1. Introduction.

The purpose of this paper is to study the maximal regularity of the solutions of the partial differential equation

$$\big(D_t^\alpha(u-u_0)\big)(t,x)+c(t,x)u_x(t,x)=f(t,x),\quad t,\,x\ge 0, \tag{1}$$

with initial and boundary conditions

$$\begin{aligned} u(0,\underline{x})&=u_0(\underline{x}),\\ u(\underline{t},0)&=u_1(\underline{t}). \end{aligned} \tag{2}$$

Here D_t^α denotes the fractional derivative of order $\alpha\in(0,1)$, see [8, p. 133], i.e.,

$$(D_t^\alpha v)(t)\stackrel{\text{def}}{=}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t g_{1-\alpha}(t-s)v(s)\,\mathrm{d}s,\quad t>0,$$
$$(D_t^\alpha v)(0)\stackrel{\text{def}}{=}\lim_{h\downarrow 0}\frac{1}{h}\int_0^h g_{1-\alpha}(h-s)v(s)\,\mathrm{d}s,$$

where

$$g_\beta(t)\stackrel{\text{def}}{=}\frac{1}{\Gamma(\beta)}t^{\beta-1},\quad t>0,\quad \beta>0,$$

and where v is (at least) continuous and satisfies $v(0)=0$. If $w(\underline{t},\underline{x})$ is a function of two variables, then $(D_t^\alpha w)(\underline{t},\underline{x})$ is the function $(t,x)\mapsto (D_t^\alpha w(\bullet,x))(t)$.

The maximal regularity results that we prove state essentially that if c and f are Hölder continuous with respect to one variable, c is strictly positive and satisfies an additional continuity assumption, and the initial/boundary data satisfy certain compatibility conditions, then there is a unique solution of (1) such that u_x and $D_t^\alpha(u-u_0)$ are Hölder continuous with respect to the same variable. In addition, appropriate Schauder estimates are obtained. By integration and subsequent interpolation we obtain additional results on the regularity of u in both variables.

Our work on the smoothness of solutions of (1) is motivated by problems related to the nonlinear fractional conservation law

$$D_t^\alpha(u-u_0)+\sigma(u)_x=f. \tag{3}$$

Equations of this type can be employed to approximate solutions of nonlinear conservation laws, [3], and the properties of their solutions have been studied in e.g. [4] and [5]. A sufficiently smooth solution of (3) satisfies (1) with $c(\underline{t},\underline{x})=\sigma'(u(\underline{t},\underline{x}))$. Thus the results we prove below can be used to show that if there is a solution of (3) that is Hölder continuous in one variable (continuously with respect to the other), then it follows, provided the functions f, u_0 and u_1 are sufficiently smooth, that it satisfies better Hölder conditions in both variables. Since this inference relies on quite strong assumptions on the smoothness of the solution it is, of course, not satisfactory. Unfortunately, there appear to be grave difficulties involved in trying to get the necessary bounds if one applies a direct bootstrapping technique. So it seems that one has to use a slightly different approach. We will return to this problem in future work.

Another motivation for studying equation (1) is that it is (when $c = 1$) the special case $\beta = 1$ of the equation

$$D_t^\alpha(u - u_0) + D_x^\beta(u - u_1) = f,$$

studied in [2] for the cases α, $\beta \in (0, 1)$.

Our proofs rely on an extension of [2, Thm. 6] to the nonconstant coefficient case and on an analogous extension of [7, Thms. 4.5, 5.5]. A key fact necessary for our analysis is that D_t^α is a positive operator with spectral angle $\alpha\pi/2 < \pi/2$. The condition $\alpha < 1$ is crucial for maximal regularity to hold. The Hölder spaces repeatedly appear in the analysis as interpolation spaces, see e.g. [1].

2. Statement of results.

Let X be a (complex) Banach space and let I be an interval. The Hölder spaces $\mathcal{C}^{(\gamma)}(I; X)$, $\gamma \in [0, 1]$ are defined by

$$\mathcal{C}^{(\gamma)}(I; X) \stackrel{\text{def}}{=} \left\{ f : I \to X \;\middle|\; \sup_{\substack{s,t \in I \\ s \neq t}} \frac{\|f(t) - f(s)\|_X}{|t - s|^\gamma} < \infty \right\},$$

with norm

$$\|f\|_{\mathcal{C}^{(\gamma)}(I)} \stackrel{\text{def}}{=} \sup_{t \in I} \|f(t)\|_X + \sup_{\substack{s,t \in I \\ s \neq t}} \frac{\|f(t) - f(s)\|_X}{|t - s|^\gamma}.$$

If $\gamma \in (1, 2]$, then $\mathcal{C}^{(\gamma)}(I; X) \stackrel{\text{def}}{=} \{ f \in \mathcal{C}^1(I; X) \mid f' \in \mathcal{C}^{(\gamma-1)}(I; X) \}$ with norm $\|f\|_{\mathcal{C}^{(\gamma)}(I)} \stackrel{\text{def}}{=} \sup_{t \in I} \|f(t)\|_X + \|f'\|_{\mathcal{C}^{(\gamma-1)}(I)}$. Observe that $\mathcal{C}^{(0)} \neq \mathcal{C}$ and $\mathcal{C}^{(1)} \neq \mathcal{C}^1$.

We consider a function of two variables to be a function of the first variable with values in a function space, that is, $f(\underline{t}, \underline{x})$ is the function $t \mapsto (x \mapsto f(t, x))$. When we have to consider the other possibility, namely the function $x \mapsto (t \mapsto f(t, x))$ we use the notation $f_\frown$ for the function, that is, we define

$$f_\frown(\underline{x}, \underline{t}) \stackrel{\text{def}}{=} f(\underline{t}, \underline{x}).$$

In addition, in each particular case below, the domains of the variables indicate how the function is to be viewed. The letters τ, T, etc. refer to the first variable t and ξ, $\mathfrak{X}$, etc. to the second variable x.

Observe that the statement $f \in \mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi];\mathbb{C}))$ is not equivalent to $f_{\frown} \in \mathcal{C}([0,\xi];\mathcal{C}^{(\mu)}([0,\tau];\mathbb{C}))$. The second implies the first, but not conversely. We do, however, have $f \in \mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}^{(\nu)}([0,\xi];\mathbb{C}))$ if and only if $f_{\frown} \in \mathcal{C}^{(\nu)}([0,\xi];\mathcal{C}^{(\mu)}([0,\tau];\mathbb{C}))$

Our first result concerns Hölder continuity in t. We will use the notation $\mathcal{C}_{0\mapsto 0}$ for functions vanishing at 0.

