathcal{D}_\phi(z) - D)u_0 \in H^2(Y)$$

is Hilbert–Schmidt. Lemma 41 implies now that $\mathcal{D}_\phi(z) - D \in H^2(\mathbf{D};\mathcal{L}(U;Y))$. This completes the proof. □

The same conclusion can be made, if A^jB is Hilbert–Schmidt, for some $j \geq 0$.

6 Factorization of the truncated Popov operator

Our main interest is in the H^∞DARE, associated to an output stable and I/O stable DLS Φ. As we have seen, this stability requirement makes some solution of DARE more interesting than others. In Section 3 we have sorted out the more interesting solutions from the less interesting.

In this section, we consider additional conditions that make the spectral DLS ϕ_P is either output stable, or I/O stable, or both, for a particular $P \in Ric(\Phi, J))$. More specifically, we introduce additional assumptions that allow us to conclude

$$P \in Ric(\Phi, J) \Rightarrow P \in ric(\Phi, J),$$

when Φ is known to be output stable and I/O stable. The basic tool to obtain the most general of these results is the factorization of the truncated Popov operator, as given in Lemma 45.

Let us first discuss the trivial cases. If Φ itself is power stable, then so are ϕ_P for all $P \in Ric(\Phi, J)$ because they have a common semigroup generator A. More generally, if the Wiener class type condition $\sum ||A^j B|| < \infty$ holds, then $\mathcal{D}_{\phi_P}$ is I/O stable for all $P \in Ric(\Phi, J)$. Now the common input structure (i.e. the common operators A and B) determine the I/O stability of both the systems Φ and ϕ_P. In the case when Φ is output stable and I/O stable, it is easy to see that ϕ_P is I/O stable (output stable) if and only if $\phi' = \left(\begin{smallmatrix} A & B \\ B^*P & 0 \end{smallmatrix} \right)$ is I/O stable (output stable, respectively) but this is just a restatement that is impossible to use in practice.

More general results are obtained by Liapunov type methods that require some type of nonnegativity, either in the cost operator J, the Popov operator $\mathcal{D}^* J \mathcal{D}$, or indicator Λ_P of the solution P. We start with discussing the case of output stability.

Proposition 43. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right]$ be an output stable DLS and $J \in \mathcal{L}(Y)$ be a self-adjoint operator. Let $P \in Ric(\Phi, J)$ such that $\Lambda_P > 0$. Then*

(i) ϕ_P is output stable if and only if the strong limit $L_{A,P} := \mathrm{s} - \lim_{j \to \infty} A^{*j} P A^j$ exists as a bounded operator. When this equivalence holds, we have

$$L_{A,P} - P = \mathcal{C}_{\phi_P}^* \Lambda_P \mathcal{C}_{\phi_P} - \mathcal{C}^* J \mathcal{C}. \tag{29}$$

(ii) In particular, if A is strongly stable, then ϕ_P is output stable.

(iii) If $P \geq 0$ and $L_{A,P} = 0$, we have

$$\mathcal{C}^* J \mathcal{C} \geq \mathcal{C}^* J \mathcal{C} - P = \mathcal{C}_{\phi_P}^* \Lambda_P \mathcal{C}_{\phi_P}$$

Proof. We prove one direction of claim (i). Assume that $\Lambda_P > 0$ and $L_{A,P} = \mathrm{s} - \lim_{j\to\infty} A^{*j}PA^j$ exists. We can iterate on the Riccati equation (19) and obtain for all $j \geq 0$

$$A^{*(j+1)}PA^{j+1} - A^{*j}PA^j = A^{*j}K_P^*\Lambda_P K_P A^j - A^{*j}C^*JCA^j.$$

Telescope summing this up to $n \geq 0$ gives for all $x_0 \in H$

$$\text{(30)} \qquad \langle x_0, (A^{*n}PA^n - P)x_0\rangle = \left\langle x_0, \sum_{j=0}^{n-1} A^{*j}K_P^*\Lambda_P K_P A^j x_0 \right\rangle - \left\langle x_0, \sum_{j=0}^{n-1} A^{*j}C^*JCA^j x_0 \right\rangle$$

By assumption, the left hand side of the previous equation converges to a finite limit $\langle x_0, (L_{A,P} - P)x_0\rangle$. On the right hand side, we have

$$\left\langle x_0, \sum_{j=0}^{n-1} A^{*j}C^*JCA^j x_0 \right\rangle = \sum_{j=0}^{n-1} \left\langle CA^j x_0, JCA^j x_0 \right\rangle = \left\langle \pi_{[0,n-1]}\mathcal{C}x_0, J\pi_{[0,n-1]}\mathcal{C}x_0 \right\rangle_{\ell^2(\mathbf{Z}_+;Y)}$$

which converges absolutely to a bounded limit $\langle x_0, \mathcal{C}^*J\mathcal{C}x_0\rangle$ as $n \to \infty$, by the assumed output stability of Φ.

Because everything else in (30) converges to a finite limit and $\Lambda_P > 0$, it follows that remaining term

$$\left\langle x_0, \sum_{j=0}^{n-1} A^{*j}K_P^*\Lambda_P K_P A^j x_0 \right\rangle = \sum_{j=0}^{n-1} \left\langle K_P A^j x_0, \Lambda_P K_P A^j x_0 \right\rangle = ||\Lambda_P^{\frac{1}{2}}\pi_{[0,n-1]}\mathcal{C}_{\phi_P}x_0\}||^2_{\ell^2(\mathbf{Z}_+;U)}$$

converges (increases) to a finite limit, equalling $||\{\Lambda_P^{\frac{1}{2}}K_P A^j x_0\}_{j\geq 0}||^2_{\ell^2(\mathbf{Z}_+;U)}$, as $n \to \infty$. Because Λ_P^{-1} is bounded and $x_0 \in H$ arbitrary, this is equivalent to the output stability of ϕ_P. This completes the proof of the first direction. The converse part in contained in the proof of Proposition 23 where also equation (29) is given. Claim (ii) follows trivially from the fact that strongly stable A implies that the strong limit operator $L_{A,P}$ always exists and equals 0. Claim (iii) is a trivial consequence of equation (29). □

Corollary 44. *Let $J \in \mathcal{L}(Y)$ be self-adjoint. Assume that ϕ is a I/O stable and output stable DLS, such that $\overline{\mathrm{range}\,(\mathcal{B})} = H$. Then $ric_{uw}(\phi, J) = ric_0(\phi, J)$.*

Proof. Trivially $ric_0(\phi, J) \subset ric_{uw}(\phi, J)$, and the converse inclusion is shown below. Because $P \in ric_{uw}(\phi, J)$, both ϕ and ϕ_P are output stable. We have for all $j \geq 1$

$$A^{*j}PA^j - P = \mathcal{C}_{\phi_P}^*\Lambda_P\pi_{[0,j-1]}\mathcal{C}_{\phi_P} - \mathcal{C}^*J\pi_{[0,j-1]}\mathcal{C},$$

as in equation (22) of Proposition 23. By the output stabilities, both $\pi_{[0,j-1]}\mathcal{C} \to \mathcal{C}$ and $\pi_{[0,j-1]}\mathcal{C}_{\phi_P} \to \mathcal{C}_{\phi_P}$ strongly. It follows that $L_{A,P}$ exists and $P \in Ric_{00}(\phi, J)$. Now claim (iv) of Proposition 22, together with the assumed approximate controllability, shows that $P \in Ric_0(\phi, J)$. □

We proceed to study the I/O stability of the spectral DLS ϕ_P. For solutions such that $\lim_{j\to\infty} \langle P\mathcal{B}\tau^{*j}\tilde{u}, \mathcal{B}\tau^{*j}\tilde{u}\rangle = 0$ for all $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$, a necessary and sufficient condition for ϕ_P to be I/O stable is the following speed estimate

$$\sum_{j\geq 0} |\, \langle x_j, Px_j\rangle - \langle x_{j+1}, Px_{j+1}\rangle \,| < \infty$$

for all trajectories $x_j = \mathcal{B}\tau^{*j}\tilde{u}$ where $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ is arbitrary, see [19, Proposition 41 with a slight modification]. The good thing in this condition is that is does not require nonnegativity of any kind, and that it has a game theoretic interpretation. Unfortunately, this condition is not practical for our purposes.

We continue by giving an unsuccessful attempt that, however, reveals something about the nature of the problem. Assume that Φ is input stable and I/O stable, and $J \geq 0$. Suppose we already know ϕ_P to be output stable. Claim (iii) of Proposition 43 implies that

$$\infty > ||\pi_-\mathcal{D}^* J \mathcal{D}\pi_-|| \geq \mathcal{B}^*\mathcal{C}^* J \mathcal{C}\mathcal{B} \geq \mathcal{B}^*_{\phi_P}\mathcal{C}^*_{\phi_P}\Lambda_P\mathcal{C}_{\phi_P}\mathcal{B}_{\phi_P},$$

if $P \geq 0$ and $L_{A,P} = 0$, because $\mathcal{B}_{\phi_P} = \mathcal{B}$. So the Hankel operator $\mathcal{C}_{\phi_P}\mathcal{B}_{\phi_P} = \bar{\pi}_+\mathcal{D}_{\phi_P}\pi_-$ is bounded in $\ell^2(\mathbf{Z}; U)$, but this does not allow us directly conclude the I/O stability of $\mathcal{D}_{\phi_P}$.

We are not far from having ϕ_P I/O stable, provided that we have the *a priori* knowledge that $\mathcal{D}_{\phi_P}(z) \in N(\mathbf{D}; \mathcal{L}(U))$ so that the nontangential limit function $\mathcal{D}_{\phi_P}(e^{i\theta})$ makes sense. More precisely, denote by Γ the bounded Hankel operator $\mathcal{C}_{\phi_P}\mathcal{B}_{\phi_P}$, and assume, for simplicity that everything is complex-valued, i.e. $U = Y = \mathbf{C}$. By [8, Theorem 3.3, Chapter IX], $\Gamma = \Gamma(Q)$, where $Q(e^{i\theta}) \in L^\infty(\mathbf{T}; d\theta)$ is a *bounded* symbol for Γ (we have omitted one unitary flip operator in the definition of the Hankel operator but this is immaterial). Write $Q(e^{i\theta})$ as the Fourier series $Q(e^{i\theta}) \sim \sum_{j\in\mathbf{Z}} q_j\, e^{ij\theta}$. Now $q_j = -K_P A^{j-1} B$ for $j \geq 1$ because $\mathcal{D}_{\phi_P}(e^{i\theta})$ is also a (possibly unbounded) symbol for Γ. It is well known that $L^\infty(\mathbf{T}; d\theta) \subset L^p(\mathbf{T}; d\theta)$ for all $1 < p < \infty$, and that the Szegö projection $\Pi : L^p \to H^p$ (zeroing the negatively indexed Fourier coefficients) is bounded for $1 < p < \infty$. But now $\mathcal{D}_{\phi_P}(e^{i\theta}) = \Pi Q(e^{i\theta}) \in \cap_{1<p<\infty} H^p(\mathbf{T}; \mathbf{C})$. Unfortunately, the inclusion $H^\infty(\mathbf{T}; \mathbf{C}) \subset \cap_{1<p<\infty} H^p(\mathbf{T}; \mathbf{C})$ is strict, and we cannot conclude $\mathcal{D}_{\phi_P}(e^{i\theta}) \in H^\infty(\mathbf{T}; \mathbf{C})$.

