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V8

Kari Eloranta*

Institute of Mathematics
Helsinki University of Technology
FIN-02015 HUT, Finland

eloranta@janus.hut.fi

Abstract

The bounded version of the eight vertex model of Statistical Mechanics is investigated. We concentrate on square and diamond domains on the square lattice and give an exact characterization to legal boundary conditions and the number of their fill-ins. Furthermore we resolve the connectivity properties of the subsets of legal configurations and find out that the best possible result holds: all configurations sharing a boundary can be transformed to each other with elementary moves. This enables an efficient configuration generation using a probabilistic cellular automaton.

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Introduction

The attempts in recent years to extend the one-dimensional theory of symbolic dynamics to higher dimensions has uncovered both challenging problems and yielded surprising successes. On one hand there are no-go results rooted to undecidability and on the other hand completely new phenomena that manifest themselves only trivially in one dimension. The former stem from the theory of tilings and in particular the fundamental work of Berger (see [GS]) whereas the latter are closely related to classical Statistical Mechanics formulations.

In this paper we consider a well known Statistical Mechanics model, the eight vertex model, but our point of view is not particularly physical. The infinite model of most physical interest has been studied earlier to a great detail ([B]). Also the toral case has received attention but because it has no boundary, none of the subtleties that are associated with the boundary dependency will show up there. We consider the bounded case here and its relative, the ice-model in the companion paper ([E2]). Perhaps the results on boundary dependency can help to clarify the long distance order in these models which in turn is the key to criticality etc. But primarily our results should be viewed as a part of a bigger program that attempts to bring unity to the theories of symbolic dynamics, tilings and classical Statistical Mechanics. The original impetus to this came mainly from a group theoretic study of polyominoes ([CL], [T]) which was later extended by others, notably in [JPS]. These studies concentrated on the tileability of a finite planar region with the given primitives, in these cases dominoes or polyominoes.

It turns out that several classical Statistical Mechanics models can be treated in this framework. Instead of polyominoes we can for example distribute arrows between neighboring lattice sites according to a fixed set of local matching rules. Models of this type include the ice-model, several color-models and the eight-vertex model ([B]). In this paper we solve the tileability and counting problems for the eight-vertex model and indicate a simple but rather general way of generating the allowed configurations. This is a consequence of a connectivity result that seems to underlie several different Statistical Mechanics models.

We do not treat the case of arbitrary domains which seems rather unwieldy. Our results deal with diamond and square domains. The reason for this is that ever since the work [JPS] and the references therein it has been known that behavior in these shapes can already reveal a great deal about a model. In particular for dominoes and the ice-model these domain shapes are of prime importance. Our results here provide a reference to the these models which are more subtle in their behavior but also less accessible to complete combinatorial analysis as well.

1. Set-up and size

In this section we first define the model and then analyze it on two different type of finite domains. This involves characterizing legal boundaries, solving the fill-in problem and computing the size of the set of legal configurations.

Consider the square lattice in two dimensions, $\mathbf{Z}^2$. Every lattice site has four nearest neighbors. Unlike in most statistical mechanics lattice models the vertex models do not have any spin etc. variables associated to the lattice points. Instead the variables are the arrows between nearest neighbor sites.

Definition 1.1.: *An arrow configuration at a lattice site in* $\mathbf{Z}^2$ *is* **legal** *for the* **eight-vertex rule** *if there are either 0, 2 or 4 incoming arrows and the rest are outgoing. A configuration in legal if it has an allowed vertex configuration at every lattice site.*

The allowed vertex configurations are illustrated in the Figure. The numbers below indicate the multiplicity of the arrangement. There are eight possibilities, hence the name of the model.

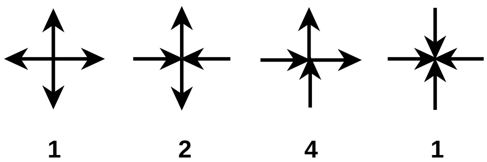

Figure 1. Vertex configurations.

The model on the infinite $\mathbf{Z}^2$-lattice as well as on a finite torus has been studied before (e.g. [B]). Both of these cases are boundaryless. In order to study the boundary dependency we need to define a suitable finite domain and the arrow configuration on the boundary.

The domains that we will consider are the diamond and square which differ in the orientation with respect to the lattice axes. The boundary arrows to be specified are obviously somewhat different. We will first derive the counting result for the diamond since it has the cleanest boundary condition of all domains.

