1. Introduction

We continue the investigation of (apparent) singularities of ordinary differential systems which was started in [12]. By a differential system we mean any system of ODEs or DAEs, of any order. This large class of systems can be handled in a uniform way using jet spaces.[1] Differential systems are then submanifolds of certain jet spaces, and their solutions are integral manifolds of certain distributions.

We will see that this geometric point of view allows us to regularise some singular systems, i.e. we can replace the original singular system by a regular one. In some cases simply formulating the problem with jets makes the singularity disappear. All the tools we use are constructive, hence useful also to more realistic problems appearing in applications.

The regularised systems thus obtained can also be used for numerical computations. The discussion of actual numerical methods, however, is outside the scope of the present article, see [13] and [2].

2. Basic definitions

We recall here briefly some notions that are needed below. For more information we refer to [11] (standard differential geometry) and [9] (jet spaces).

We will use the convention that the components of the vector are indicated by superscripts and the derivatives (or jet coordinates) by subscripts. All maps and manifolds are assumed to be smooth, i.e. infinitely differentiable. The differential (or Jacobian) of a map f is denoted by df. Let M be a manifold. The tangent (resp. cotangent) space of M at $p \in M$ is denoted by TM_p (resp. T^*M_p) and the tangent (resp. cotangent) bundle of M by TM (resp. T^*M). Let M and N be manifolds and let α be a section of T^*N (i.e. one-form). Given a map $f : M \mapsto N$ we can define a section of T^*M by

$$f^*\alpha(V_p) = \alpha(df(V_p))$$

where $V_p \in TM_p$ and $f^*\alpha$ is called the pull-back of α. Let $\mathcal{E} = \mathbb{R} \times \mathbb{R}^n$ and let us denote the q'th order jet bundle of $\mathcal{E}$ by $J_q(\mathcal{E})$. The coordinates of $J_q(\mathcal{E})$ are denoted by $(x, y, y_1, \ldots, y_q)$. Let us define the one forms

$$\alpha_{(i,j)} = dy^i_j - y^i_{j+1}dx \qquad i = 1, \ldots, n \qquad j = 0, \ldots, q-1 \tag{1}$$

Each $\alpha_{(i,j)}$ is then a section of $T^*J_q(\mathcal{E})$. Let us further define

$$C_p = \left\{ v_p \in (TJ_q(\mathcal{E}))_p \,\middle|\, \alpha_{(i,j)}(v_p) = 0 \right\}$$

Hence C is a $n+1$ dimensional distribution on $J_q(\mathcal{E})$. Now let us consider the differential system

$$f(x, y, y_1, \ldots, y_q) = 0 \tag{2}$$

where $f : J_q(\mathcal{E}) \simeq \mathbb{R}^{(q+1)n+1} \mapsto \mathbb{R}^k$. Let $M = f^{-1}(0)$ and let us define a distribution $\mathcal{D}$ on M by

$$\mathcal{D}_p = TM_p \cap C_p$$

Now supposing that the system is involutive and that $\mathcal{D}$ is one-dimensional we can define the solutions as follows:

[1]In particular there is no difference between ODEs and DAEs, so the term DAE is superfluous in jet space context.

Definition 1. *Solutions of the involutive system (2) are integral manifolds of $\mathcal{D}$.*

Since one-dimensional distributions always have integral manifolds, we conclude that there always exist solutions to our problem.

Note that it is absolutely essential that the system is involutive: otherwise it is possible that the distribution $\mathcal{D}$ is one-dimensional, although (some of) the corresponding integral manifolds are not really solutions in any reasonable sense. Since the involutivity is not explicitly needed in the sequel we refer to [10], [8], [5] and [13] for a further discussion of this important concept. Some elementary examples of applications of jets to differential equations can be found in [1].

3. Resolution of singularities

Basically we consider a differential system (or equation) as a submanifold of a jet space, but in general any manifold M with one-dimensional distribution $\mathcal{D}$ on it, may be interpreted as a (system of) differential equation(s). The word singular(ity) can be used at least in the following senses.

1. M is not everywhere smooth manifold.
2. $\dim(\mathcal{D}_p) > 1$ at some points of M.
3. Solution curves (or their projections) are not everywhere smooth.
4. Classical distinction of general versus singular solution.