After one impractical and another unsuccessful attempt, we approach the I/O stability problem of ϕ_P from a third direction. We begin with factorization lemma of the truncated Popov operator for strongly H^2 stable DLSs. Recall that impulse response operator $\mathcal{D}\pi_0 : U \to \ell^2(\mathbf{Z}_+; Y)$ of a strongly H^2 stable DLS is bounded, by definition. It then immediately follows, by the shift invariance, that the truncated Toeplitz operators $\mathcal{D}\pi_{[0,m]}$ are bounded, for all $m \geq 0$.

Lemma 45. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ strongly H^2 stable. Let $P \in Ric_{uw}(\Phi, J)$; i.e.*

$$\left\langle PA^j x_0, A^j x_0 \right\rangle \to 0 \quad \textit{for all} \quad x_0 \in \mathrm{range}\,(\mathcal{B}) \tag{31}$$

as $j \to \infty$. Assume also that the spectral DLS ϕ_P is strongly H^2 stable.

Then $\mathcal{D}\pi_{[0,m]} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$ and $\mathcal{D}_{\phi_P}\pi_{[0,m]} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ are bounded, and the truncated Popov operator has the factorization

$$(\mathcal{D}\pi_{[0,m]})^* J \mathcal{D}\pi_{[0,m]} = (\mathcal{D}_{\phi_P}\pi_{[0,m]})^* \Lambda_P \mathcal{D}_{\phi_P}\pi_{[0,m]} \tag{32}$$

for all $m \geq 0$.

Proof. Let $x_0 \in H$ and $\{u_j\}_{j \geq 0} = \tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ be arbitrary. Denote $x_j = x_j(x_0, \tilde{u}) = A^j x_0 + \mathcal{B}\tau^{*j}\tilde{u}$ the trajectory of Φ with this given initial state and input. We have in [19, claim (i) of Proposition 36] for all $n > 0$

(33)

$$\begin{aligned} &\langle Px_0, x_0\rangle - \langle Px_n, x_n\rangle \\ &= \sum_{j=0}^{n-1} \langle J(Cx_j + Du_j), Cx_j + Du_j\rangle - \sum_{j=0}^{n-1} \langle \Lambda_P(-K_P x_j + u_j), -K_P x_j + u_j)\rangle. \end{aligned}$$

Consider now the special case when the input is otherwise arbitrary, but of form $\tilde{u} = \pi_{[0,m]}\tilde{u}$, for $m \geq 0$. Then, for $n > m$,

$$\begin{aligned} x_n &= x_n(x_0, \pi_{[0,m]}\tilde{u}) = A^{n-m-1} \cdot x_{m+1}(x_0, \pi_{[0,m]}\tilde{u}), \\ x_{m+1}(x_0, \pi_{[0,m]}\tilde{u}) &= A^{m+1}x_0 + \mathcal{B}\tau^{*(m+1)}\pi_{[0,m]}\tilde{u}. \end{aligned}$$

Let $x_0 = 0$. Because now $x_{m+1}(0, \pi_{[0,m]}\tilde{u}) \in \mathrm{range}\,(\mathcal{B})$, it follows from the residual cost condition (31) that $\langle Px_n, x_n\rangle \to 0$ as $n \to \infty$. It follows that the left hand side of (33) vanishes as $n \to \infty$.

We must now consider the right hand side of (33). Because both the operators $\mathcal{D}\pi_{[0,m]}$ and $\mathcal{D}_{\phi_P}\pi_{[0,m]}$ are bounded, by the H^2-stability assumption of ϕ_P, it is not difficult to see that the limit of the left hand side of (33) is actually

$$\left\langle J\mathcal{D}\pi_{[0,m]}\tilde{u}, \mathcal{D}\pi_{[0,m]}\tilde{u}\right\rangle_{\ell^2(\mathbf{Z}_+;Y)} - \left\langle \Lambda_P \mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u}, \mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u}\right\rangle_{\ell^2(\mathbf{Z}_+;Y)},$$

as $n \to \infty$. Adjoining this gives

$$\left\langle \tilde{u}, \left((\mathcal{D}\pi_{[0,m]})^* J \mathcal{D}\pi_{[0,m]} - (\mathcal{D}_{\phi_P}\pi_{[0,m]})^* \Lambda_P \mathcal{D}_{\phi_P}\pi_{[0,m]}\right) \tilde{u}\right\rangle_{\ell^2(\mathbf{Z}_+;Y)} = 0$$

for all $\tilde{u} \in \ell^2(\mathbf{Z}; U)$. Now an application of [35, Theorem 12.7] completes the proof. □

The result of the previous lemma can be translated to the frequency plane by Corollary 42, provided that the input operator is Hilbert–Schmidt. With this additional structure, further conclusions can be drawn.

Proposition 46. *Let J be a self-adjoint cost operator. Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}_\tau{}^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ output stable, such that the input operator B is Hilbert–Schmidt and the input space U is separable. Let $P \in Ric_{uw}(\Phi, J)$ be such that ϕ_P is output stable.*

Then the adjoints of the boundary traces $\mathcal{D}(e^{i\theta})^*$ and $\mathcal{D}_{\phi_P}(e^{i\theta})^*$ exists a.e. $e^{i\theta} \in \mathbf{T}$, and belong to $L^2(\mathbf{T}; \mathcal{L}(U;Y))$, $L^2(\mathbf{T}; \mathcal{L}(U))$, respectively. Both the self-adjoint operator-valued functions

$$\begin{aligned} &\mathbf{T} \ni e^{i\theta} \mapsto \mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta}) \in \mathcal{L}(U), \quad \textit{and} \\ &\mathbf{T} \ni e^{i\theta} \mapsto \mathcal{D}_{\phi_P}(e^{i\theta})^* \Lambda_P \mathcal{D}_{\phi_P}(e^{i\theta}) \in \mathcal{L}(U) \end{aligned}$$

are in $L^1(\mathbf{T}; \mathcal{L}(U))$. We have the factorization

$$\mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta}) = \mathcal{D}_{\phi_P}(e^{i\theta})^* \Lambda_P \mathcal{D}_{\phi_P}(e^{i\theta}) \quad a.e. \quad e^{i\theta} \in \mathbf{T}.$$

Proof. Recall that the output stability implies strong H^2-stability. So we can apply Lemma 45. Equation (32) implies for all $\tilde{u}_1, \tilde{u}_2 \in \ell^2(\mathbf{Z}_+; U)$

$$\left\langle \mathcal{D}\pi_{[0,m]}\tilde{u}_1, J\mathcal{D}\pi_{[0,m]}\tilde{u}_2 \right\rangle_{\ell^2(\mathbf{Z}_+;Y)} = \left\langle \mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u}_1, \Lambda_P \mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u}_2 \right\rangle_{\ell^2(\mathbf{Z}_+;Y)}.$$

Because both Φ and ϕ_P are output stable, the transfer functions $\mathcal{D}(z)$ and $\mathcal{D}_{\phi_P}(z)$ are analytic in the whole of $\mathbf{D}$, by Proposition 11. We have also $\mathcal{D}(z)\tilde{p}(z) \in H^2(\mathbf{D}; Y)$, $\mathcal{D}_{\phi_P}(z)\tilde{p}(z) \in H^2(\mathbf{D}; U)$ for all U-valued trigonometric polynomials $p(z) \in H^\infty(\mathbf{D}; U)$. Now we can put the factorization in form

$$\langle \mathcal{D}(z)p_1(z), J\mathcal{D}(z)p_2(z) \rangle_{H^2(\mathbf{D};Y)} = \langle \mathcal{D}_{\phi_P}(z)p_1(z), \Lambda_P \mathcal{D}_{\phi_P}(z)p_2(z) \rangle_{H^2(\mathbf{D};U)}$$

where $p_1(z), p_2(z)$ are polynomials as above. This is as far as we get without assuming that B is Hilbert–Schmidt.

Because B is Hilbert–Schmidt, we can state the factorization in terms of the boundary traces $\mathcal{D}(e^{i\theta}) \in H^2(\mathbf{T}; \mathcal{L}(U;Y))$ and $\mathcal{D}_{\phi_P}(e^{i\theta}) \in H^2(\mathbf{T}; \mathcal{L}(U))$, by Corollary 42. By choosing the trigonometric polynomials $p_1(e^{i\theta}) = e^{ip_1\theta}u_1$ and $p_2(e^{i\theta}) = e^{ip_2\theta}u_2$, $p_1, p_2 \in \mathbf{Z}$, $u_1, u_2 \in U$, we obtain

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \left\langle u_1, \mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta}) e^{ip\theta} u_2 \right\rangle_U \, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle \mathcal{D}(e^{i\theta}) e^{ip_1\theta} u_1, J\mathcal{D}(e^{i\theta}) e^{ip_2\theta} u_2 \right\rangle_Y \, d\theta \end{aligned}$$

$$= \left\langle \mathcal{D}(e^{i\theta})e^{ip_1\theta}u_1, J\mathcal{D}(e^{i\theta})e^{ip_2\theta}u_2 \right\rangle_{H^2(\mathbf{T};Y)}$$
$$= \left\langle \mathcal{D}_{\phi_P}(e^{i\theta})e^{ip_1\theta}u_1, \Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta})e^{ip_2\theta}u_2 \right\rangle_{H^2(\mathbf{T};U)}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \left\langle \mathcal{D}(e^{i\theta})e^{ip_1\theta}u_1, J\mathcal{D}(e^{i\theta})e^{ip_2\theta}u_2 \right\rangle_Y \, d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \left\langle u_1, \mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta})e^{ip\theta}u_2 \right\rangle_U \, d\theta,$$

where $p = p_2 - p_1$. Let us stop for a moment to see that previous is true integration theoretically. The functions $\mathbf{T} \ni e^{i\theta} \mapsto \mathcal{D}(e^{i\theta})^* \in \mathcal{L}(Y;U)$, $\mathbf{T} \ni e^{i\theta} \mapsto \mathcal{D}_{\phi_P}(e^{i\theta})^* \in \mathcal{L}(U)$ are weakly measurable and also in the respective L^2-spaces, by a trivial argument involving adjoining. Now the products $\mathcal{D}(e^{i\theta})^*J\mathcal{D}(e^{i\theta})$ and $\mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta})$ are weakly measurable, and they both are in $L^1(\mathbf{T};U)$, by the Hölder inequality; some of this detail and further references have been discussed immediately after Definition 13.