N**-diamond** is a subset of $\mathbf{Z}^2$ which has N arrows along each of its four diagonal sides, $N \geq 2$, even. The total number of arrows is hence N^2. One can think it to be made of $N^2/2 - N + 1$ unit squares each of which has four arrows as sides and the neighboring squares sharing an arrow. Such domain contains $N^2/2 + N$ lattice sites.

The boundary configuration of the N-diamond, which consists of $4N - 4$ arrows, is fixed. It can in principle be chosen arbitrarily but our first problem is to solve when a given boundary configuration can be extended to a complete configuration of in the interior. To this end it is useful to partition the configuration into **shells** as indicated in Figure 2 (the boundary is distinguished by bold arrows and the next smaller shell by light arrows).

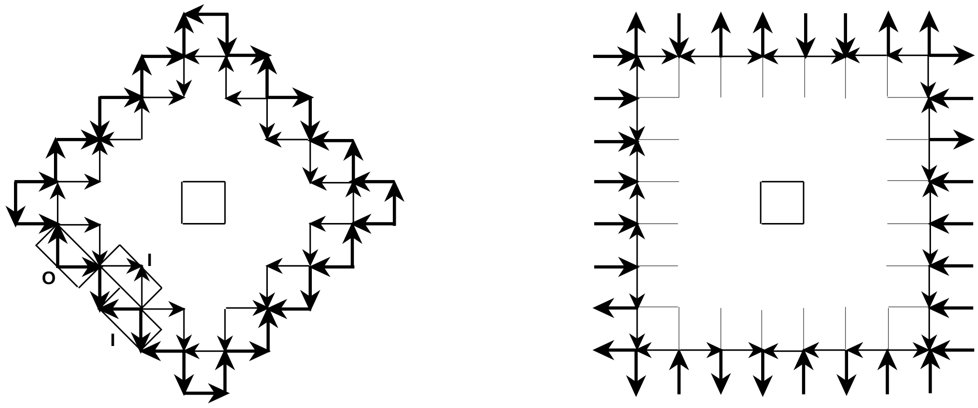

Figure 2. Diamond and square domains.

On a shell we distinguish two types of arrow pairs. If two neighboring arrows on the shell point to or away from the common lattice point we say that they form a **switch block** and call the lattice point a **switch point.** It one of the two neighboring arrows on a shell point in and one out of the common lattice lattice point we say that they form to a **neutral block.** Furthermore if the the switch point is on the inside of the shell we call the block an **inside switch block** and **outside** otherwise. These are marked with "I"and "O"n the Figure.

With these definitions we are ready to formulate a few basic observations:

1. By the eight vertex rule the existence of an inner switch block on a shell implies the existence of an outer switch block at the next smaller shell. The switch blocks share a common switch point. However the inner switch block does not force the type of the outer switch block: it can be both arrows in or both out.

2. The total number of switch points on a shell must be even. Here we add switch points both in inner and outer switch blocks. This is just a parity count - when we traverse the shell the direction of the arrows changes at every time we cross a switch point. Hence when we arrive back to the initial arrow we must have seen an even number of switch points.

3. 1. and 2. immediately imply that if the boundary has an even (odd) number of inner switch points, then all the inner shells must have an even (odd resp.) number of switch points, inner and outer.

4. The smallest shell (little square in the Figure) can filled in iff the next larger shell has an even number of inner switch blocks.

Define the **boundary flux**, F, as the quantity obtained when we subtract from the number of inward pointing arrows the number of outward pointing ones and ignore the four corner arrows (ambiguous ones). The facts 1.-4. imply that the fill-in shell by shell from the boundary is successful iff on the boundary there is an even number of inner switch points. Every switch point contributes ± 2 to the boundary flux. Neutral blocks contribute 0. Hence the fill-in is possible iff F is divisible by four. Call a boundary arrow arrangement that has such property a **legal boundary.**

The boundary minus the four corner arrows is determined by its partition into inner switch or neutral blocks. There are $2N-4$ of them in an N-diamond. Using an elementary binomial identity we find that the total number of ways that the boundary blocks can be chosen legally is

$$\sum_{\substack{k=0\\ k\ \mathrm{even}}}^{2N-4} \binom{2N-4}{k} 2^k 2^{2N-4-k} = 2^{2N-5} \sum_{k=0}^{2N-4} \binom{2N-4}{k} = 2^{4N-9}.$$

Together with the corner arrows this gives total of 2^{4N-5} choices.