We will not consider the last one of these; for an extensive discussion of this problem we refer to a recent thesis [7]. Usually in any problem there are some sets where $\dim(\mathcal{D}_p) > 1$ and the problem is then to analyse the qualitative behavior of solutions near these sets. This is related to (but more general than) analysing the solutions near zeros of vector fields. This problem is outside the scope of the present article and we refer to [1] for some examples.

Below we will consider cases 1 and 3, and also 2 when $\dim(\mathcal{D}_p) > 1$ 'unnecessarily'. We will show that sometimes it is possible to resolve the singularity in the sense that the original problem can be replaced by a regular one whose solutions correspond to the solutions of the original problem.

3.1. **Regularisation by introducing jets.** Let us consider the following problem

$$f(x, y, y_1) = x(y_1)^2 - 2yy_1 - x = 0$$

From the traditional point of view there is a singularity at $x = 0$. However, putting $\mathcal{E} = \mathbb{R} \times \mathbb{R}$ and $M = f^{-1}(0) \subset J_1(\mathcal{E})$ we see that M is everywhere smooth. The distribution $\mathcal{D}$ whose integral manifolds are solutions is given by the nullspace of

$$A = \begin{pmatrix} (y_1)^2 - 1 & -2y_1 & 2xy_1 - 2y \\ -y_1 & 1 & 0 \end{pmatrix}$$

Obviously the rank of A is two on M; hence $\mathcal{D}$ is everywhere one-dimensional and the problem is regular. So in this case simply introducing the jet spaces made the (apparent) singularity disappear. The projections of the solutions to $\mathcal{E}$ (a family of parabolas) give the lines of curvature around an umbilic, see [11, vol. 3] or any book on classical differential geometry. Note also that $M \setminus \{y_1\text{–axis}\}$ covers doubly $\mathcal{E} \setminus \{0\}$. Projecting the distribution from one of the sheets to $\mathcal{E} \setminus \{0\}$ gives an elementary example of the fact that singularities of distributions are more general than singularities of vector fields. Indeed, there is no vector field around origin which would span the projected distribution.

3.2. **Regularisation by prolongation.** Consider again the general system (2) and let $M = f^{-1}(0) \subset J_q(\mathcal{E})$.

Definition 2. *The prolongation of f, denoted by $\rho_1(f)$, is a system obtained by putting together f and its total derivatives. The zero set of $\rho_1(f)$ is the prolongation of M, denoted by $\rho_1(M) \subset J_{q+1}(\mathcal{E})$.*

Let us start with a simple problem

$$f(x, y, y_1) = xy_1 - y = 0$$

whose solutions are $y = cx$. Again let $\mathcal{E} = \mathbb{R} \times \mathbb{R}$ and $M = f^{-1}(0) \subset J_1(\mathcal{E})$. Obviously M is smooth, but the distribution is not one-dimensional at $x = 0$. Note that the codimension of the set where the distribution is not one-dimensional should be two, see [12] for a discussion why this is so. Here the codimension is only one. Now taking the total derivative of f we obtain xy_2 and retaining only the relevant factor we get the system

$$\begin{cases} xy_1 - y = 0 \\ y_2 = 0 \end{cases} \tag{3}$$

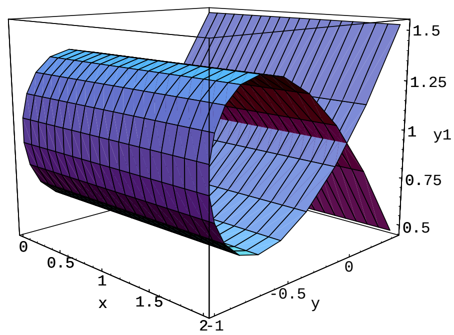
This defines a regular problem in $J_2(\mathcal{E})$. The distribution can in this case readily be projected back to $J_1(\mathcal{E})$ and to M, so we get a regular problem on M.

Taking another point of view, one can say that the 'wrong' codimension indicates that the problem is not generic. Indeed, considering the equation $xy + ay_1 = 0$, it is easily checked that the codimension is two (for $a \neq 0$), except for the value $a = -1$. Hence it depends on the intended application whether it is more appropriate to produce a regular system (3) or to perturb the coefficients to get a generic problem.