We can now calculate the weak Fourier coefficients of the difference of these two functions (which lies in $L^1(\mathbf{T};\mathcal{L}(U))$) as follows:

$$\left\langle u_1, \left(\int_0^{2\pi} \left[\mathcal{D}(e^{i\theta})^*J\mathcal{D}(e^{i\theta}) - \mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta})\right] e^{ip\theta}\, d\theta\right) u_2 \right\rangle_U$$
$$= \int_0^{2\pi} \left\langle u_1, \left[\mathcal{D}(e^{i\theta})^*J\mathcal{D}(e^{i\theta}) - \mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta})\right] e^{ip\theta}u_2 \right\rangle_U \, d\theta = 0$$

for all $u_1, u_2 \in U$ and $p \in \mathbf{Z}$. Proposition 17 implies that

$$\left[\mathcal{D}(e^{i\theta})^*J\mathcal{D}(e^{i\theta}) - \mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta})\right] u = 0,$$

for all $u \in U$ and $e^{i\theta} \in \mathbf{T} \setminus E_u$, where $mE_u = 0$. Choose a countable dense subsequence $\{u_j\} \in U$, and define the exceptional set $E := \cup_j E_{u_j}$ of measure zero. Because $\mathcal{D}(e^{i\theta})^*J\mathcal{D}(e^{i\theta}) - \mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta}) \in \mathcal{L}(U)$ for all $e^{i\theta} \in \mathbf{T} \setminus E'$, $mE' = 0$, we conclude now that

$$\mathcal{D}(e^{i\theta})^*J\mathcal{D}(e^{i\theta}) - \mathcal{D}_{\phi_P}(e^{i\theta})^*\Lambda_P\mathcal{D}_{\phi_P}(e^{i\theta}) = 0$$

$e^{i\theta} \in \mathbf{T} \setminus (E' \cup E)$, by the density of the sequence $\{u_j\}$. This completes the proof. □

Corollary 47. *Let J be a self-adjoint cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau{}^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an output stable and I/O stable DLS. Furthermore, assume that the input operator B is Hilbert–Schmidt and the input space U is separable. Let $P \in Ric_{uw}(\Phi, J)$ be such that ϕ_P is output stable.*

If $\Lambda_P > 0$ then ϕ_P is I/O stable, and we can write $P \in ric(\Phi, J)$. Furthermore, we have the inclusion

$$\{P \in Ric_0(\Phi, J) \mid \Lambda_P > 0\} \subset ric_0(\Phi, J) \tag{34}$$

Proof. By Proposition 46, $\mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta}) = \mathcal{D}_{\phi_P}(e^{i\theta})^* \Lambda_P \mathcal{D}_{\phi_P}(e^{i\theta})$ a.e. $e^{i\theta} \in \mathbf{T}$. By the assumed I/O stability of Φ, $\operatorname{ess\,sup}_{e^{i\theta} \in T} ||\mathcal{D}(e^{i\theta})|| < \infty$. We conclude that $\operatorname{ess\,sup}_{e^{i\theta} \in \mathbf{T}} ||\Lambda_P^{\frac{1}{2}} \mathcal{D}_{\phi_P}(e^{i\theta})|| < \infty$. The output stability of ϕ_P and the Hilbert–Schmidt compactness of B imply that $\Lambda_P^{\frac{1}{2}} \mathcal{D}_{\phi_P}(e^{i\theta}) \in H^2(\mathbf{T}; \mathcal{L}(U))$, by Corollary 42. Now [33, Theorem 4.7A], as used in Lemma 36, implies that $\Lambda_P^{\frac{1}{2}} \mathcal{D}_{\phi_P}(e^{i\theta}) \in H^\infty(\mathbf{T}; \mathcal{L}(U))$. Because Λ_P has a bounded inverse, $\mathcal{D}_{\phi_P}(e^{i\theta}) \in H^\infty(\mathbf{T}; \mathcal{L}(U))$.

To verify inclusion (34), note that Proposition 43 implies that ϕ_P is output stable. Because $L_{A,P} = 0$, then $P \in Ric_{uw}(\Phi, J)$. Now the first part of this Corollary implies that ϕ_P is I/O stable, and so $P \in ric(\Phi, J)$. The proof is now complete. □

A slight modification of the proof verifies also

$$\{P \in Ric_{00}(\Phi, J) \cap Ric_{uw}(\Phi, J) \mid \Lambda_P > 0\} \subset ric_{00}(\Phi, J) \cap ric_{uw}(\Phi, J) \tag{35}$$

under the assumptions of the previous corollary. If $\overline{\operatorname{range}(\mathcal{B})} = H$, then this reduces to inclusion (34), by claim (iv) of Proposition 22. We also have:

Corollary 48. *Let $J \geq 0$ be a cost operator, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau{}^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an output stable and I/O stable DLS. Furthermore, assume that the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt, and the input space U is separable.*

(i) The set $ric_0(\Phi, J)$ of regular H^∞ solutions is downward complete in the sense that if $\tilde{P} \in Ric_0(\Phi, J)$, $\tilde{P} \geq 0$, then

$$\{P \in Ric(\Phi, J) \mid 0 \leq P \leq \tilde{P}\} \subset ric_0(\Phi, J).$$

(ii) In particular, if a regular critical solution $P_0^{\text{crit}} \in ric_0(\phi, J)$ exists, then

$$\{P \in Ric(\Phi, J) \mid 0 \leq P \leq P_0^{\text{crit}}\} \subset ric_0(\Phi, J). \tag{36}$$

Proof. To prove claim (i), let $\tilde{P} \in Ric_0(\Phi, J)$, $P \geq 0$ be arbitrary. But then for any $P \in Ric(\Phi, J)$ such that $0 \leq P \leq \tilde{P}$ and $x_0 \in H$ we have

$$||P^{\frac{1}{2}} A^j x_0||_H^2 = \langle P A^j x_0, A^j x_0 \rangle \leq \left\langle A^{*j} \tilde{P} A^j x_0, x_0 \right\rangle \leq ||A^{*j} \tilde{P} A^j x_0||_H \cdot ||x_0||_H,$$

which approaches zero as $j \to \infty$, because $L_{A,\tilde{P}} = 0$ by assumption. Thus $L_{A,P}$ exists and vanishes. Because $J \geq 0$, it follows that $\Lambda_P > 0$ for all nonnegative $P \in Ric(\Phi, J)$. An application of Corollary 47 proves now claim (i). The other claim (ii) is just a particular case. □

In [27, Theorem 96], we consider the converse inclusion of formula (36). This gives us a full order-theoretic characterization of nonnegative regular H^∞ solutions, under the indicated technical assumptions. Another result in this direction is [27, Lemma 99], showing that the set $ric_0(\phi, J)$ is order-convex.

We complete this section by the following lemma about the "inertia" of the indicators Λ_P.

Lemma 49. *Let J be a self-adjoint cost operator. Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ I/O stable and output stable, such that the input operator B is Hilbert–Schmidt and the input space U is separable. Assume that the H^∞ solution set $ric_{uw}(\Phi, J)$ is nonempty, and let $P \in Ric_{uw}(\Phi, J)$ be such that ϕ_P is output stable.*

Then there is a decomposition of U as a orthogonal direct sum $U = U_+ \oplus U_-$ such that for each $P \in Ric_{uw}(\Phi, J)$, there is a boundedly invertible operator $V_P \in \mathcal{L}(U)$ such that

$$\Lambda_P = V_P^* \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix} V_P,$$

where I_+, (I_-) is the identity of U_+, (U_-, respectively). If particular, if $\Lambda_{P_0} > 0$ for some $P_0 \in ric_{uw}(\Phi, J)$, then $\Lambda_P > 0$ for all $P \in Ric_{uw}(\Phi, J)$ with an output stable ϕ_P.

Proof. Let $P_0 \in ric_{uw}(\Phi, J)$ be fixed, and $P_0 \in Ric_{uw}(\Phi, J)$ be arbitrary, such that ϕ_{P_0} is output stable. Because Λ_{P_0} is self-adjoint and invertible, we can work with the spectral projections of Λ_{P_0}, on the disjoint spectral sets on negative and positive real axes. This gives $\Lambda_{P_0} = \Lambda_+ - \Lambda_-$, where $\Lambda_+ \in \mathcal{L}(U_+)$, $\Lambda_- \in \mathcal{L}(U_-)$, and both are positive invertible operators in their respective spectral subspaces that are reducing. Now

$$\Lambda_{P_0} = V^* \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix} V,$$

where $V^* := \begin{bmatrix} \Lambda_+^{\frac{1}{2}} & \Lambda_-^{\frac{1}{2}} \end{bmatrix} : U_+ \oplus U_- \to U$ has a bounded inverse. By Lemma 45, we can choose $e^{i\theta_0} \in \mathbf{T}$ from a set of full Lebesque measure, such that

$$\mathcal{D}_{\phi_{P_0}}(e^{i\theta_0})^* \Lambda_{P_0} \mathcal{D}_{\phi_{P_0}}(e^{i\theta_0}) = \mathcal{D}_{\phi_P}(e^{i\theta_0})^* \Lambda_P \mathcal{D}_{\phi_P}(e^{i\theta_0}).$$

By claim (ii) of Proposition 38 and the fact that $\mathcal{D}_{\phi_{P_0}}(0) = I$ has a bounded inverse, $\mathcal{D}_{\phi_{P_0}}(e^{i\theta})^{-1}$ exists a.e. $e^{i\theta} \in \mathbf{T}$, and in fact $\mathcal{D}_{\phi_{P_0}}(e^{i\theta})^{-1} \in L^\infty(\mathbf{T}; \mathcal{L}(U))$. Thus we can assume that $\mathcal{D}_{\phi_{P_0}}(e^{i\theta_0})$ has a bounded inverse, and

$$V^* \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix} V = \left(\mathcal{D}_{\phi_P}(e^{i\theta_0}) \mathcal{D}_{\phi_{P_0}}(e^{i\theta_0})^{-1}\right)^* \cdot \Lambda_P \cdot \left(\mathcal{D}_{\phi_P}(e^{i\theta_0}) \mathcal{D}_{\phi_{P_0}}(e^{i\theta_0})^{-1}\right).$$

This proves the claim, with $V_P = \mathcal{D}_{\phi_P}(e^{i\theta_0}) \mathcal{D}_{\phi_{P_0}}(e^{i\theta_0})^{-1} V^{-1}$. □

By dimension counting, we immediately see that if either of the spaces U_+, U_- is finite dimensional, then the dimension will be an invariant of all the solutions $P \in Ric_{uw}(\Phi, J)$, whose spectral DLS ϕ_P is output stable. If the indicators Λ_P are positive, the output stability requirement of ϕ_P could be replaced by the requiring $P \in Ric_{00}(\Phi, J)$, see Proposition 43. The special case of positive indicators is discussed in Lemma 53 where the input operator is not required to be Hilbert–Schmidt, and the proof is not based upon the study of nontangential boundary traces. For an analogous matrix result, see [15, Corollary 12.2.4].

7 Factorization of the Popov operator

Let Φ be an output stable and I/O stable DLS, and J a self-adjoint cost operator. In this section we show that there is a one-to-one correspondence between certain factorizations of the Popov operator $\mathcal{D}^* J \mathcal{D}$ and certain solutions of the H^∞DARE $ric(\Phi, J)$. It is worth noting that these factorizations do not depend on the nonnegativity of the cost operator J.