Theorem 1. *Assume that* $\alpha \in (0,1)$, $\tau > 0$, $\xi > 0$, $\mu \in (0,\alpha)$, *and that*

(i) $c_{\frown} \in \mathcal{C}([0,\xi];\mathcal{C}^{(\mu)}([0,\tau];\mathbb{R}))$, *and*

$$c(t,x) > 0, \quad (t,x) \in [0,\tau]\times[0,\xi];$$

(ii) $f \in \mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi];\mathbb{C}))$ *and* $f(0,0) = 0$;

(iii) $u_0 \in \mathcal{C}^1([0,\xi];\mathbb{C})$, $u_0(0) = u_0'(0) = 0$ *and*

$$c(0,\underline{x})u_0'(\underline{x}) - f(0,\underline{x}) \in \mathcal{C}^{(\mu/\alpha)}([0,\xi];\mathbb{C});$$

(iv) $u_1 \in \mathcal{C}_{0\mapsto 0}([0,\tau];\mathbb{C})$ *and* $D_t^\alpha u_1 \in \mathcal{C}^{(\mu)}_{0\mapsto 0}([0,\tau];\mathbb{C})$.

Then there exists a unique solution $u \in \mathcal{C}([0,\tau]\times[0,\xi];\mathbb{C})$ *of* (1) *on* $[0,\tau]\times[0,\xi]$ *such that*

(a) $u_x \in \mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi];\mathbb{C}))$;

(b) $u \in \mathcal{C}^{(\alpha)}([0,\tau];\mathcal{C}([0,\xi];\mathbb{C}))$;

(c) $u \in \mathcal{C}^{(\theta\mu+(1-\theta)\alpha)}([0,\tau];\mathcal{C}^{(\theta)}([0,\xi];\mathbb{C}))$ *for every* $\theta \in (0,1)$.

Moreover, there is a constant M *that depends on* α, μ, τ, ξ, *and* c *such that*

$$\begin{aligned} &\|u_x\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi]))} + \|u\|_{\mathcal{C}^{(\alpha)}([0,\tau];\mathcal{C}([0,\xi]))} + \|u\|_{\mathcal{C}^{(\theta\mu+(1-\theta)\alpha)}([0,\tau];\mathcal{C}^{(\theta)}([0,\xi]))} \\ &\quad \le M\Big(\|f\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi]))} + \|c(0,\underline{x})u_0'(\underline{x}) - f(0,\underline{x})\|_{\mathcal{C}^{\mu/\alpha}([0,\xi])} \\ &\qquad + \|D_t^\alpha u_1\|_{\mathcal{C}^{\mu}([0,\tau])}\Big). \end{aligned} \tag{4}$$

The claim (b) (and therefore also (c)) can be improved as follows: What we actuallly have is that

$$u(\underline{t},\underline{x}) - u_0(\underline{x}) + \tfrac{1}{\Gamma(\alpha+1)}\underline{t}^\alpha(c(0,\underline{x})u_0'(\underline{x}) - f(0,\underline{x})) \in \mathcal{C}^{(\alpha+\mu)}([0,\tau];\mathcal{C}([0,\xi];\mathbb{C})),$$

(except in the case where $\alpha + \mu = 1$; then the space of Lipschitz functions should be replaced by the Zygmund class). In order to use interpolation to get an improved version of (c) one must make extra assumptions on the smoothness of the term $c(0,\underline{x})u_0'(\underline{x}) - f(0,\underline{x})$ or not use the full force of (a).

Our second result concerns Hölder continuity in x.

Theorem 2. *Assume that* $\alpha \in (0,1)$, $\tau > 0$, $\xi > 0$, $\nu \in (0,1)$, *and that*

(i) $c \in \mathcal{C}([0,\tau], \mathcal{C}^{(\nu)}([0,\xi];\mathbb{R}))$ *and*

$$c(t,x) > 0, \quad (t,x) \in [0,\tau] \times [0,\xi];$$

(ii) $f_{\frown} \in \mathcal{C}^{(\nu)}([0,\xi];\mathcal{C}([0,\tau];\mathbb{C}))$ *and* $f(0,0) = 0$;

(iii) $u_0 \in \mathcal{C}^{(1+\nu)}([0,\xi];\mathbb{C})$, $u_0(0) = u_0'(0) = 0$;

(iv) $u_1 \in \mathcal{C}_{0\mapsto 0}([0,\tau];\mathbb{C})$ *and* $\frac{1}{c(\underline{t},0)}\big((D_t^\alpha u_1)(\underline{t}) - f(\underline{t},0)\big) \in \mathcal{C}_{0\mapsto 0}^{(\nu\alpha)}([0,\tau];\mathbb{C})$.

Then there exists a unique solution $u \in \mathcal{C}([0,\tau]\times[0,\xi];\mathbb{C})$ *of* (1) *such that*

(a) $(u_x)_{\frown} \in \mathcal{C}^{(\nu)}([0,\xi];\mathcal{C}([0,\tau];\mathbb{C}))$;

(b) $u \in \mathcal{C}^{(\alpha)}([0,\tau];\mathcal{C}^{(\nu)}([0,\xi];\mathbb{C}))$;

(c) $u \in \mathcal{C}^{((1-\theta)\alpha)}([0,\tau];\mathcal{C}^{(\theta+\nu)}([0,\xi];\mathbb{C}))$ *for each* $\theta \in (0,1)$.

Moreover, there is a constant M *that depends on* α, ν, τ, ξ *and* c *such that*

$$\begin{aligned} &\|u_x\|_{\mathcal{C}^{(0)}([0,\tau];\mathcal{C}^{(\nu)}([0,\xi]))} + \|u\|_{\mathcal{C}^{(\alpha)}([0,\tau];\mathcal{C}^{(\nu)}([0,\xi]))} + \|u\|_{\mathcal{C}^{((1-\theta)\alpha)}([0,\tau];\mathcal{C}^{(\theta+\nu)}([0,\xi]))} \\ &\qquad \le M\Bigg(\|f\|_{\mathcal{C}^{(0)}([0,\tau];\mathcal{C}^{(\nu)}([0,\xi]))} + \|u_0'\|_{\mathcal{C}^{\nu}([0,\xi])} \\ &\qquad\qquad + \left\|\frac{1}{c(\underline{t},0)}\Big((D_f^\alpha u_1)(\underline{t}) - f(\underline{t},0)\Big)\right\|_{\mathcal{C}^{\nu\alpha}([0,\tau])}\Bigg). \end{aligned} \tag{5}$$

We need the following simple lemma which is an extension of [6, Prop. 1.1.5, 2.2.12]. For the definition of the spaces $(X,Y)_{\theta,\infty}$ and the K-method, see, e.g., [1] or [6].