Let us now examine the fill-in choices. For that purpose we number the shells from outside in such a way that the boundary is the first shell the next largest is the second and so on. By the preceding argument the i^{th} shell partitions into $2N-4i$ inner blocks, switch or neutral.

The key fact that enables the counting is

5. The locations of the inner switch blocks on a shell can be chosen independently of the the locations of inner switch blocks on other shells. Or equivalently the location of inner switch blocks on a shell is independent of the location of outer switch blocks on the shell.

The equivalence follows immediately from Fact 1. above. Note that the statement does only refer to location and not to type.

Given the shell $i-1$, the counting argument above slightly refined gives that the i^{th} shell can be chosen in

$$2 \sum_{\substack{k=0,\\ k\ \mathrm{even}}}^{2N-4i} \binom{2N-4i}{k} = 2^{2N-4i}$$

different ways. The factor 2 in front is due to the fact that aside from the switch point locations we can choose the direction of exactly one arrow on the shell.

By 5. the total number is then obtained by multiplying the shell contributions

$$2 \prod_{i=2}^{\frac{N}{2}-1} 2^{2N-4i} = 2^{N^2/2-3N+5}$$

where the 2 in front comes from choosing one arrow direction in the smallest shell (the only choice there).

We can summarize the above as the first existence and counting result.

Theorem 1.2.: *An arrow configuration on a diamond boundary can be extended to an arrow configuration on the entire set iff* $F \equiv 0 \pmod 4$. *There are* 2^{4N-5} *such legal boundaries for an* N*-diamond. Each of these extends in* $2^{\frac{N^2}{2}-3N+5}$ *ways to a complete arrow configuration of the interior. The total number of* N*-diamond configurations is* $2^{\frac{N(N+2)}{2}}$.

We'll postpone interpreting this until we have analyzed the square domain case as well.

The N**-square** is the domain that consists of N^2 lattice points and $2N^2+2N$ arrows as indicated on the right of Figure 2. The $4N$ arrows that have been rendered bold have to be specified as a boundary condition. For simplicity let N be even.

It is again useful to distinguish a shell. In Figure 2. the first shell is is the one marked with light arrows. The smallest shell (the $(N/2)^{\text{th}}$, here unoriented) is shown as well. The reason for this shell choice is evident; given the boundary arrows, once we choose the arrows on the neighboring shell a new inner boundary is uniquely determined (the unoriented arrows on the inside of the first shell in the Figure) and we can proceed inductively.

The flux across a loop around any lattice point is either 0 or ± 4. A loop around a set of lattice points is a sum of such loops hence the flux across it has to be divisible by four. Therefore the boundary flux in the square case, the total flux along the arrows on the boundary, has to be divisible by four. Note that the diamond tileability condition follows from this simple argument as well. The boundary flux definitions for the two domains agree. This flux condition equals to the requirement that there is an even number of arrows pointing in. Hence there are total of $\sum_{k=0,\ k \text{ even}}^{4N} \binom{4N}{k} = 2^{4N-1}$ legal boundary conditions.

Compatible with Fact 2. in the diamond context we must record an even number of arrow direction reversals on the shell as we traverse it once. Call the lattice points where this happens again switch points. The location and number of corner switch points we cannot choose as they are determined by the next larger shell. But others on the shell we can among the $4(N-2i)$ possible locations on the i^{th} shell. Depending on whether there is an even or odd number of corner switch points we have to pick even or odd number of off-corner switch points on each shell. But in either case there are the total of $2^{4(N-2i)}$, $1 \le i \le N/2-1$ choices. Here we have also accounted the choice of one arrow orientation after which the shell is completely determined. For $i = N/2$ (the center shell) there are two choices as in the diamond case.

The shells were chosen the given way to have the independence of the choices as in the diamond case. Now the locations (hence also the count) of the off-corner switch points on neighboring shells are independent. Therefore we can compute the totality of choices as

$$2\prod_{i=1}^{\frac{N}{2}-1} 2^{4(N-2i)} = 2^{(N-1)^2}.$$

Theorem 1.3.: *An arrow configuration on a square boundary can be extended to an arrow configuration on the entire set iff* $F \equiv 0 \pmod 4$. *There are* 2^{4N-1} *such legal boundaries for an* N*-square. Each of these extends in* 2^{N^2-2N+1} *ways to a complete arrow configuration of the interior. The total number of* N*-square configurations is* $2^{N(N+2)}$.