The factorizations of the Popov operator have a number of useful consequences. In Lemma 53 and its Corollary 54, we show that sometimes all interesting solutions of DARE have a positive indicator. Proposition 55 gives results of the $(\Lambda_P, \Lambda_{P^{\mathrm{crit}}})$-inner-outer factorization for the I/O-map of the spectral DLS ϕ_P.

In Definition 26, the Popov operator was defined to be the Toeplitz operator $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+$. We call the bounded shift-invariant (but noncausal) operator $\mathcal{D}^* J \mathcal{D}$ (the *symbol* of the Toeplitz operator $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+$) Popov operator, too.

Theorem 50. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ *be an I/O stable and output stable DLS. Let* $J \in \mathcal{L}(Y)$ *be a self-adjoint operator*

(i) To each solution $P \in ric_{uw}(\Phi, J)$*, we can associate the following factorization of the Popov operator*

$$\mathcal{D}^* J \mathcal{D} = \mathcal{D}^*_{\phi_P} \Lambda_P \mathcal{D}_{\phi_P}, \tag{37}$$

where ϕ_P *is the spectral DLS (of* Φ *and* J*), centered at* P*.*

(ii) Assume, in addition that $\overline{\mathrm{range}\,(\mathcal{B})} = H$*. Assume that the Popov operator has a factorization of form*

$$\mathcal{D}^* J \mathcal{D} = \mathcal{D}^*_{\phi'} \Lambda \mathcal{D}_{\phi'}, \tag{38}$$

where

$$\phi' := \begin{pmatrix} A & B \\ -K & I \end{pmatrix}, \quad K \in \mathcal{L}(H, U), \quad \Lambda = \Lambda^*, \Lambda^{-1} \in \mathcal{L}(U),$$

is an I/O stable and output stable DLS. Then $\phi' = \phi_P$ *and* $\Lambda = \Lambda_P$ *for a* $P \in ric_0(\Phi, J)$*.*

Proof. We prove claim (i). Let $P \in ric_{uw}(\Phi, J)$. By Lemma 45, we have for all $m \geq 0$

$$\pi_{[0,m]} \mathcal{D}^* J \mathcal{D} \pi_{[0,m]} = \pi_{[0,m]} \mathcal{D}^*_{\phi_P} \Lambda_P \mathcal{D}_{\phi_P} \pi_{[0,m]}, \tag{39}$$

where using the adjoints is legal because both $\mathcal{D}$ and $\mathcal{D}_{\phi_P}$ are assumed to be bounded. Let $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ be arbitrary. Then

$$\begin{aligned}
&||\pi_{[0,m]} \mathcal{D}^* J \mathcal{D} \pi_{[0,m]} \tilde{u} - \bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+ \tilde{u}|| \\
&\leq ||\pi_{[0,m]} \mathcal{D}^* J \mathcal{D} (\pi_{[0,m]} \tilde{u} - \bar{\pi}_+ \tilde{u})|| + ||(\pi_{[0,m]} - \bar{\pi}_+) \mathcal{D}^* J \mathcal{D} \pi_{[0,m]} \bar{\pi}_+ \tilde{u}|| \\
&\leq ||\pi_{[0,m]} \mathcal{D}^* J \mathcal{D}|| \cdot ||\pi_{[m+1,\infty]} \tilde{u}|| + ||\pi_{[m+1,\infty]} \cdot \bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+ \tilde{u}||
\end{aligned}$$

Because both $\tilde{u}$ and $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+\tilde{u}$ are in $\ell^2(\mathbf{Z}_+;U)$, it follows that $\mathrm{s}-\lim_{m\to\infty}\pi_{[0,m]}\mathcal{D}^*J\mathcal{D}\pi_{[0,m]} = \bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+$. Similarly we obtain the limit $\mathrm{s}-\lim_{m\to\infty}\pi_{[0,m]}\mathcal{D}^*_{\phi_P}J\mathcal{D}_{\phi_P}\pi_{[0,m]} = \bar{\pi}_+\mathcal{D}^*_{\phi_P}J\mathcal{D}_{\phi_P}\bar{\pi}_+$. The uniqueness of the strong limit, together with equation (39), gives now factorization (37).

To prove the other claim (ii), we show that there is a conjugate symmetric sesquilinear form $P(\, , \,)$ such that for all $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$, $x_0 \in H$

$$J(x_0,\tilde{u}) = P(x_0,x_0) + \langle \Lambda(\mathcal{C}_{\phi'}x_0 + \mathcal{D}_{\phi'}\bar{\pi}_+\tilde{u}), (-, \, , -)\rangle \,, \tag{40}$$

assuming that the factorization (38) exists. Here $J(x_0,\tilde{u}) := \langle J(\mathcal{C}x_0 + \mathcal{D}\bar{\pi}_+\tilde{u}), (-, \, , -)\rangle$ is a cost functional, see [19, Section 3]. Suppose that such a sesquilinear form $P(\, , \,)$ exists and try to find an expression for it. By expanding (40) we obtain

$$\begin{aligned} &\langle \mathcal{C}^*J\mathcal{C}x_0, x_0\rangle + \overbrace{2Re\,\langle \bar{\pi}_+\mathcal{D}^*J\mathcal{C}x_0, \tilde{u}\rangle}^{(i)} + \overbrace{\langle \bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+\tilde{u}, \tilde{u}\rangle}^{(ii)} \\ &= P(x_0,x_0) + \left\langle \mathcal{C}^*_{\phi'}\Lambda\mathcal{C}_{\phi'}x_0, x_0\right\rangle + \\ &\overbrace{2Re\left\langle \bar{\pi}_+\mathcal{D}^*_{\phi'}\Lambda\mathcal{C}_{\phi'}x_0, \tilde{u}\right\rangle}^{(iii)} + \overbrace{\left\langle \bar{\pi}_+\mathcal{D}^*_{\phi'}\Lambda\mathcal{D}_{\phi'}\bar{\pi}_+\tilde{u}, \tilde{u}\right\rangle}^{(iv)} \end{aligned} \tag{41}$$

for all $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$ and $x_0 \in H$ because both Φ and ϕ' are I/O stable and output stable. By equation (38), parts (ii) and (iv) are equal. To compare parts (i) and (iii), note that for $x := \mathcal{B}\tilde{w}$, $\tilde{w} \in \mathrm{dom}\,(\mathcal{B})$, we have, because $\mathcal{B} = \mathcal{B}_{\phi'}$

$$\begin{aligned} &\bar{\pi}_+\mathcal{D}^*J\mathcal{C}x - \bar{\pi}_+\mathcal{D}^*_{\phi'}\Lambda\mathcal{C}_{\phi'}x = \bar{\pi}_+\mathcal{D}^*J\bar{\pi}_+\mathcal{D}\pi_-\tilde{w} - \bar{\pi}_+\mathcal{D}^*_{\phi'}\Lambda\bar{\pi}_+\mathcal{D}_{\phi'}\pi_-\tilde{w} \\ &= \bar{\pi}_+(\mathcal{D}^*J\mathcal{D} - \mathcal{D}^*_{\phi'}\Lambda\mathcal{D}_{\phi'})\pi_-\tilde{w} = 0 \end{aligned} \tag{42}$$

by (38), and the anticausality of $\mathcal{D}^*$ and $\mathcal{D}^*_{\phi'}$. Because $\overline{\mathrm{range}\,(\mathcal{B})} = H$ it follows that $\bar{\pi}_+\mathcal{D}^*J\mathcal{C}x - \bar{\pi}_+\mathcal{D}^*_{\phi'}\Lambda\mathcal{C}_{\phi'}x = 0$, for all $x \in H$, by I/O stability and output stability of Φ and ϕ'.

So the parts (i), (ii), (iii) and (iv) cancel each other out in equation (41). What remains allows us to conclude that the sesquilinear form of equation (40) exists and equals

$$P(x_0,x_0) = \left\langle \left(\mathcal{C}^*J\mathcal{C} - \mathcal{C}^*_{\phi'}\Lambda\mathcal{C}_{\phi'}\right)x_0, x_0\right\rangle =: \langle Px_0, x_0\rangle \,,$$

which gives us a unique self-adjoint operator $P \in \mathcal{L}(H)$. We note that for all $x_0 \in H$

$$\begin{aligned} \left\langle A^{*j}PA^jx_0, x_0\right\rangle &= \left\langle J\mathcal{C}A^jx_0, \mathcal{C}A^jx_0\right\rangle - \left\langle \Lambda\mathcal{C}_{\phi'}A^jx_0, \mathcal{C}_{\phi'}A^jx_0\right\rangle \\ &= \left\langle J\pi_{[j,\infty]}\mathcal{C}x_0, \pi_{[j,\infty]}\mathcal{C}x_0\right\rangle - \left\langle \Lambda\pi_{[j,\infty]}\mathcal{C}_{\phi'}x_0, \pi_{[j,\infty]}\mathcal{C}_{\phi'}x_0\right\rangle . \end{aligned}$$

By the output stabilities of Φ and ϕ', both $\pi_{[j,\infty]}\mathcal{C}x_0 \to 0$ and $\pi_{[j,\infty]}\mathcal{C}_{\phi'}x_0 \to 0$ in $\ell^2(\mathbf{Z}_+;Y)$, $\ell^2(\mathbf{Z}_+;U)$, respectively. Thus $\langle PA^jx_0, A^jx_0\rangle \to 0$ for all $x_0 \in H$, by the boundedness of Λ^{-1}.

We complete the proof by showing that $P \in Ric(\Phi, J)$, and that $K = K_P$, $\Lambda = \Lambda_P$. We have for Λ_P

$$\begin{aligned}\Lambda_P &= D^*JD + B^*PB \\ &= (D^*JD + (\mathcal{C}B)^*J(\mathcal{C}B)) - (I^*\Lambda I + (\mathcal{C}_{\phi'}B)^*\Lambda(\mathcal{C}_{\phi'}B)) + \Lambda \\ &= (D\pi_0 + \tau\mathcal{C}B)^*J(D\pi_0 + \tau\mathcal{C}B) - (\pi_0 + \tau\mathcal{C}_{\phi'}B)^*\Lambda(\pi_0 + \tau\mathcal{C}_{\phi'}B) + \Lambda \\ &= \bar{\pi}_+\mathcal{D}^*J\mathcal{D}\pi_0 - \bar{\pi}_+\mathcal{D}_{\phi'}^*\Lambda\mathcal{D}_{\phi'}\pi_0 + \Lambda = \Lambda,\end{aligned}$$

where the second to the last equality has been written with the identification of spaces U and range (π_0), allowing us to write $\mathcal{D}\pi_0 = D\pi_0 + \tau\mathcal{C}B$. The last identity follows directly from the factorization (38), and so $\Lambda_P = \Lambda$.