Lemma 3. *Let* X *and* Y *be Banach spaces that are continuously injected in a Hausdorff locally convex topological vector space. If* I *is an interval and* $f \in \mathcal{C}^{(\alpha)}(I;X) \cap \mathcal{C}^{(\beta)}(I;Y)$ *where* α, $\beta \in (0,2]$, *then* $f \in \mathcal{C}^{((1-\theta)\alpha+\theta\beta)}(I;(X,Y)_{\theta,\infty})$ *for each* $\theta \in (0,1)$ *and*

$$\begin{aligned} \|f\|_{\mathcal{C}^{((1-\theta)\alpha+\theta\beta)}(I;(X,Y)_{\theta,\infty})} &\le 2\big(\|f\|_{\mathcal{C}^{(\alpha)}(I;X)} + \|f\|_{\mathcal{C}^{(\beta)}(I;Y)}\big) \\ &\quad + \begin{cases} 2^\alpha |I|^{1-\alpha} \sup_{t\in I}\|f'(t)\|_X, & \text{if } \alpha > 1, \beta \le 1, \\ 2^\beta |I|^{1-\beta} \sup_{t\in I}\|f'(t)\|_Y, & \text{if } \beta > 1, \alpha \le 1. \end{cases} \end{aligned}$$

Here $|I|$ denotes the length of the interval I.

3. Proofs.

Proof of Lemma 3. First we recall that by [1, (2.2.2)] we have

$$\|v\|_{(X,Y)_{\theta,\infty}} \le \|v\|_X^{1-\theta}\|v\|_Y^\theta, \quad v \in (X,Y)_{\theta,\infty}. \tag{6}$$

Using this inequality we get

$$\sup_{t\in I}\|f(t)\|_{(X,Y)_{\theta,\infty}} \le \sup_{t\in I}\|f(t)\|_X + \sup_{t\in I}\|f(t)\|_Y.$$

Similarly, if α and $\beta > 1$ then

$$\sup_{t\in I}\|f'(t)\|_{(X,Y)_{\theta,\infty}} \le \sup_{t\in I}\|f'(t)\|_X + \sup_{t\in I}\|f'(t)\|_Y.$$

Furthermore, in the case where $0 < \alpha, \beta \le 1$, we get

$$\sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f(t)-f(s)\|_{(X,Y)_{\theta,\infty}}}{|t-s|^{(1-\theta)\alpha+\theta\beta}} \le \sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f(t)-f(s)\|_X}{|t-s|^{\alpha}} + \sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f(t)-f(s)\|_Y}{|t-s|^{\beta}},$$

and, in the case where $1 < \alpha, \beta \le 2$, we have

$$\sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f'(t)-f'(s)\|_{(X,Y)_{\theta,\infty}}}{|t-s|^{(1-\theta)\alpha+\theta\beta-1}} \le \sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f'(t)-f'(s)\|_X}{|t-s|^{\alpha-1}} + \sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f'(t)-f'(s)\|_Y}{|t-s|^{\beta-1}}.$$

Next we consider the case where $(\alpha-1)(\beta-1) < 0$ and we assume first that $\alpha > 1$ and $\beta < 1$. Furthermore, let us assume that $\theta = (\alpha-1)/(\alpha-\beta)$ so that $(1-\theta)\alpha + \theta\beta = 1$. This case is made more difficult by the fact that the estimates we obtain depend on the length of the interval. First we derive an estimate on f'.

Let $t \in I$ and let $h \neq 0$ be such that $t + h \in I$. Now we have

$$f'(t) = \int_0^1 \big(f'(t) - f'(t+\sigma h)\big)\,\mathrm{d}\sigma + \frac{f(t+h)-f(t)}{h},$$

and we see that the first term belongs to X and the second to Y. If $\rho > 0$, then

$$K(\rho, f'(t), X, Y) \le |h|^{\alpha-1} \sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f'(t)-f'(s)\|_X}{|t-s|^{\alpha-1}} + \rho|h|^{\beta-1} \sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f(t)-f(s)\|_Y}{|t-s|^{\beta}}.$$

If I is unbounded we can for any $\rho > 0$ choose h such that $|h| = \rho^{1/(\alpha-\beta)}$. Then

$$\rho^{-\theta}K(\rho, f'(t), X, Y) \le \sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f'(t)-f'(s)\|_X}{|t-s|^{\alpha-1}} + \sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f(t)-f(s)\|_Y}{|t-s|^{\beta}}.$$

If I has length $|I|$ we can only be certain to be able to have $|h| < |I|/2$ and this means that we get the inequality above for $\rho < (|I|/2)^{\alpha-\beta}$. For larger values of ρ we have

$$\rho^{-\theta} K(\rho, f'(t), X, Y) \leq \left(\frac{|I|}{2}\right)^{1-\alpha} \sup_{t\in I} \|f'(t)\|_X.$$

Thus we have shown that $f'(t) \in (X,Y)_{\theta,\infty}$ and, since $f(t)-f(s) = \int_s^t f'(\sigma)\,\mathrm{d}\sigma$, and as we can treat $f'(\sigma)$ as above, we conclude that $f \in \mathcal{C}^{(1)}(I;(X,Y)_{\theta,\infty})$ and that we have the estimate

$$\|f\|_{\mathcal{C}^{(1)}(I;(X,Y)_{\theta,\infty})} \leq \sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f'(t)-f'(s)\|_X}{|t-s|^{\alpha-1}} + \sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f(t)-f(s)\|_Y}{|t-s|^{\beta}} + \left(\frac{|I|}{2}\right)^{1-\alpha} \sup_{t\in I} \|f'(t)\|_X.$$

If $\alpha < 1$ and $\beta > 1$, then we use the fact that $(X,Y)_{\theta,\infty} = (Y,X)_{1-\theta,\infty}$ and obtain a similar result.

Next we consider the case where $\alpha > 1$ and $\beta = 1$. Let $t \in I$. Since f is differentiable in X we see that $\{\frac{1}{h}(f(t+h)-f(t))\}$ is Cauchy sequence in X as $h \to 0$ (such that $t+h \in I$) and then it follows from (6) that it is a Cauchy sequence in $(X,Y)_{\theta,\infty}$ as well. Therefore it converges in this space and this shows that f is differentiable in $(X,Y)_{\theta,\infty}$. Applying inequality (6) to $\frac{1}{h}\big(f(t+h)-f(t)\big) - \frac{1}{k}\big(f(s+k)-f(s)\big)$ and then letting h and $k \to 0$ we get

$$\sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f'(t)-f'(s)\|_{(X,Y)_{\theta,\infty}}}{|t-s|^{(1-\theta)\alpha+\theta\beta}} \leq \sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f'(t)-f'(s)\|_X}{|t-s|^{\alpha}} + 2 \sup_{\substack{t,s\in I\\ s\neq t}} \frac{\|f(t)-f(s)\|_Y}{|t-s|^{\beta}},$$

which is the desired result. If $\alpha = 1$ and $\beta > 1$ we again use the fact that $(X,Y)_{\theta,\infty} = (Y,X)_{1-\theta,\infty}$.