Remarks: 1. Although the geometry of the domain forces a somewhat different argument in the two cases it does not alter the number of choices in a significant way. One has to remember that the square domain has approximately twice as many lattice points and arrows but essentially the same amount of boundary arrows. In particular the asymptotics like topological entropy agree. This quantity for a vertex model is the maximal "uncertainty per arrow". More formally it is

$$h_{top} = \lim_{M\to\infty} \frac{1}{M} \log\left(\{\text{total number of } M\text{−arrow configurations}\}\right).$$

Theorems imply immediately the lower bound $\frac{1}{2}\log 2$ for the topological entropy of the infinite model. In fact the bound is exact since we are imposing no boundary condition in the last statements of the Theorems. The number is approximately 0.346574. For comparisons sake we mention that for the infinite free model $h_{top} = \log 2 \approx 0.69315$ and for the (more restrictive) six-vertex model $h_{top} = \frac{3}{4}\log\frac{4}{3} \approx 0.215761$.

2. These results indicates a striking homogeneity in the model: all legal boundary conditions in the given geometry have equal number of fill-ins. It reminds of the situation to the one encountered in finite groups, the fill-ins corresponding to the cosets of a group. At the end we will see what the action generating each **coset** is.

3. The results extend immediately to a rectangle standing on its corner and a lattice rectangle.

2. Irreducibility

In this section we investigate the "perturbations" of the allowed configurations. This yields a simple characterization of the topological structure of the set of configurations. Moreover from it we obtain a constructive method to generate the configurations and approximate the measure of maximal entropy.

The first observation is that at any vertex we can simultaneously flip the directions of two arrows. This is illustrated on the left in Figure 3. For any legal vertex configuration we fix two arrows, say a and b, and then flipping the non-bold arrows yields another legal vertex configuration. This obviously holds for all of the eight possible vertex configurations.

In the flipping of a single vertex configuration we violate the rule on two of its neighbors. But if we reverse the arrows along a closed arrow loop (or in the infinite model along a path from infinity to infinity) in the resulting configuration all vertex configurations are again legal. Note that while this loop/path consists of arrows it is not usually directed as a whole.

Let us call the simplest such action, the reversal of the arrows in a **1-loop** an **elementary move**. The Figure illustrates it with the notation where the symbol refers to the arrow heading and its negative to its reverse ($a, \ldots, d$ are arbitrary).

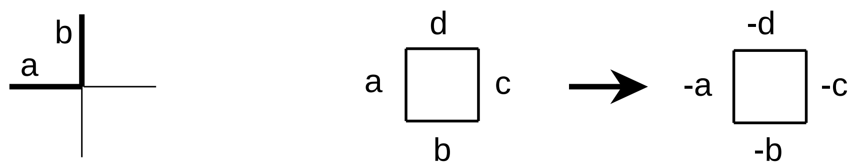

Figure 3. Vertex perturbation and elementary move.

Reversal of 1-loops is an equivalence relation on some set of configurations. The natural question then is to characterize this set i.e. the configurations that can by constructed from a given configuration using a finite sequence of elementary moves. Note that in the case of a bounded domain with a fixed boundary a loop reversal can never reach the other cosets as the path to be reversed cannot contain any boundary arrows.

Call the action of 1-loop reversals **irreducible** if the connected set is the set of all the configurations with the given boundary.

Theorem 2.1.: *The action of the elementary moves is irreducible on the set of diamond configurations with identical boundary arrows. Maximum number of elementary moves needed for a N-diamond is $N^2/2 - 3N + 5$, $N \geq 4$.*

Proof: Consider two legal arrow covers of a N-diamond with same arrow configuration on the boundary. Call these configurations A and B. Call the diagonal subsets of the configuration columns if they are oriented NW-SE and rows if oriented SW-NE. Number the rows and columns as shown in Figure 4 with indices running from 0 to $N-1$.

We compare the two configurations lexicographically, change B locally if needed, and at the end of the comparison A and B will be identical.

If arrows at $(1,1)$ in both configurations agree they agree at $(1,2)$ as well since they have two common boundary arrows at that lattice point (as usual row index is first and column second). We do nothing but move on to $(1,3)$ to compare the arrows at that location.