For $K_P = \Lambda_P^{-1}(-D^*JC - B^*PA)$ we calculate similarly

$$\begin{aligned}(43)\qquad & - D^*JC - B^*PA \\ &= -(D^*J\mathcal{C} + (\mathcal{C}B)^*J\mathcal{C}A) + (-I^*\Lambda K + (\mathcal{C}_{\phi'}B)^*\Lambda\mathcal{C}_{\phi'}A) + \Lambda K\end{aligned}$$

Now $D^*J\mathcal{C} + (\mathcal{C}B)^*J\mathcal{C}A = (D\pi_0 + \tau\mathcal{C}B)^*J\mathcal{C} = (\mathcal{D}\pi_0)^*J\mathcal{C} = \pi_0\mathcal{D}^*J\mathcal{C}$. Quite similarly $-\Lambda K + (\mathcal{C}_{\phi'}B)^*\Lambda\bar{\pi}_+\tau^*\mathcal{C}_{\phi'} = (\mathcal{D}_{\phi'}\pi_0)^*\Lambda\mathcal{C}_{\phi'} = \pi_0\mathcal{D}_{\phi'}^*\Lambda\mathcal{C}_{\phi'}$. Then we obtain from (43)

$$(44)\qquad -D^*JC - B^*PA = -\pi_0(\mathcal{D}^*J\mathcal{C} - \mathcal{D}_{\phi'}^*\Lambda\mathcal{C}_{\phi'}) + \Lambda K,$$

with the identification of spaces U and range (π_0).

For all $x = \mathcal{B}\tilde{w} = \mathcal{B}_{\phi'}\tilde{w}$, $\tilde{w} \in \operatorname{dom}(\mathcal{B}) = \operatorname{dom}(\mathcal{B}_{\phi'})$, we have

$$\pi_0(\mathcal{D}^*J\mathcal{C} - \mathcal{D}_{\phi'}^*\Lambda\mathcal{C}_{\phi'})x = \pi_0(\mathcal{D}^*J\mathcal{D} - \mathcal{D}_{\phi'}^*\Lambda\mathcal{D}_{\phi'})\pi_-\tilde{w} = 0,$$

by the factorization (38). Because $\overline{\operatorname{range}(\mathcal{B})} = H$, and $\pi_0(\mathcal{D}^*J\mathcal{C} - \mathcal{D}_{\phi'}^*\Lambda\mathcal{C}_{\phi'})$ is continuous in H, it follows that vanishes in the whole of H. From (44) it now follows that $K = \Lambda^{-1}(-D^*JC - B^*PA) = \Lambda_P^{-1}(-D^*JC - B^*PA) = K_P$ because $\Lambda = \Lambda_P$ has been shown earlier.

It is now straightforward to show that $P \in Ric(\Phi, J)$:

$$\begin{aligned}& P(Ax_0, Ax_0) - P(x_0, x_0) \\ &= \langle \pi_+\mathcal{C}x_0, J\pi_+\mathcal{C}\rangle - \langle \pi_+\mathcal{C}_{\phi'}x_0, \Lambda\pi_+\mathcal{C}_{\phi'}\rangle - \langle \mathcal{C}x_0, J\mathcal{C}\rangle + \langle \mathcal{C}_{\phi'}x_0, \Lambda\mathcal{C}_{\phi'}\rangle \\ &= \langle -Kx_0, -\Lambda\ Kx_0\rangle - \langle Cx_0, JCx_0\rangle = \langle K_P^*\Lambda_P K_P x_0, x_0\rangle - \langle C^*JCx_0, x_0\rangle .\end{aligned}$$

Because ϕ' is output stable and I/O stable, by assumption, and $\phi_P = \phi'$, it follows that P is a H^∞ solution: $P \in ric(\Phi, J)$.

It remains to prove the final claim about the residual cost operator. Because Φ and ϕ' are output stable by assumption, we have

$$\begin{aligned}A^{*j}PA^j &= A^{*j}\mathcal{C}^*J\mathcal{C}A^j - A^{*j}\mathcal{C}_{\phi'}^*\Lambda\mathcal{C}_{\phi'}A^j \\ &= (\bar{\pi}_+\tau^{*j}\mathcal{C})^*J(\bar{\pi}_+\tau^{*j}\mathcal{C}) - (\bar{\pi}_+\tau^{*j}\mathcal{C}_{\phi'})^*\Lambda(\bar{\pi}_+\tau^{*j}\mathcal{C}_{\phi'}) \\ &= \mathcal{C}^*J\pi_{[j,\infty]}\mathcal{C} - \mathcal{C}_{\phi'}^*\Lambda\pi_{[j,\infty]}\mathcal{C}_{\phi'}.\end{aligned}$$

Now $\mathrm{s}-\lim_{j\to\infty}\pi_{[j,\infty]}\mathcal{C} = \mathrm{s}-\lim_{j\to\infty}\pi_{[j,\infty]}\mathcal{C}_{\phi'} = 0$, and immediately $L_{A,P} = \mathrm{s}-\lim_{j\to\infty}A^{*j}PA^j = 0$. This completes the proof. □

For analogous spectral factorization results, see [15, Chapter 19], [13, Theorem 4.6] and [10] together with its references. In claim (ii) of Theorem 50, a requirement has been imposed on the spectral factor $\mathcal{D}_{\phi'}$ of the Popov operator: it must be realizable by using the same input structure as the original DLS Φ and all the spectral DLSs ϕ_P. It is necessary to make such an apriori requirement explicitly. To see this, consider the trivial case when $\mathcal{D} = \mathcal{I}$, the identity operator of $\ell^2(\mathbf{Z};U)$. Then the Popov operator satisfies $\mathcal{D}^* J \mathcal{D} = \mathcal{I}$, if $J = I$, the identity operator of U. Each inner from the left operator $\mathcal{N}'$ is, by definition, a spectral factor of the Popov operator $\mathcal{I}$. There is a multitude of such inner operators; if $U = \mathbf{C}$, then these are parameterized by sequences in $\mathbf{D}$ satisfying the Blaschke condition and the singular positive measures on $\mathbf{T}$. However, the DLS $\Phi = \phi$ can be very trivial, say $\phi = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$. The DARE $Ric(\phi, I)$ is trivially $I = I$, and all (self-adjoint) operators $P \in \mathcal{L}(H)$ are its solution. However, each of the spectral DLSs equal $\phi_P = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$, and only one spectral factor of the Popov operator is covered by a solution of the DARE.

In the proof of Theorem 50, we never wrote down a state space realization for the Popov function $\mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta})$. Suppose $\mathcal{D}(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U))$ would be analytic in an open set $\Omega \subset \mathbf{C}$, such that $\mathbf{D} \subset \Omega$ and $\mathbf{T} \setminus (\mathbf{T} \cap \Omega)$ is, say, a finite set of points. Then the Popov function $\mathcal{D}(e^{i\theta})^* J \mathcal{D}(e^{i\theta})$ would have an analytic continuation to a neighbourhood of each $e^{i\theta_0} \in \mathbf{T} \cap \Omega$. This analytic continuation is given by $\widetilde{\mathcal{D}}(z^{-1}) J \mathcal{D}(z)$, and its realization ϕ^{Popov} can be formed by using the formula for the product realization. Now, the connection between the DARE and the spectral factorization of the Popov function can be studied by using ϕ^{Popov}, even for certain classes of unstable transfer functions $\mathcal{D}(z)$. However, a general $\mathcal{D}(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U))$ does not allow this approach; there is a function in the complex-valued disk algebra $f(z) \in A(\mathbf{D})$ that does not allow analytic continuation to any set larger than $\mathbf{D}$, and in fact the boundary trace $f(e^{i\theta})$ can be smooth. Such a function is constructed in [34, Example 16.7]. Then $f(z)$ and $\widetilde{f}(z^{-1})$ are bounded analytic functions in open sets $\mathbf{D}$ and $(\overline{\mathbf{D}})^c$, with an empty intersection.

In a later result [27, Lemma 101], we shall need a different spectral factorization result, associated to solutions $P \in ric(\Phi, J)$ that need not satisfy the strong residual cost condition. The nonvanishing residual cost is included in the Popov operator. To achieve this, we must first define analogues (in I/O-form) to the residual cost operator $L_{A,P} := \mathrm{s} - \lim_{j\to\infty} A^{*j} P A^j$.

Definition 51. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ C & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS, and $P \in Ric(\Phi, J)$. Let $n, m \geq 0$ be arbitrary. Define the linear operators in $\ell^2(\mathbf{Z}_+;U)$*

$$\mathcal{L}^{(m,n)}_{\Phi,P} := \left(\mathcal{B}\tau^{*n}\pi_{[0,m]}\right)^* P \left(\mathcal{B}\tau^{*n}\pi_{[0,m]}\right),$$

and

$$\mathcal{L}^{(m)}_{\Phi,P} := \mathrm{s} - \lim_{n\to\infty} \mathcal{L}^{(m,n)}_{\Phi,P}, \quad \mathcal{L}_{\Phi,P} := \mathrm{s} - \lim_{m\to\infty} \mathcal{L}^{(m)}_{\Phi,P},$$

provided that the strong limits exists. The operator $\mathcal{L}_{\Phi,P}$ is the residual cost operator (in I/O-form), and the operator $\mathcal{L}^{(n)}_{\Phi,P}$ is the truncated residual cost operator (in I/O-form).

The operator $\mathcal{B}\tau^{*n}\pi_{[0,m]} : \ell^2(\mathbf{Z}_+; U) \to H$ is a finite sum of products of the bounded operators A, B, the orthogonal projections π_j, and the unitary shift τ^* in $\ell^2(\mathbf{Z}_+; U)$. Thus it is bounded for all $m, n \geq 0$, and it follows that $\mathcal{L}_{\Phi,P}^{(m,n)}$ always exists as a bounded operator.

Lemma 52. *Let $J \in \mathcal{L}(Y)$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, and $P \in ric(\Phi, J)$. Then*

(i) Both the residual cost operators $L_{A,P} \in \mathcal{L}(H)$ and $\mathcal{L}_{\Phi,P} \in \mathcal{L}(\ell^2(\mathbf{Z}_+; U))$ exist.

(ii) We have the spectral factorization identity

$$\mathcal{L}_{\phi,P} + \bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+ = \bar{\pi}_+ \mathcal{D}_{\phi_P}^* \Lambda_P \mathcal{D}_{\phi_P} \bar{\pi}_+.$$

The residual cost operator $\mathcal{L}_{\Phi,P}$ is a self-adjoint Toeplitz operator.

(iii) Assume, in addition, that $\overline{\text{range}\,(\mathcal{B})} = H$. Then both $B^* L_{A,P} A = 0$ and $B^* L_{A,P} B = 0$ if and only if $\mathcal{L}_{\Phi,P} = 0$ if and only if $L_{A,P} = 0$.