The remaining cases can be obtained by combining the results already obtained and using the reiteration theorem.

Lemma 4. *Let $\tau > 0$, $a \in \mathcal{C}([0,\tau];\mathbb{R})$ with $a(\underline{t}) > 0$, and let $Y = \mathcal{C}_{0\mapsto 0}([0,\tau];\mathcal{C}([0,\xi];\mathbb{C}))$. Define the operator A_a in Y by $A_a u(\underline{t}) = \frac{1}{a(\underline{t})}(D_t^\alpha u)(\underline{t})$. Then there is for each $\eta > 0$ a constant N that depends on η, α, τ, $\inf_{t\in[0,\tau]} a(t)$, $\sup_{t\in[0,\tau]} a(t)$, and on the modulus of continuity of a, such that*

$$\|(\lambda I + A_a)^{-1}\|_{\mathcal{L}(Y)} \leq \frac{N}{|\lambda|+1}, \quad \text{when } |\arg\lambda| \leq \pi(1-\frac{\alpha}{2}) - \eta.$$

Proof of Lemma 4. If we can prove this lemma with $\mathcal{C}([0,\xi];\mathbb{C})$ replaced by $\mathbb{C}$, then we get pointwise estimates for functions depending on a second variable, and these estimates also imply continuity, so we get the original claim. Therefore we shall consider this case only.

Let

$$a_0 \stackrel{\text{def}}{=} \min\{1, \inf_{t\in[0,\tau]} a(t)\} \quad \text{and} \quad a_1 \stackrel{\text{def}}{=} \sup_{t\in[0,\tau]} a(t).$$

For every $h \in Y$ we have to show that there is a solution g of the equation $\lambda g + A_a g = h$ such that $\|g\|_Y \le \frac{N}{|\lambda|+1}\|h\|_Y$. We can rewrite this equation as

$$(D_t^\alpha g)(\underline{t}) + \lambda a(\underline{t})g(\underline{t}) = a(\underline{t})h(\underline{t}). \tag{7}$$

By [2, Lemma 11.(b)], there is a constant M_1 depending on η, α, and τ such that

$$\|(D_t^\alpha + \mu I)^{-1}\|_{\mathcal{L}(Y)} \le \frac{M_1}{|\mu|+1}, \quad |\arg\mu| \le \pi(1-\frac{\alpha}{2}) - \eta. \tag{8}$$

Let

$$\epsilon = \frac{a_0}{2M_1},$$

and let T_ϵ be a positive number such that

$$|a(t) - a(s)| \le \epsilon, \quad t, s \in [0,\tau], \quad |t-s| \le T_\epsilon.$$

Suppose that $T \in [0,\tau)$ and that there is a continuous solution g of (7) on $[0,T]$ such that for some constant N_T, depending on η, α, τ, and on the sup, inf, and modulus of continuity of a, one has

$$\sup_{t\in[0,T]} |g(t)| \le \frac{N_T}{|\lambda|+1} \sup_{t\in[0,T]} |h(t)|, \tag{9}$$

for all $\lambda \in \mathbb{C}$ satisfying $|\arg\lambda| \le \pi(1-\frac{\alpha}{2}) - \eta$. If $T = 0$ we take $g = 0$. Now we rewrite (7) in the form

$$(D_t^\alpha w)(\underline{t}) + \lambda a(T)w(\underline{t}) = a(\underline{t})h(\underline{t}) + \lambda\big(a(T) - a(\underline{t})\big)v(\underline{t}), \tag{10}$$

and we see that if $v = w$, then we have a solution of (7).

We define $\hat{T} = \min\{\tau, T + T_\epsilon\}$ and

$$U = \big\{\, v \in \mathcal{C}_{0\mapsto 0}([0,\hat{T}];\mathbb{C}) \;\big|\; v(t) = g(t), \quad t \in [0,T] \,\big\}.$$

Since $D_t^\alpha + \lambda a(T)I$ is invertible in $\mathcal{C}_{0\mapsto 0}([0,\hat{T}];\mathbb{C})$ there is a solution of (10) for each $v \in U$ and the uniqueness guarantees that we have $w \in U$. Denote

$w = G(v)$. If v_1 and v_2 are two functions in U, then we have by (8) and our choices of T_ϵ and ϵ and by the fact that $v_1(t) - v_2(t) = 0$ when $t \in [0, T]$ that

$$\sup_{t\in[0,\hat{T}]} |G(v_1)(t) - G(v_2)(t)| \leq \frac{M_1}{|\lambda|a(T)+1} \sup_{t\in[0,\hat{T}]} |\lambda||a(T)-a(t)||v_1(t)-v_2(t)|$$
$$\leq \frac{M_1}{a_0} \sup_{t\in[T,\hat{T}]} |a(T)-a(t)| \sup_{t\in[0,\hat{T}]} |v_1(t)-v_2(t)| \leq \frac{1}{2} \sup_{t\in[0,\hat{T}]} |v_1(t)-v_2(t)|.$$

Thus G is a contraction on U and we get a unique fixed-point g that is the solution of (7) on $[0, \hat{T}]$. If we let $v_0 \in U$ be such that $v_0(t) = g(T)$ for $t \in [T, \hat{T}]$, then we get from (8) and (10) the estimate

$$\sup_{t\in[0,\hat{T}]} |G(v_0)(t)| \leq \frac{2M_1 a_1 |\lambda|}{|\lambda|a(T)+1} \sup_{t\in[0,T]} |g(t)| + \frac{M_1 a_1}{|\lambda|a(T)+1} \sup_{t\in[0,\hat{T}]} |h(t)|.$$

Because $\sup_{t\in[0,\hat{T}]}|v_0(t)| = \sup_{t\in[0,T]}|g(t)|$ and the contraction factor is $\frac{1}{2}$ we get the following estimate for the norm of the fixed-point,

$$\sup_{t\in[0,\hat{T}]} |g(t)| \leq \Big(3 + \frac{4M_1 a_1}{a_0}\Big) \sup_{t\in[0,T]} |g(t)| + \frac{2M_1 a_1}{a_0(|\lambda|+1)} \sup_{t\in[0,\hat{T}]} |h(t)|,$$

and we conclude that (9) holds with $[0, T]$ replaced by $[0, \hat{T}]$ and N_T replaced by $N_{\hat{T}}$, where

$$N_{\hat{T}} = N_T \Big(3 + \frac{4M_1 a_1}{a_0}\Big) + \frac{2M_1 a_1}{a_0}.$$

Since the choice of T_ϵ only depends on M_1, the lower bound of a and on the modulus of continuity of a, we get the desired conclusion by induction.