Let us then consider another example where prolongation resolves the singularity.

$$f(x, y, y_1) = (y_1)^2 - 2y_1 - y^3 - y^2 + 1 = 0 \tag{4}$$

This defines a curve in the (y, y_1)–plane with a double point at $(0, 1)$. We denote this curve by $\mathcal{K}$ and hence $M = f^{-1}(0) = \mathbb{R}\times\mathcal{K}$. The set of singular points, i.e. the points at which df vanishes, is $S = \mathbb{R} \times \{(0, 1)\}$. Now the geometry of M suggests that through each point of S there could pass two smooth solutions, see the picture on the right.

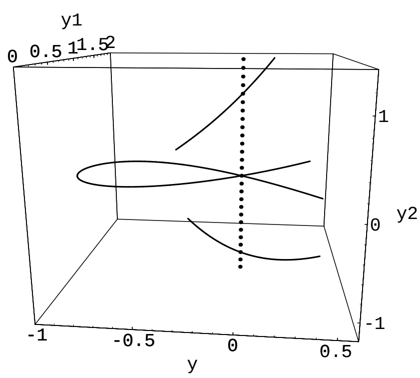


In $\mathcal{E}$ this would mean that at each point of $\mathbb{R} \times \{0\}$, there are two solutions which meet tangentially. Now if two curves intersect tangentially, one expects that in general their second derivatives would not coincide (note that M intersects itself transversally) and hence we would obtain a regular situation if the information on second derivatives were also available.

To this end we prolong f and get the system

$$\rho_1(f) \ : \ \begin{cases} (y_1)^2 - 2y_1 - y^3 - y^2 + 1 = 0 \\ 2y_1y_2 - 2y_2 - 3y^2y_1 - 2yy_1 = 0 \end{cases}$$

Let us denote the zero set of these equations in (y, y_1, y_2)–space by $\rho_1(\mathcal{K})$ so $\rho_1(M) = \mathbb{R} \times \rho_1(\mathcal{K}) \subset J_2(\mathcal{E})$. Let $\pi \ : \ \rho_1(\mathcal{K}) \mapsto \mathcal{K}$ be the projection induced by the projection $(y, y_1, y_2) \mapsto (y, y_1)$. Now we would like π to be bijective in a neighborhood of $(0, 1) \in \mathcal{K}$, except that the fiber of $(0, 1)$ should consist of two points. However, the fiber of $(0, 1)$ is the entire dotted line $(0, 1, y_2)$ in the picture. We would like to eliminate this line somehow.



To proceed we need some elementary algebraic tools, see for example [4]. Note that the components of $\rho_1(f)$ can be interpreted as elements of the ring $\mathbb{Q}[y, y_1, y_2]$. Let $\mathcal{I}$ be the ideal generated by $\rho_1(f)$. We are interested in the solution sets and therefore we computed the prime components of the radical of $\mathcal{I}$, denoted by $\sqrt{\mathcal{I}}$.[2] This gave $\sqrt{\mathcal{I}} = \mathcal{I}_1 \cap \mathcal{I}_2$, where $\mathcal{I}_1 = (y, y_1 - 1)$ and

$$\begin{aligned} \mathcal{I}_2 = \Big(\ & (y_1)^2 - 2y_1 - y^3 - y^2 + 1 \ , \ 2y_1y_2 - 2y_2 - 3y^2y_1 - 2yy_1 \ , \\ & 2y(y+1)y_2 - (3y+2)(y_1 - 1)y_1 \ , \\ & (4y+4)(y_2)^2 - (9y^2 + 12y + 4)(y_1)^2 \ \Big) \end{aligned}$$

Because $\mathcal{I}_1$ is known a priori, the same result can be obtained by computing the radical of the saturation of $\mathcal{I}$ with respect to $\mathcal{I}_1$, i.e. $\mathcal{I}_2 = \sqrt{\mathcal{I} : \mathcal{I}_1^\infty}$. Note that one does not automatically get the 'right' number of equations, i.e. the variety corresponding to $\mathcal{I}_2$ is not a complete intersection. However, we need only the 'piece' of the variety in the neighborhood of the singular set and by inspection it is seen that the second and fourth generators give the desired representation. It would be nice if this could be done algorithmically, i.e. given a (radical) ideal and a regular point p of the corresponding variety M, compute a representation of M as a complete intersection in a neighborhood of p. Perhaps the solution to this problem is well-known; however, we did not find it in the literature.