Proof. Because $P \in ric(\Phi, J)$, the residual cost operator $L_{A,P}$ exists by Proposition 23. We prove the rest of claim (i) and claim (ii) simultaneously. Let $x_0 \in H$ and $\{u_j\}_{j\geq 0} = \tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ be arbitrary. Denote $x_j = x_j(x_0, \tilde{u}) = A^j x_0 + \mathcal{B}\tau^{*j}\tilde{u}$ the trajectory of the DLS Φ with this given initial state and input. We have in [19, claim (i) of Proposition 36] for all $n > 0$

$$\begin{aligned} &\langle P x_0, x_0 \rangle - \langle P x_n, x_n \rangle \\ &= \sum_{j=0}^{n-1} \langle J(Cx_j + Du_j), Cx_j + Du_j \rangle \\ &- \sum_{j=0}^{n-1} \langle \Lambda_P(-K_P x_j + u_j), -K_P x_j + u_j) \rangle. \end{aligned} \tag{45}$$

We now set $x_0 = 0$ and assume that the inputs are of form $\pi_{[0,m]}\tilde{u}$ for some fixed $m \geq 0$ and arbitrary $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$. In this case, $\langle P x_0, x_0 \rangle = 0$ and equation (45) takes now the form

$$\begin{aligned} &\left\langle P\mathcal{B}\tau^{*n}\pi_{[0,m]}\tilde{u}, \mathcal{B}\tau^{*n}\pi_{[0,m]}\tilde{u} \right\rangle + \left\langle J\mathcal{D}\pi_{[0,m]}\tilde{u}, \pi_{[0,n-1]}\mathcal{D}\pi_{[0,m]}\tilde{u} \right\rangle_{\ell^2(\mathbf{Z}_+;Y)} \\ &= \left\langle \Lambda_P \mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u}, \pi_{[0,n-1]}\mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u} \right\rangle_{\ell^2(\mathbf{Z}_+;Y)}, \end{aligned}$$

because $x_n = \mathcal{B}\tau^{*n}\pi_{[0,m]}\tilde{u}$.

Both the operators $\mathcal{D}\pi_{[0,m]}$ and $\mathcal{D}_{\phi_P}\pi_{[0,m]}$ are bounded, because Φ and ϕ_P are I/O stable DLSs by assumptions. Also the operators $\mathcal{B}\tau^{*n}\pi_{[0,m]}$ are bounded, as has been discussed after Definition 51. So the adjoints $\left(\mathcal{B}\tau^{*n}\pi_{[0,m]}\right)^*$, $\mathcal{D}^*$ and $\mathcal{D}_{\phi_P}^*$ make sense, and we can write

$$\begin{aligned} &\left\langle \mathcal{L}_{\Phi,P}^{(m,n)}\tilde{u}, \tilde{u} \right\rangle + \left\langle \pi_{[0,m]}\mathcal{D}^* J \pi_{[0,n-1]}\mathcal{D}\pi_{[0,m]}\tilde{u}, \tilde{u} \right\rangle_{\ell^2(\mathbf{Z}_+;Y)} \\ &= \left\langle \pi_{[0,m]}\mathcal{D}_{\phi_P}^* \Lambda_P \pi_{[0,n-1]}\mathcal{D}_{\phi_P}\pi_{[0,m]}\tilde{u}, \tilde{u} \right\rangle_{\ell^2(\mathbf{Z}_+;Y)}, \end{aligned}$$

by Definition 51. Because $\tilde{u}$ is arbitrary, and all the operators $\mathcal{L}_{\Phi,P}^{(m,n)}$, $\mathcal{D}$ and $\mathcal{D}_{\phi_P}$ are bounded, [35, Theorem 12.7] implies that

$$\mathcal{L}_{\Phi,P}^{(m,n)} = -\pi_{[0,m]}\mathcal{D}^* J \cdot \pi_{[0,n-1]}\mathcal{D}\pi_{[0,m]} + \pi_{[0,m]}\mathcal{D}_{\phi_P}^* \Lambda_P \cdot \pi_{[0,n]}\mathcal{D}_{\phi_P}\pi_{[0,m]} \tag{46}$$

for all $m, n \geq 0$. Because $\mathcal{D}$ is bounded, $\mathrm{s}-\lim_{n\to\infty} \pi_{[0,n-1]}\mathcal{D}\pi_{[0,m]} = \mathcal{D}\pi_{[0,m]}$ and $\mathrm{s}-\lim_{n\to\infty} \pi_{[0,n-1]}\mathcal{D}_{\phi_P}\pi_{[0,m]} = \mathcal{D}_{\phi_P}\pi_{[0,m]}$. But then, the strong limit in the right hand side of (46) exists, and we conclude that the residual cost operator $\mathcal{L}_{\Phi,P}^{(m)} \in \mathcal{L}(\ell^2(\mathbf{Z}_+; U))$ exists as a bounded operator. We obtain

$$\mathcal{L}_{\Phi,P}^{(m)} = -\pi_{[0,m]}\mathcal{D}^* J\mathcal{D}\pi_{[0,m]} + \pi_{[0,m]}\mathcal{D}_{\phi_P}^* \Lambda_P \mathcal{D}_{\phi_P}\pi_{[0,m]} \tag{47}$$

for all $m \geq 0$. We proceed to show that $\mathrm{s}-\lim_{m\to\infty} \pi_{[0,m]}\mathcal{D}^* J\mathcal{D}\pi_{[0,m]}$ exists and equals the Popov operator $\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+$. For all $m \geq 0$ and $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$, we have

$$\begin{aligned}
&||\pi_{[0,m]}\mathcal{D}^* J\mathcal{D}\pi_{[0,m]}\tilde{u} - \bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} \\
&\leq ||\pi_{[0,m]}\mathcal{D}^* J\mathcal{D}\pi_{[m+1,\infty]}\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} + ||\pi_{[m+1,\infty]}\mathcal{D}^* J\mathcal{D}\bar{\pi}_+\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} \\
&\leq ||J||_{\mathcal{L}(Y)} \cdot ||\mathcal{D}||_{\ell^2(\mathbf{Z};U)\to\ell^2(\mathbf{Z};Y)} \cdot ||\pi_{[m+1,\infty]}\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} \\
&+ ||\pi_{[m+1,\infty]} \cdot \bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)}.
\end{aligned}$$

Because both $\tilde{u}$ and $\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+\tilde{u}$ belong to $\ell^2(\mathbf{Z}_+; U)$, the right hand side of the previous equation converges to zero as $m \to \infty$. It follows that $\mathrm{s}-\lim_{m\to\infty} \pi_{[0,m]}\mathcal{D}^* J\mathcal{D}\pi_{[0,m]} = \bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+$ and similarly $\mathrm{s}-\lim_{m\to\infty} \pi_{[0,m]}\mathcal{D}_{\phi_P}^* J\mathcal{D}_{\phi_P}\pi_{[0,m]} = \bar{\pi}_+\mathcal{D}_{\phi_P}^* J\mathcal{D}_{\phi_P}\bar{\pi}_+$. Because the right hand side of equation (47) converges strongly as $m \to \infty$, we obtain the spectral factorization

$$\mathcal{L}_{\Phi,P} = -\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+ + \bar{\pi}_+\mathcal{D}_{\phi_P}^* \Lambda_P \mathcal{D}_{\phi_P}\bar{\pi}_+ \tag{48}$$

where $\mathcal{L}_{\Phi,P}$ is the residual cost operator in I/O-form, as introduced in Definition 51. Clearly $\mathcal{L}_{\Phi,P}$ is a self-adjoint Toeplitz operator, because the right hand side of equation (48) is such an operator. This proves claims (i) and (ii).

We proceed to prove claim (iii). We first calculate the block matrix elements $(\mathcal{L}_{\Phi,P})_{j_1,j_2} := \pi_{j_2}\mathcal{L}_{\Phi,P}\pi_{j_1}$ of $\mathcal{L}_{\Phi,P}$ for $j_1, j_2 \geq 0$. Let $\tilde{u}, \tilde{w} \in \ell^2(\mathbf{Z}_+; U)$ be arbitrary. Then

$$\begin{aligned}
\left\langle (\mathcal{L}_{\Phi,P})_{j_1,j_2}\tilde{u}, \tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)} &= \left\langle \left(\mathrm{s}-\lim_{m\to\infty} \mathcal{L}_{\Phi,P}^{(m)}\right) \cdot \pi_{j_1}\tilde{u}, \pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)} \\
= \left\langle \lim_{m\to\infty} \left(\mathcal{L}_{\Phi,P}^{(m)}\pi_{j_1}\tilde{u}\right), \pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)} &= \lim_{m\to\infty} \left\langle \mathcal{L}_{\Phi,P}^{(m)}\pi_{j_1}\tilde{u}, \pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)}.
\end{aligned}$$

But if $m \geq j_1$, then $\mathcal{L}_{\Phi,P}^{(m)}\pi_{j_1}\tilde{u} = \mathcal{L}_{\Phi,P}^{(j_1)}\pi_{j_1}\tilde{u}$. It follows that the sequence in the right hand side of the previous equation stabilizes, and for $m \geq \max(j_1, j_2)$

we get

$$\begin{aligned}
&\left\langle (\mathcal{L}_{\Phi,P})_{j_1,j_2}\,\tilde{u},\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)}\\
&= \left\langle \mathcal{L}^{(m)}_{\Phi,P}\pi_{j_1}\tilde{u},\pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)} = \left\langle \left(\mathrm{s}-\lim_{n\to\infty}\mathcal{L}^{(m,n)}_{\Phi,P}\right)\cdot\pi_{j_1}\tilde{u},\pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)}\\
&= \left\langle \lim_{n\to\infty}\left(\mathcal{L}^{(m,n)}_{\Phi,P}\pi_{j_1}\tilde{u}\right),\pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)} = \lim_{n\to\infty}\left\langle \mathcal{L}^{(m,n)}_{\Phi,P}\pi_{j_1}\tilde{u},\pi_{j_2}\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)}\\
&= \lim_{n\to\infty}\left\langle P\mathcal{B}\tau^{*(n-j_1-1)}\pi_{-1}\tau^{*(j_1+1)}\tilde{u},\mathcal{B}\tau^{*(n-j_2-1)}\pi_{-1}\tau^{*(j_2+1)}\tilde{w}\right\rangle_H.
\end{aligned}$$

But now $\mathcal{B}\tau^{*(n-j-1)}\pi_{-1} = \mathcal{B}\tau^{*(n-j-1)}\pi_{-}\cdot\pi_{-1} = A^{n-j-1}\mathcal{B}\pi_{-1} = A^{n-j-1}B\pi_{-1}$, where be have used $\mathcal{B}\pi_{-1} = B\pi_{-1}$. Now, if $j := \max(j_1,j_2)$, then

$$\begin{aligned}
&\left\langle (\mathcal{L}_{\Phi,P})_{j_1,j_2}\,\tilde{u},\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)}\\
&= \lim_{n\to\infty}\left\langle A^{*(n-j-1)}PA^{n-j-1}\cdot A^{j-j_1}B\pi_{-1}\tau^{*(j_1+1)}\tilde{u},A^{j-j_2}B\pi_{-1}\tau^{*(j_2+1)}\tilde{w}\right\rangle_H\\
&= \left\langle \left(\mathrm{s}-\lim_{n\to\infty}A^{*(n-j-1)}PA^{n-j-1}\right)\cdot A^{j-j_1}B\pi_{-1}\tau^{*(j_1+1)}\tilde{u},A^{j-j_2}B\pi_{-1}\tau^{*(j_2+1)}\tilde{w}\right\rangle_H\\
&= \left\langle L_{A,P}\cdot A^{j-j_1}B\pi_{-1}\tau^{*(j_1+1)}\tilde{u},A^{j-j_2}B\pi_{-1}\tau^{*(j_2+1)}\tilde{w}\right\rangle_H.
\end{aligned}$$

This gives for the block matrix elements of $\mathcal{L}_{\Phi,P}$ the expression

$$\left\langle (\mathcal{L}_{\Phi,P})_{j_1,j_2}\,\tilde{u},\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)} \tag{49}$$

$$= \left\langle \pi_{j_2}B^*A^{*(j-j_2)}L_{A,P}A^{j-j_1}B\pi_{j_1}\cdot\tilde{u},\tilde{w}\right\rangle_{\ell^2(\mathbf{Z}_+;U)}, \tag{50}$$

where $j = \max(j_1,j_2)$ and $\tilde{u},\tilde{w}\in\ell^2(\mathbf{Z}_+;U)$ are arbitrary.