Proof of Theorem 1.

By the linearity of the problem it can easily be split into two parts, where in the first part we have $u_1 = 0$ and $f(\underline{t}, 0) = 0$ and in the second part we have $u_0 = 0$ and $f(0, \underline{x}) = 0$. We start by considering the case where $u_1 = 0$ and $f(\underline{t}, 0) = 0$.

Let

$$c_0 = \inf_{\substack{t\in[0,\tau]\\ x\in[0,\xi]}} c(t,x), \qquad c_1 = \sup_{\substack{t\in[0,\tau]\\ x\in[0,\xi]}} c(t,x). \tag{11}$$

We begin by studying the following equation:

$$\big(D_t^\alpha (u - u_0)\big)(t,x) + b(x)u_x(t,x) = g(t,x), \quad t \in [0,\tau], \quad x \in [0,\xi], \tag{12}$$

with boundary condition $u(\underline{t},0)=0$ under the following assumption on the function b:

$$b\in\mathcal{C}(\mathbb{R}^+;\mathbb{R}) \text{ and } 0<c_0\le b(\underline{x})\le c_1<\infty. \tag{13}$$

We denote by B_b the linear operator in $\mathcal{C}_{0\mapsto 0}([0,\xi];\mathbb{C})$ with domain

$$\mathcal{D}_{B_b}=\left\{\, v\in\mathcal{C}^1([0,\xi];\mathbb{C}) \;\middle|\; v(0)=v'(0)=0 \,\right\}$$

and

$$(B_b v)(x)=b(x)v'(x),\quad x\in[0,\xi].$$

We denote by B the corresponding operator with $b(\underline{x})=1$ and ξ replaced by $\xi_0=\xi/c_0$.

Thus (12) can be written as

$$D_t^\alpha(u-u_0)+B_b u=g. \tag{14}$$

Next, perform a change of variable $y=\int_0^x \frac{1}{b(s)}\,\mathrm{d}s$, so that equation (14) is replaced by

$$D_t^\alpha(u^b-u_0^b)+Bu^b=g^b, \tag{15}$$

where

$$\begin{aligned} g^b(\underline{t},\underline{y})&=g(\underline{t},\rho(\underline{y})),\\ u_0^b(\underline{y})&=u_0(\rho(\underline{y})),\end{aligned}\quad y\in[0,\xi_b]\quad\text{and}\quad \begin{aligned} g^b(\underline{t},\underline{y})&=g(\underline{t},\xi),\\ u_0^b(\underline{y})&=u_0(\xi),\end{aligned}\quad y\in(\xi_b,\xi_0],$$

where $\xi_b=\int_0^\xi \frac{1}{b(s)}\,\mathrm{d}s$ and ρ is the inverse of the function $x\mapsto\int_0^x\frac{1}{b(s)}\,\mathrm{d}s$. By [2, Thm. 6.(a)] equation (15) has a unique solution u^b which satisfies the bound

$$\begin{aligned}&\|Bu^b(\underline{t})-g^b(0)\|_{\mathcal{C}^\mu([0,\tau];\mathcal{C}_{0\mapsto 0}([0,\xi_0]))}\\ &\qquad\le M_2\left(\|Bu_0^b-g^b(0)\|_{\mathcal{C}^{\mu/\alpha}([0,\xi_0])}+\|g^b(\underline{t})-g^b(0)\|_{\mathcal{C}^\mu([0,\tau];\mathcal{C}_{0\mapsto 0}([0,\xi_0]))}\right),\end{aligned}$$

where M_2 depends on α, μ, τ and ξ_0. Now we change variables back again, that is, we define

$$u(\underline{t},x)=u^b\left(\underline{t},\int_0^x\frac{1}{b(s)}\,\mathrm{d}s\right),\qquad\text{for } x\in[0,\xi].$$

We can therefore conclude that there is a unique solution u of (12) such that

$$\begin{aligned}\|u_x\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi];\mathbb{C}))}\le M_3\Big(&\|b(\underline{x})u_0'(\underline{x})-g(0,\underline{x})\|_{\mathcal{C}^{\mu/\alpha}([0,\xi])}\\ &+\|g\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi]))}\Big),\end{aligned}\tag{16}$$

where (with some crude estimates) $M_3 = \frac{1}{c_0}(M_2 \max\{1, c_1^{\mu/\alpha}\} + 2)$. Note that since M_2 depends on ξ_0, M_3 depends on c_0 as well.

Choose ϵ to be so small that

$$\frac{M_3 \epsilon}{c_0} \leq \frac{1}{2}, \tag{17}$$

and choose $T_\epsilon > 0$ such that

$$\|c\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi]))} T_\epsilon^\mu \leq \frac{\epsilon}{2}. \tag{18}$$

Suppose $T \in [0, \tau)$ and that we have found a solution of (1) on $[0, T] \times [0, \xi]$ that satisfies the inequality

$$\begin{aligned} &\|u_x\|_{\mathcal{C}^{(\mu)}([0,T];\mathcal{C}([0,\xi]))} \\ &\quad \leq N_T \Big(\|f\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi]))} + \|c(0,\underline{x})u_0'(\underline{x}) - f(0,\underline{x})\|_{\mathcal{C}^{\mu/\alpha}([0,\xi])} \Big), \end{aligned} \tag{19}$$

for some constant N_T depending only on α, μ, τ, ξ, c_0, and c_1. If $T = 0$ we take this solution to be $u(0, \underline{x}) = u_0(\underline{x})$. We let $\hat{T} = \min\{\tau, T + T_\epsilon\}$ and let U denote the set

$$\begin{aligned} U \stackrel{\text{def}}{=} \{\, v \in \mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi];\mathbb{C})) \mid v(t,\underline{x}) = u_x(t,\underline{x}), \quad 0 \leq t \leq T, \\ v(\underline{t},0) = 0 \,\}. \end{aligned}$$

For each $v \in U$ we proceed to find a solution w of the equation

$$D_t^\alpha (w - u_0)(\underline{t},\underline{x}) + c(T,\underline{x}) w_x(\underline{t},\underline{x}) = f(\underline{t},\underline{x}) + \big(c(T,\underline{x}) - c(\underline{t},\underline{x})\big) v(\underline{t},\underline{x}), \tag{20}$$

with boundary condition $w(\underline{t}, 0) = 0$. Observe that the right-hand side of (20) evaluated at $t = 0$ is

$$f(0,x) + (c(T,x) - c(0,x))u_0'(x),$$

and therefore the element $b(x)u_0'(\underline{x}) - f(0,\underline{x})$ appearing in (16) when $b(\underline{x}) = c(T,\underline{x})$ is $c(0,x)u_0'(x) - f(0,x)$. Thus we conclude from (iii) and from the results above that we can find a solution w of (20) such that $w_x \in \mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi];\mathbb{C}))$ and the uniqueness guarantees that we have $w_x \in U$.