If arrows at $(1,1)$ disagree, reverse the 1-cycle centered at $(3/2,3/2)$ (marked by c in the Figure). After this the configurations agree at $(1,1)$ and $(1,2)$.

Continue by comparing the arrows at $(1,2n+1)$, $\forall n=1,\ldots,N/2-2$. As above flip the 1-cycles at the sites where there is disagreement. So all the reversed cycles are on the first white row in the Figure. After this the boundary and all the arrows on on the first row and the arrows at $(2,1)$ and $(2,N-2)$ agree in A and B.

Next row is treated similarly except that only the sites $(2,2n)$, $\forall n=1,\ldots,N/2-2$ need to be checked. Now the reversed 1-cycles are on the second grey row.

This procedure clearly can be continued down to the last row. At the end of it the configurations agree at every arrowsite.

Once we subtract the 1-cycles which have boundary arrows in them there are $(N/2-1)^2+(N/2-2)^2=N^2/2-3N+5$ 1-cycles left in the configuration. By the construction above reversing all of these once suffices to convert any legal configuration to another. ∎

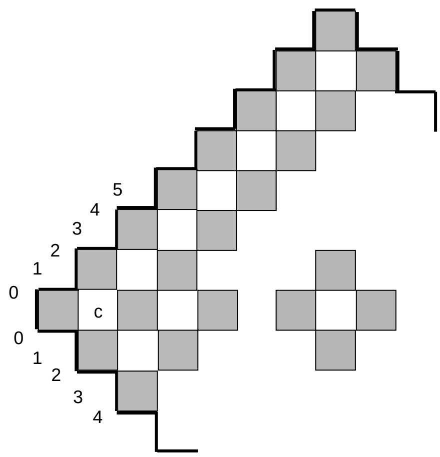

Figure 4. N-diamond, notation for the proof.

The proof above is immediately transferable to the square case. One again sweeps the domain diagonally correcting the errors encountered by reversing any of the $(N-1)^2$ 1-cycles.

Corollary 2.2.: *The action of the elementary moves is irreducible on the set of square configurations with identical boundary arrows. Maximum number of elementary moves needed for a* N*-square is* $(N-1)^2,\ N \geq 2$.

Remarks: 1. One can view the statements above as a graph result. By identifying the configurations with a common boundary as vertices in a graph the result simply says that the graph is connected and that the longest path (diameter of the graph) is at most $N^2/2 - 3N + 5$ or $N^2 - 2N + 1$ edges respectively.
2. Note that since an elementary move transforms two configurations to each other and that these moves can be made independently at distinct sites the diameter results above have to agree with the coset size exponents formulated in Theorems 1.2. and 1.3.
3. The proofs works verbatim for a rectangle tilted 45 degrees and a lattice rectangle.
4. This type of connectivity result seems to hold with some generality once the correct elementary moves have been identified. It has been shown to the ice model in [E2] and to dominoes in greater generality in [STCR]. Sometimes it almost holds, failing in an interesting way for a small subset of "exotic" configurations ([E1]).

The Proof of Theorem 2.1. points immediately to a natural way of generating the arrow configurations with a given boundary. Suppose that we are given one configuration from that coset. From that we form the **even configuration** in the following way. Since every 1-cycle consists of four arrows there are 16 different ones. Let the symbol set be $S = \{0, \ldots, 15\}$. Given a N-diamond record the symbols at the grey squares in Figure 4. in an array $C^{(e)}$.

The local **rule** is simply to read off from four adjoining 1-cycles (as in Figure 4., bottom right) the white symbol in the center and reverse it with probability p. This local operation performed at every neighborhood centered at a white 1-cycle gives the new **odd configuration** $C^{(o)}$. Hence we have defined a **probabilistic cellular automaton** $F_p : C^{(e)} \rightarrow C^{(o)}$ Essentially the same map also maps $C^{(o)} \rightarrow C^{(e)}$. Note that we have to augment the image with the grey boundary cycles on the boundary. They cannot ever be reversed since the boundary is fixed.

Alternating the two maps generates the infinite forward orbit of even and odd configurations all of which correspond do configurations with the given boundary. If the local updates are done independently and non-trivially i.e. $0 < p < 1$ this orbit reaches every allowed configuration almost surely in finite time. The automaton relaxes from a legal initial configuration to the equilibrium distribution on all legal configurations sharing the boundary with the initial configuration. This distribution is uniform (the measure of maximal entropy). At $p = 1/2$ the relaxation rate is maximal.

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