If both $B^*L_{A,P}A = 0$ and $B^*L_{A,P}B = 0$, then all the block matrix elements $(\mathcal{L}_{\Phi,P})_{j_1,j_2}$ vanish, by equation (49). By a straightforward density argument, the bounded operator $\mathcal{L}_{\Phi,P}$ is seen to vanish.

Assume that $\mathcal{L}_{\Phi,P} = 0$. Then all the block matrix elements $(\mathcal{L}_{\Phi,P})_{j_1,j_2}$ for $j_1,j_2\geq 0$ vanish by their definition, and equation (48) implies that $B^*L_{A,P}A^kB = 0$ for all $k\geq 0$. It follows that $B^*L_{A,P}\mathcal{B}\tilde{u} = 0$ for all $\tilde{u}\in\mathrm{dom}(\mathcal{B})$, and thus $B^*L_{A,P}x = 0$ for all $x\in\mathrm{range}(\mathcal{B})$. Because B and $L_{A,P}$ are bounded, and $\overline{\mathrm{range}(\mathcal{B})} = H$, it follows that $B^*L_{A,P} = 0$, and also $L_{A,P}B = 0$ because $L_{A,P}$ is self-adjoint.

It is easy to see that $A^{*j}L_{A,P}A^j = L_{A,P}$ for all $j\geq 0$. Thus $A^{*j}L_{A,P}A^jB = L_{A,P}B = 0$ and immediately $B^*A^{*k}L_{A,P}A^jB = B^*A^{*(k-j)}\cdot A^{*j}L_{A,P}A^jB = 0$ for all $k\geq j$. By adjoining, we see that $B^*A^{*k}L_{A,P}A^jB = 0$ for arbitrary $j,k\geq 0$. But this implies that $\langle L_{A,B}\mathcal{B}\tilde{u},\mathcal{B}\tilde{u}\rangle_H = 0$, for all $\tilde{u}\in\mathrm{dom}(\mathcal{B})$. By the assumed approximate controllability $\overline{\mathrm{range}(\mathcal{B})} = H$, boundedness of $L_{A,B}$, and [35, Theorem 12.7], it follows that $L_{A,B} = 0$.

Trivially, if $L_{A,B} = 0$ then both $B^*L_{A,P}A = 0$ and $B^*L_{A,P}B = 0$. This completes the proof. □

Recall that in Propositions 24 and 25 we asked whether the indicator Λ_P and the DLS ϕ_P uniquely determine the solution $P\in Ric(\phi,J)$. Under the indicated additional assumptions, claim (iii) of Lemma 52 provides an answer to this. Under the approximate controllability $\overline{\text{range}(\mathcal{B})} = H$, it is exactly the solutions $P \in ric_0(\phi, J)$ (in the set $ric(\phi, J)$) that give us a spectral factorization of the Popov operator $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+$.

We proceed to consider the inertia of the indicator operator. The following is another variant of Lemma 49:

Lemma 53. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an output stable, I/O stable and J-coercive DLS. Assume that that the input space U is separable and there exists $P_0 \in ric_{uw}(\Phi, J)$ such that $\Lambda_{P_0} > 0$.*

Then for all $P \in ric_{uw}(\Phi, J)$, we have $\Lambda_P > 0$.

Proof. By Theorem 50, $\mathcal{D}^*J\mathcal{D} = \mathcal{D}_{\phi_{P_0}}^*\Lambda_{P_0}\mathcal{D}_{\phi_{P_0}} = \mathcal{D}_{\phi_P}^*\Lambda_P\mathcal{D}_{\phi_P}$ for all $P \in ric_{uw}(\Phi, J)$. By Proposition 38, the noncausal inverse $\mathcal{D}_{\phi_P}^{-1} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ is exists and is bounded, because $\mathcal{D}_{\phi_P}(0) = I$ has a bounded inverse. Then we have

$$(\mathcal{D}_{\phi_P}^{-1})^*\mathcal{D}_{\phi_{P_0}}^*\Lambda_{P_0}\mathcal{D}_{\phi_{P_0}}\mathcal{D}_{\phi_P}^{-1} = \left(\mathcal{D}_{\phi_{P_0}}\mathcal{D}_{\phi_P}^{-1}\right)^*\Lambda_{P_0}\left(\mathcal{D}_{\phi_{P_0}}\mathcal{D}_{\phi_P}^{-1}\right) = \Lambda_P,$$

which represents a shift-invariant, bounded and self-adjoint operator in $\ell^2(\mathbf{Z}; U)$. Because $\Lambda_{P_0} > 0$, it follows that $\Lambda_P \geq 0$, regarded as a static shift invariant operator on $\ell^2(\mathbf{Z}; U)$. But then, quite trivially, $\Lambda_P > 0$ as an element of $\mathcal{L}(U)$. ☐

Corollary 54. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an output stable and I/O stable DLS, with a separable input space U. Then the following are equivalent*

(i) $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ for some $\epsilon > 0$.

(ii) The solution set $ric_{uw}(\Phi, J)$ is not empty, and for all $P \in ric_{uw}(\Phi, J)$, $\Lambda_P > 0$.

When these equivalent conditions hold, the regular critical solution $P_0^{\text{crit}} := \left(\mathcal{C}^{\text{crit}}\right)^* J\mathcal{C}^{\text{crit}} \in ric_0(\Phi, J)$ exists.

Proof. Assume (i). Corollary 32 implies that a critical $P^{\text{crit}} \in Ric_{uw}(\Phi, J)$ exists. Proposition 29 implies that we have a regular critical H^∞ solution $P_0^{\text{crit}} \in ric_0(\Phi, J) \subset ric_{uw}(\Phi, J)$. Thus the solution set $ric_{uw}(\Phi, J)$ is not empty. By Theorem 50, $\bar{\pi}_+\mathcal{X}^*\Lambda_{P_0^{\text{crit}}}\mathcal{X}\bar{\pi}_+ = \bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ where $\mathcal{X} := \mathcal{D}_{\phi_{P_0^{\text{crit}}}}$ is outer with a bounded inverse. By shift invariance, also $\mathcal{X}^*\Lambda_{P_0^{\text{crit}}}\mathcal{X} \geq \epsilon\mathcal{I}$, and then $\Lambda_{P_0^{\text{crit}}} \geq \epsilon\mathcal{X}^{-*}\mathcal{X}^{-1} = \epsilon\left(\mathcal{X}\mathcal{X}^*\right)^{-1} > 0$, where $\Lambda_{P_0^{\text{crit}}}$ is regarded as a static multiplication operator on $\ell^2(\mathbf{Z}; U)$. Immediately, $\Lambda_{P_0^{\text{crit}}} > 0$ as an element of $\mathcal{L}(U)$, too. An application of Lemma 53 gives now claim (ii). The converse direction and the final comment are given in Proposition 31. ☐

There is a one-to-one correspondence between (J, S)-inner-outer factorizations of $\mathcal{D} = \mathcal{N}\mathcal{X}$ (with the outer part having a bounded inverse $\mathcal{X}^{-1}$) and S-spectral factorizations of the Popov operator $\mathcal{D}^*J\mathcal{D}$, see [19, Proposition 20]. Applying this to the spectral DLSs gives the proposition:

Proposition 55. *Let* $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O stable and output stable DLS. Let* J *be a self-adjoint operator. Assume that the equivalent conditions of Theorem 27 hold, and by* $P_0^{\text{crit}} := (\mathcal{C}^{\text{crit}})^* J \mathcal{C}^{\text{crit}} \in ric_0(\Phi, J)$ *denote the regular critical solution. Let* $P \in ric_{uw}(\Phi, J)$ *be arbitrary. Then*

(i) $\mathcal{D}_{\phi_P}$ *has an* $(\Lambda_P, \Lambda_{P_0^{\text{crit}}})$*-inner-outer factorization given by*

$$\mathcal{D}_{\phi_P} = \mathcal{N}_P \mathcal{X},$$

where $\mathcal{X} = \mathcal{D}_{\phi_{P_0^{\text{crit}}}}$ *is I/O stable, and* $\mathcal{N}_P := \mathcal{D}_{\phi_P} \mathcal{D}^{-1}_{\phi_{P_0^{\text{crit}}}}$*. The equivalent conditions of Theorem 27 hold for the DLS* ϕ_P *and the cost operator* Λ_P*. The outer factor does not depend upon the solution* P*. Both* range $(\mathcal{D}_{\phi_P} \bar{\pi}_+)$ *and* range $(\mathcal{D}_{\phi_P})$ *are closed. If the input space* U *is separable, then* range $(\mathcal{D}_{\phi_P}) = \ell^2(\mathbf{Z}; U)$.

(ii) $\mathcal{X}$ $(\mathcal{X}^{-1})$ *is the I/O-map of the spectral DLS* $\phi_{P_0^{\text{crit}}}$ $(\phi^{-1}_{P_0^{\text{crit}}}$*, respectively), with the realizations*

$$\phi_{P_0^{\text{crit}}} = \begin{pmatrix} A & B \\ -K_{P_0^{\text{crit}}} & I \end{pmatrix}, \quad \phi^{-1}_{P_0^{\text{crit}}} = \begin{pmatrix} A_{P_0^{\text{crit}}} & B \\ K_{P_0^{\text{crit}}} & I \end{pmatrix},$$

and $\mathcal{N}_P$ *is the I/O-map of the DLS*

$$\phi_P \phi^{-1}_{P_0^{\text{crit}}} = \begin{pmatrix} A_{P_0^{\text{crit}}} & B \\ K_{P_0^{\text{crit}}} - K_P & I \end{pmatrix},$$

where $A_{P_0^{\text{crit}}} := A + BK_{P_0^{\text{crit}}}$.