Let us denote the mapping $v \to w_x$ by $w_x = G(v)$. Using the linearity of equation (20), and (16) with $b(\underline{x}) = c(T,\underline{x})$ once more, we conclude that

$$\begin{aligned} &\|(G(v_1) - G(v_2))(\underline{t},\underline{x})\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi]))} \\ &\qquad \leq M_3 \|(c(T,\underline{x}) - c(\underline{t},\underline{x}))(v_1 - v_2)(\underline{t},\underline{x})\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi]))}. \end{aligned}$$

Let $v_\Delta = v_1 - v_2$ and $c_\Delta(\underline{t},\underline{x}) = c(T,\underline{x}) - c(\underline{t},\underline{x})$. Since v_1 and $v_2 \in U$ it follows that $v_\Delta(t,\underline{x}) = 0$ for $t \in [0,T]$ and therefore we can, when analyzing the term $(c(T,\underline{x}) - c(\underline{t},\underline{x}))(v_1 - v_2)(\underline{t},\underline{x})$ assume that $c(t,\underline{x}) = c(T,\underline{x})$ for $t \in [0,T]$. Thus we conclude from (18) that

$$\sup_{\substack{t\in[0,\hat{T}]\\ x\in[0,\xi]}} |c_\Delta(t,x)v_\Delta(t,x)| \le \epsilon \sup_{\substack{t\in[0,\hat{T}]\\ x\in[0,\xi]}} |v_\Delta(t,x)|.$$

Furthermore, if we write $c_\Delta(t,x)v_\Delta(t,x) - c_\Delta(s,x)v_\Delta(s,x) = c_\Delta(t,x)(v_\Delta(t,x) - v_\Delta(s,x)) + (c_\Delta(t,x) - c_\Delta(s,x))(v_\Delta(s,x) - v_\Delta(T,x))$, using the fact that $v_\Delta(T,\underline{x}) = 0$, and use (18) once again, we conclude that

$$\begin{aligned}\|(c(T,x) - c(t,x))(v_1 - v_2)(t,x)\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi]))}&\\ \le \epsilon\|(v_1 - v_2)(t,x)\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi]))}&.\end{aligned}$$

Hence we have

$$\|(G(v_1) - G(v_2))(t,x)\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi]))} \le \frac{M_3\epsilon}{c_0}\|(v_1 - v_2)(t,x)\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi]))},$$

and we see that the mapping G is a contraction and there is a unique fixed-point. Thus we get a solution on the interval $[0,\hat{T}]$.

If we take $v_0 \in U$ to be such that $v_0(t,\underline{x}) = u(T,\underline{x})$ for $t \in [T,\hat{T}]$ then $\|v_0\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi]))} = \|u\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi]))}$ and using (16) to estimate $\|G(v_0)\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi]))}$ we conclude from (19) that (19) holds with T replaced by $\hat{T}$ and N_T replaced by

$$N_{\hat{T}} = \Big(3 + \frac{4M_2c_1}{c_0}\Big)N_T + 2M_2.$$

Since this procedure can be repeated with the same T_ϵ, we find a solution on $[0,\tau]$ that satisfies the desired bounds.

Next we consider the case where $u_0 = 0$ and $f(0,\underline{x}) = 0$. We proceed in the same manner as above but instead of using [2, Thm. 6] we use [7, Thm. 5.5] combined with Lemma 4. The conclusion we can draw is that for every $a \in \mathcal{C}^{(\mu)}([0,\tau];\mathbb{C})$ with $a(\underline{t}) > 0$ there is a constant M_4 depending on α, μ, τ, and on $\inf_{t\in[0,\tau]} a(t)$ and $\|a\|_{\mathcal{C}^{(\mu)}([0,\tau])}$ such that if $g \in \mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi];\mathbb{C}))$ with $g(0,\underline{x}) = 0$ and $D_t^\alpha u_1 \in \mathcal{C}^{(\mu)}_{0\mapsto 0}([0,\tau];\mathbb{C})$, then there is a unique solution u of the equation

$$\begin{aligned}&\frac{1}{a(t)}(D_t^\alpha u)(t,x) + u_x(t,x) = g(t,x),\\ &u(t,0) = u_1(t), \quad t \in [0,\tau],\end{aligned} \tag{21}$$

such that

$$\|D_t^\alpha u\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi]))} \le M_4\Big(\|g\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi]))} + \|D_t^\alpha u_1\|_{\mathcal{C}^{(\mu)}([0,\tau])}\Big). \quad (22)$$

Choose ϵ to be so small that

$$\epsilon M_4 \left\|\frac{1}{c(\underline{t},\underline{x})}\right\|^2_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi]))} \le \frac{1}{2},$$

and choose $\mathfrak{X}_\epsilon > 0$ so that

$$\|c(\underline{t},x) - c(\underline{t},y)\|_{\mathcal{C}^{(\mu)}([0,\tau])} \le \epsilon, \quad |x-y| \le \mathfrak{X}_\epsilon, \quad x,y \in [0,\xi].$$

We assume that there is a solution of (1) on $[0,\tau]\times[0,\mathfrak{X}]$ for some $\mathfrak{X} \in [0,\xi)$ such that there is a constant $N_{\mathfrak{X}}$ satisfying

$$\|D_t^\alpha u\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\mathfrak{X}]))} \le N_{\mathfrak{X}}\Big(\|f\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\mathfrak{X}]))} + \|D_t^\alpha u_1\|_{\mathcal{C}^{(\mu)}([0,\tau])}\Big). \quad (23)$$

If $\mathfrak{X} = 0$ this is certainly the case. Let $\hat{\mathfrak{X}} = \min\{\xi, \xi + \mathfrak{X}_\epsilon\}$. In analogy with the previous case, we consider the equation

$$\begin{aligned} &\frac{1}{c(\underline{t},\mathfrak{X})}(D_t^\alpha w)(\underline{t},\underline{x}) + w_x(\underline{t},\underline{x}) = \frac{f(t,x)}{c(\underline{t},x)} + \left(\frac{1}{c(\underline{t},\mathfrak{X})} - \frac{1}{c(\underline{t},\underline{x})}\right) v(\underline{t},\underline{x}), \\ &w(\underline{t},0) = u_1(\underline{t}), \end{aligned} \quad (24)$$

and we see immediately that if $v = D_t^\alpha w$, then we have equation (1). We define the set U to be

$$U \stackrel{\text{def}}{=} \big\{ v \in \mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\hat{\mathfrak{X}}];\mathbb{C})) \;\big|\; v(\underline{t},x) = (D_t^\alpha u)(\underline{t},x), \quad 0 \le x \le \mathfrak{X}, \quad v(0,\underline{x}) = 0 \big\}.$$

If $v_1, v_2 \in U$, then $v_\Delta = v_1 - v_2$ satisfies $v_\Delta(\underline{t},x) = 0$ when $x \in [0,\mathfrak{X}]$. Therefore we can use the fact that $\mathcal{C}^{(\mu)}$ is a Banach algebra with pointwise multiplication to see that

$$\begin{aligned} &\left\|\left(\frac{1}{c(\underline{t},\mathfrak{X})} - \frac{1}{c(\underline{t},\underline{x})}\right) v_\Delta(\underline{t},\underline{x})\right\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\hat{\mathfrak{X}}]))} \\ &\qquad \le \epsilon \left\|\frac{1}{c(\underline{t},\underline{x}))}\right\|^2_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi]))} \|v_\Delta\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\hat{\mathfrak{X}}]))}. \end{aligned}$$

By (22) this shows that the mapping $v \mapsto w_x$ is a contraction and therefore we get a fixed-point which is the solution u_x on $[0,\tau]\times[0,\hat{\mathfrak{X}}]$. We can take

v_0 in U such that $v_0(\underline{t}, x) = (D_t^\alpha u)(\underline{t}, \mathfrak{X})$, for $x \in [\mathfrak{X}, \hat{\mathfrak{X}}]$, and then we get the estimate (23) with $\mathfrak{X}$ replaced by $\hat{\mathfrak{X}}$ and $N_{\mathfrak{X}}$ replaced by $N_{\hat{\mathfrak{X}}} = N_{\mathfrak{X}}(3 + 4M_4\|1/c\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi]))}) + 2M_4 \max\{1, \|1/c\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi]))}\}$. Since we can choose $\mathfrak{X}_\epsilon$ independent of $\mathfrak{X}$ we get the desired result by iteration.

From the results established so far we know that (a) holds and that $D_t^\alpha(u - u_0) \in \mathcal{C}^{(\mu)}([0,\tau]; \mathcal{C}([0,\xi]; \mathbb{C}))$. The inverse of D_t^α is given by $(D_t^{-\alpha} h)(\underline{t}) = \int_0^{\underline{t}} g_\alpha(\underline{t} - s)h(s)\,\mathrm{d}s$ and so it follows from a straightforward calculation that $u - u_0 \in \mathcal{C}^{(\alpha)}([0,\tau]; \mathcal{C}([0,\xi]; \mathbb{C}))$. By assumption $u_0 \in \mathcal{C}^1([0,\xi]; \mathbb{C})$, and therefore we have (b). Since (a) is equivalent to having $u \in \mathcal{C}^{(\mu)}([0,\tau]; \mathcal{C}^1([0,\xi]; \mathbb{C}))$, we get (c) from Lemma 3. The estimates in (4) follow immediately.

Finally observe that the arguments in the above paragraph can be used to validate the remark after Theorem 1. □

Proof of Theorem 2.

First we consider the case where $u_1 = 0$ and $f(\underline{t}, 0) = 0$. We proceed in the same way as in the proof of Theorem 1 and by applying [2, Thm. 6.(c)] we conclude that there is a unique solution of (12) such that for some constant M_5 (depending on c_0 and c_1 and $\|b\|_{\mathcal{C}^\nu([0,\xi])}$ but not otherwise on b)

$$\|u_x\|_{\mathcal{C}([0,\tau];\mathcal{C}^{(\nu)}([0,\xi]))} \leq M_5\Big(\|u_0'\|_{\mathcal{C}^{(\nu)}([0,\xi])} + \|g\|_{\mathcal{C}([0,\tau];\mathcal{C}^{(\nu)}([0,\xi]))}\Big). \tag{25}$$

Choose ϵ to be so small that

$$M_5\epsilon \leq \frac{1}{2},$$

and choose T_ϵ so that

$$\|c(t,\underline{x}) - c(s,\underline{x})\|_{\mathcal{C}^{(\nu)}([0,\xi])} \leq \epsilon, \quad |t - s| \leq T_\epsilon, \quad t, s \in [0,\tau]. \tag{26}$$

Suppose that we have found a solution of (1) on $[0,T] \times [0,\xi]$ that satisfies the inequality

$$\|u_x\|_{\mathcal{C}([0,T];\mathcal{C}^{(\nu)}([0,\xi]))} \leq N_T\Big(\|f\|_{\mathcal{C}([0,\tau];\mathcal{C}^{(\nu)}([0,\xi]))} + \|u_0'(\underline{x})\|_{\mathcal{C}^{(\nu)}([0,\xi])}\Big), \tag{27}$$

If $T = 0$ we take this solution to be $u(0,\underline{x}) = u_0(\underline{x})$. Let $\hat{T} = \min\{\tau, T + T_\epsilon\}$ and let U denote the set

$$U \stackrel{\text{def}}{=} \{\, v_\frown \in \mathcal{C}^{(\nu)}([0,\xi]; \mathcal{C}([0,\hat{T}]; \mathbb{C})) \mid v(t,\underline{x}) = u_x(t,\underline{x}), \quad 0 \leq t \leq T, \\ v(\underline{t}, 0) = 0 \,\}.$$

For each $v \in U$, we can by the results above find a solution w of (20) such that $(w_x)_\frown \in \mathcal{C}^{(\nu)}([0,\xi]; \mathcal{C}([0,\hat{T}]; \mathbb{C}))$ and we can apply inequality (25). The uniqueness of the solution guarantees that we have $w_x \in U$.