Proof. To prove claim (i), we note that we have the factorization of the Popov operator, for all $P \in ric_{uw}(\Phi, J)$

$$\mathcal{D}^* J \mathcal{D} = \mathcal{D}^*_{\phi_P} \Lambda_P \mathcal{D}_{\phi_P} = \mathcal{D}^*_{\phi_{P_0^{\text{crit}}}} \Lambda_{P_0^{\text{crit}}} \mathcal{D}_{\phi_{P_0^{\text{crit}}}},$$

by claim (i) of Theorem 50. But then, $\mathcal{X} := \mathcal{D}_{\phi_{P_0^{\text{crit}}}}$ is a $\Lambda_{P_0^{\text{crit}}}$-spectral factor of $\mathcal{D}^*_{\phi_P} \Lambda_P \mathcal{D}_{\phi_P}$, and then, by [19, Proposition 20], $\mathcal{D}_{\phi_P} = \mathcal{N}_P \mathcal{X}$, where $\mathcal{N}_P := \mathcal{D}_{\phi_P} \mathcal{X}^{-1}$ is a $(\Lambda_P, \Lambda_{P_0^{\text{crit}}})$-inner-outer factorization, and the outer part has a bounded inverse. Both range $(\mathcal{D}_{\phi_P})$ and range $(\mathcal{D}_{\phi_P})$ are closed because ϕ_P is Λ_P-coercive, by [19, Proposition 6]. Finally, claim (ii) of Proposition 38 implies that range $(\mathcal{D}_{\phi_P}) = \ell^2(\mathbf{Z}; U)$ if U is separable, because $\mathcal{D}_{\phi_P}(0) = I$ has a full range.

To prove claim (ii), Proposition 2 is used. Only the claim concerning $\mathcal{N}_P$ is somewhat nontrivial, and the outlines are given below. For a more complete presentation using the same technique, see the proof of (ii) of Proposition 56. First, the product DLS $\phi_P \phi^{-1}_{P_0^{\text{crit}}}$ is written

$$\phi_P \phi^{-1}_{P_0^{\text{crit}}} = \begin{pmatrix} \begin{bmatrix} A & BK_{P_0^{\text{crit}}} \\ 0 & A_{P_0^{\text{crit}}} \end{bmatrix} & \begin{bmatrix} B \\ B \end{bmatrix} \\ \begin{bmatrix} -K_P & K_{P_0^{\text{crit}}} \end{bmatrix} & I \end{pmatrix}.$$

Its the semigroup generator is seen to satisfy

$$\begin{bmatrix} A & BK_{P_0^{\rm crit}} \\ 0 & A_{P_0^{\rm crit}} \end{bmatrix}^j = \begin{bmatrix} A^j & A^j_{P_0^{\rm crit}} - A^j \\ 0 & A^j_{P_0^{\rm crit}} \end{bmatrix}.$$

Finally, looking at the Taylor coefficients of the I/O-map, we see

$$\begin{bmatrix} -K_P & K_{P_0^{\rm crit}} \end{bmatrix} \begin{bmatrix} A^j & A^j_{P_0^{\rm crit}} - A^j \\ 0 & A^j_{P_0^{\rm crit}} \end{bmatrix} \begin{bmatrix} B \\ B \end{bmatrix} = (K_{P_0^{\rm crit}} - K_P) A^j_{P_0^{\rm crit}} B.$$

We consider this claim to be proved. □

Let $P \in ric_{uw}(\Phi, J)$ be arbitrary. To the spectral DLS ϕ_P, we can associate a minimax cost optimization problem with the cost operator Λ_P, see [19, Section 3]. It follows from Proposition 55 and Theorem 27 that if one of these problems is solvable (in the sense of Theorem 27), then they all are, together with the original minimax problem associated to Φ and J. This is true just because all the I/O-maps have the same outer factor $\mathcal{X}$, if they have such factorization at all.

In Proposition 55, a particular fixed regular critical solution $P_0^{\rm crit} \in ric_0(\Phi, J)$ was picked and the proposition was formulated relative to this solution. One should ask whether we would have obtained another factorization $\mathcal{D}_{\phi_P} = \mathcal{N}'_P \mathcal{X}'$ for another critical solution, say $P_2^{\rm crit} \in Ric_{uw}(\phi, J)$. The answer in negative. In the proof of Corollary 30, we have seen that the indicators of the critical solutions are all the same: $\Lambda_{P^{\rm crit}} = \Lambda_{P_2^{\rm crit}}$. Then we might have two different $(\Lambda_P, \Lambda_{P^{\rm crit}})$ -inner-outer factorizations $\mathcal{D}_{\phi_P} = \mathcal{N}_P \mathcal{X} = \mathcal{N}'_P \mathcal{X}'$. However, the feed-through parts of both $\mathcal{X}$ and $\mathcal{X}'$ are normalized to identity operator I, and this implies by [19, Proposition 21] that $\mathcal{X} = \mathcal{X}'$ as the I/O-maps. It now follows that the factor $\mathcal{N}_P$ does not depend on the choice of the critical solution. However, the realizations $\phi_{P^{\rm crit}}$, $\phi_{P_2^{\rm crit}}$ for $\mathcal{X}$, $\mathcal{X}'$ might be different, because the feedback operators $K_{P_0^{\rm crit}}$, $K_{P_2^{\rm crit}}$ might differ. However, this can happen only in the orthogonal complement of $\overline{\text{range}\,(\mathcal{B})}$. So, if $\overline{\text{range}\,(\mathcal{B})} = H$, then $K_{P_0^{\rm crit}} = K_{P_2^{\rm crit}}$ as in the proof of Proposition 30, and the possible nonuniqueness of the realizations disappears.

The following proposition gives us realizations for chains of certain I/O-maps. It is instructive to compare the DLS ϕ_{P_1,P_2} to the realization of $\mathcal{N}_P$, given in claim (ii) of Proposition 55. We remark that the following tedious calculations depend on the properties of the Riccati equation only in a very implicit manner, if at all.

Proposition 56. *Let Φ be an DLS, and J self-adjoint. Let $P_1, P_2, P_3 \in ric(\Phi, J)$ be arbitrary. Define the DLS*

$$\phi_{P_1,P_2} = \begin{pmatrix} A_{P_2} & B \\ K_{P_2} - K_{P_1} & I \end{pmatrix}$$

and we denote $\mathcal{N}_{P_1,P_2} := \mathcal{D}_{\phi_{P_1,P_2}}$. Then *(i)* $\mathcal{N}_{P_1,P_2}^{-1} = \mathcal{N}_{P_2,P_1}$,

(ii) $\mathcal{N}_{P_1,P_2}\mathcal{N}_{P_2,P_3} = \mathcal{N}_{P_1,P_3}$,

(iii) Assume, in addition, that the conditions of Theorem 27 hold. Then $\mathcal{N}_{P_1,P_0^{\mathrm{crit}}} = \mathcal{N}_{P_1}$ *is the* $(\Lambda_{P_1}, \Lambda_{P_0^{\mathrm{crit}}})$*-inner factor of* $\mathcal{D}_{\phi_P}$. *Also* $\mathcal{N}_{P_1}\mathcal{N}_{P_2}^{-1} = \mathcal{N}_{P_1,P_2}$.

Proof. To prove claim (i), use claim (i) of Proposition 2. A direct calculation gives

$$\phi_{P_1,P_2}^{-1} = \begin{pmatrix} A_{P_2} - B(K_{P_2} - K_{P_1}) & B \\ -(K_{P_2} - K_{P_1}) & I \end{pmatrix} = \begin{pmatrix} A_{P_1} & B \\ K_{P_1} - K_{P_2} & I \end{pmatrix} = \phi_{P_2,P_1},$$

proving claim (i). To verify claim (ii), claim (ii) of Proposition 2 is now used. We obtain

$$\phi_{P_1,P_2}\phi_{P_2,P_3} = \begin{pmatrix} A_{P_2} & B \\ K_{P_2} - K_{P_1} & I \end{pmatrix} \begin{pmatrix} A_{P_3} & B \\ K_{P_3} - K_{P_2} & I \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} A_{P_2} & B(K_{P_3} - K_{P_2}) \\ 0 & A_{P_3} \end{bmatrix} & \begin{bmatrix} B \\ B \end{bmatrix} \\ \begin{bmatrix} (K_{P_2} - K_{P_1}) & (K_{P_3} - K_{P_2}) \end{bmatrix} & I \end{pmatrix}. \tag{51}$$

Now we have to consider the I/O-map of the product DLS $\phi_{P_1,P_2}\phi_{P_2,P_3}$. We first see that its feed-through operator I is that of $\mathcal{D}_{\phi_{P_1,P_3}}$. The rest is studied by applying the Taylor series formula (7) for the I/O-map of a DLS on the right hand side of (51). The whole trick lies in noting that the semigroup generator satisfies $\begin{bmatrix} A_{P_2} & B(K_{P_3}-K_{P_2}) \\ 0 & A_{P_3} \end{bmatrix} = \begin{bmatrix} A_{P_2} & A_{P_3}-A_{P_2} \\ 0 & A_{P_3} \end{bmatrix}$, and we have for the block matrices of this kind

$$A^j_{(\phi_{P_1,P_2}\phi_{P_2,P_3})} = \begin{bmatrix} A_{P_2} & A_{P_3} - A_{P_2} \\ 0 & A_{P_3} \end{bmatrix}^j = \begin{bmatrix} A^j_{P_2} & A^j_{P_3} - A^j_{P_2} \\ 0 & A^j_{P_3} \end{bmatrix}$$

for all $j \geq 0$, as can easily be shown by induction. We now obtain for all $j \geq 0$

$$\begin{aligned} & C_{(\phi_{P_1,P_2}\phi_{P_2,P_3})} A^j_{(\phi_{P_1,P_2}\phi_{P_2,P_3})} B_{(\phi_{P_1,P_2}\phi_{P_2,P_3})} \\ &= \begin{bmatrix} (K_{P_2} - K_{P_1}) & (K_{P_3} - K_{P_2}) \end{bmatrix} \begin{bmatrix} A^j_{P_2} & A^j_{P_3} - A^j_{P_2} \\ 0 & A^j_{P_3} \end{bmatrix} \begin{bmatrix} B \\ B \end{bmatrix} \\ &= (K_{P_2} - K_{P_1})A^j_{P_2}B + (K_{P_2} - K_{P_1})(A^j_{P_3} - A^j_{P_2})B \\ &+ (K_{P_3} - K_{P_2})A^j_{P_3}B \\ &= (K_{P_2} - K_{P_1})A^j_{P_3}B + (K_{P_3} - K_{P_2})A^j_{P_3}B \\ &= (K_{P_3} - K_{P_1})A^j_{P_3}B. \end{aligned}$$

But these equal the corresponding coefficients of ϕ_{P_1,P_3}, and claim (ii) is proved. Claim (iii) follows immediately from claim (ii) of Proposition 55. The last claim follows from the previous claims: $\mathcal{N}_{P_1}\mathcal{N}_{P_2}^{-1} = \mathcal{N}_{P_1,P_0^{\mathrm{crit}}}\mathcal{N}_{P_2,P_0^{\mathrm{crit}}}^{-1} = \mathcal{N}_{P_1,P_0^{\mathrm{crit}}}\mathcal{N}_{P_0^{\mathrm{crit}},P2} = \mathcal{N}_{P_1,P2}$. □

The I/O-maps of the DLSs ϕ_{P_1,P_2} will play crucial role in [25].

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ISSN 0784-3143
Edita, Espoo, 1999