Let us denote the mapping $v \to w_x$ by $w_x = G(v)$. Using the linearity of the equation and (25) we conclude that

$$\|(G(v_1) - G(v_2))(\underline{t}, \underline{x})\|_{\mathcal{C}([0,\hat{T}];\mathcal{C}^{(\nu)}([0,\xi]))} \\ \leq M_5 \|(c(T, \underline{x}) - c(\underline{t}, \underline{x}))(v_1 - v_2)(\underline{t}, \underline{x})\|_{\mathcal{C}([0,\hat{T}];\mathcal{C}^{(\nu)}([0,\xi]))}.$$

Again we use the fact that since v_1 and $v_2 \in U$ it follows that $(v_1 - v_2)(t, \underline{x}) = 0$ for $t \in [0, T]$, together with the fact that $\mathcal{C}^{(\nu)}([0, \xi]; \mathcal{C}([0, \hat{T}]; \mathbb{C}))$ is a Banach algebra, to conclude that

$$\|(c(T, \underline{x}) - c(\underline{t}, \underline{x}))(v_1 - v_2)(\underline{t}, \underline{x})\|_{\mathcal{C}([0,\hat{T}];\mathcal{C}^{(\nu)}([0,\xi]))} \\ \leq \epsilon \|(v_1 - v_2)\|_{\mathcal{C}([0,\hat{T}];\mathcal{C}^{(\nu)}([0,\xi]))}.$$

Hence we see that the mapping G is a contraction and there is a unique fixed-point. Therefore we get a solution on the interval $[0, \hat{T}]$ and since this procedure can be repeated with the same T_ϵ, we find a solution on $[0, \tau]$.

To see that the desired bounds hold, we observe that we can take $v_0 \in U$ such that $v_0(t, \underline{x}) = u_x(T, \underline{x})$ so that we have a bound on $\|v_0\|_{\mathcal{C}([0,\hat{T}];\mathcal{C}^{(\nu)}([0,\xi]))}$.

We proceed to consider the case where $u_0 = 0$ and $f(0, \underline{x}) = 0$. Now we apply [7, Thm 4.5] with A as in Lemma 4 and the x and t variables interchanged. The conclusion we can draw is that there is a constant M_6 depending on α, ν, τ, and on c such that if $g_\frown \in \mathcal{C}^{(\nu)}([0, \xi]; \mathcal{C}([0, \tau]; \mathbb{C}))$ with $g(0, \underline{x}) = 0$ and $g(\underline{t}, 0) - \frac{1}{a(\underline{t})}(D_t^\alpha u_1)(\underline{t}) \in \mathcal{C}_{0 \mapsto 0}^{(\alpha\nu)}([0, \tau]); \mathbb{C})$ then there is a unique solution u of equation (21) such that

$$\|D_t^\alpha u\|_{\mathcal{C}([0,\tau];\mathcal{C}^{(\nu)}([0,\xi]))} \leq M_6 \Big(\|g\|_{\mathcal{C}([0,\tau];\mathcal{C}^{(\nu)}([0,\xi]))} \\ + \Big\| g(\underline{t}, 0) - \frac{1}{a(\underline{t})}(D_t^\alpha u_1)(\underline{t}) \Big\|_{\mathcal{C}^{(\alpha\nu)}([0,\tau])} \Big).$$

Due to the uniqueness, the same inequality certainly holds with the same constant M_6 if τ and ξ are replaced by some smaller numbers.

Let ϵ be such that

$$M_6 \epsilon \leq \frac{1}{2},$$

and let $\mathfrak{X}_\epsilon$ satisfy

$$\left\| \frac{1}{c(\underline{t}, \underline{x})} \right\|_{\mathcal{C}([0,\tau];\mathcal{C}^{(\nu)}([0,\xi]))} \mathfrak{X}_\epsilon^\nu \leq \frac{\epsilon}{2}.$$

Again we suppose that we have found a solution on $[0, \tau] \times [0, \mathfrak{X}]$ and then consider (24). Observe that the term $g(\underline{t}, 0) - \frac{1}{a(\underline{t})}(D_t^\alpha u_1)(\underline{t})$ becomes $\frac{f(\underline{t},0)}{c(\underline{t},0)} - \frac{1}{c(\underline{t},0)} D_t^\alpha u_1(\underline{t})$.

Let $\hat{\mathfrak{X}} = \min\{\xi, \mathfrak{X} + \mathfrak{X}_\epsilon\}$. In this case we take the set U to be

$$U = \big\{\, w_{\frown} \in \mathcal{C}^{(\nu)}([0,\hat{\mathfrak{X}}];\mathcal{C}([0,\tau];\mathbb{C})) \;\big|\; v(\underline{t},x) = (D_t^\alpha u)(\underline{t},x), \quad x \in [0,\mathfrak{X}], \\ v(0,\underline{x}) = 0 \,\big\}.$$

If v_1 and v_2 are two functions in U and $v_\Delta = v_1 - v_2$, then we get, using the fact that $v_\Delta(\underline{t},x) = 0$ for $x \in [0,\mathfrak{X}]$ and the argument used in the first part of the proof of Theorem 1 that

$$\left\| \left(\frac{1}{c(\underline{t},\mathfrak{X})} - \frac{1}{c(\underline{t},\underline{x})} \right) v_\Delta(\underline{t},\underline{x}) \right\|_{\mathcal{C}([0,\tau];\mathcal{C}^{(\nu)}([0,\hat{\mathfrak{X}}]))} \le \epsilon.$$

The desired conclusion follows.

From what we have shown so far and from the assumptions it follows that we have $(D_t^\alpha(u-u_0))_{\frown}$ and $(u_x)_{\frown} \in \mathcal{C}^{(\nu)}([0,\xi];\mathcal{C}([0,\tau];\mathbb{C}))$. The inverse of D_t^α is given by $(D_t^{-\alpha}h)(\underline{t}) = \int_0^{\underline{t}} g_\alpha(\underline{t}-s)h(s)\,\mathrm{d}s$; therefore a straightforward caluculation gives that $u - u_0 \in \mathcal{C}^{(\alpha)}([0,\tau];\mathcal{C}^{(\nu)}([0,\xi];\mathbb{C}))$. By assumption $u_0 \in \mathcal{C}^1([0,\xi];\mathbb{C})$ and we get (b). Since (a) is equivalent to having $u_{\frown} \in \mathcal{C}^{(1+\nu)}([0,\xi];\mathcal{C}([0,\tau];\mathbb{C}))$ and (b) is equivalent to having $u_{\frown} \in \mathcal{C}^{(\nu)}([0,\xi];\mathcal{C}^{(\alpha)}([0,\tau];\mathbb{C}))$ we obtain (c) from Lemma 3. The estimates in (5) follow immediately. □

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