Finally let us note that no point of $\rho_1(\mathcal{K})$ projects to $(-1, 1) \in \mathcal{K}$. This is related to the fact that if the solution is projected from $J_1(\mathcal{E})$ to $\mathcal{E}$, then the projected curve is not smooth at $(x, -1)$. However, this is not problematic in our intended application, since we need to prolong only in a neighborhood S. Hence if we follow numerically a

[2] The computations were performed with SINGULAR, which has been developed in the university of Kaiserslautern by G.-M. Greuel, G. Pfister, H. Schönemann and their coworkers, see the web site http://www.mathematik.uni-kl.de/~wwwagag/E/Singular.html

particular solution, we can do the computations in $J_1(\mathcal{E})$ until we get too close to S; then we pass S in $J_2(\mathcal{E})$ and then switch back to $J_1(\mathcal{E})$ again.

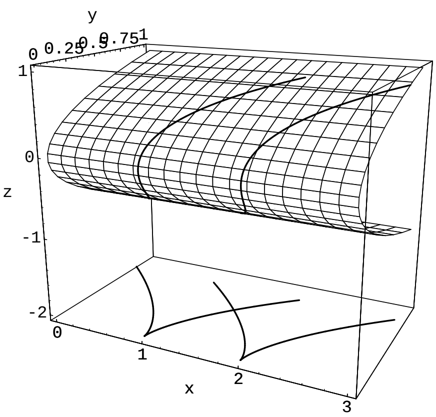
3.3. **Regularisation by pull-back.** We will finally treat two examples where a convenient pull-back produces a regular problem. First consider the problem

$$f(x, y, y_1) = y(y_1)^2 - 1 = 0$$

whose solutions are given by $y^3 = \frac{9}{4}(x + c)^2$. Let $M = f^{-1}(0)$ as usual. In this case introducing jet spaces does not directly allow one to pass the singularity. Indeed following the solutions in M one would never reach the singular point; one could compute only one branch of the solution at a time.

Now it is well-known that this type of singular curves are usually obtained by projecting a smooth curve on a fold. So the idea is to introduce a new manifold with a fold in such a way that the original solutions can be recovered from this new setting by projection.

In this simple case the solution is rather immediate. Consider the map $\varphi : (x, y, z) \mapsto (x, y, 1/z)$ and let $M^* = \varphi^{-1}(M)$. Taking the closure of M^*, i.e. adding the x–axis, we get a smooth manifold $\tilde{M}$ which is evidently the zero set of $g(x, y, z) = y - z^2$, see the figure on the right.



Then taking the pull-back of $\alpha = dy - y_1 dx$ we get $\varphi^*\alpha = dy - \frac{1}{z}\,dx$. Now recall that we are not interested in one forms as such, only in distributions defined by them. Hence we can multiply forms by non-zero functions and in particular we can replace $\varphi^*\alpha$ by $\beta = zdy - dx$. This can be smoothly extended to $\tilde{M}$.

So the original manifold is replaced by $\tilde{M}$ and the distribution $\tilde{\mathcal{D}}$ on $\tilde{M}$ is the nullspace of

$$A = \begin{pmatrix} 0 & 1 & -2z \\ -1 & z & 0 \end{pmatrix}$$

Clearly $\tilde{\mathcal{D}}$ is one-dimensional on $\tilde{M}$, so the problem is regular, and the original solution curves can be obtained with projection $(x, y, z) \mapsto (x, y)$. In the previous figure there are some solution curves as well as their projections.

As our final example consider the following problem which is taken from [3].

$$\begin{aligned}
&y^1_1 + y^1\lambda - (x - 1/2)e^x - 1 = 0 \\
&y^2_1 + y^2\lambda - (x^2 - 1/4)e^x - 2x = 0 \\
&(y^1)^2 + (y^2)^2 - (x - 1/2)^2 - (x^2 - 1/4)^2 = 0
\end{aligned}$$

Of course we could denote λ by y^3 and treat it in the same way as other variables. However, the form of the system resembles the form of mechanical systems with holonomic constraints and Lagrange multipliers, and in those cases, like in the present one, λ is not really needed in computations. Let $\mathcal{E} = \mathbb{R} \times \mathbb{R}^2$; since the system is linear in derivatives, we do not need to work with $J_1(\mathcal{E})$, because the distribution in $J_1(\mathcal{E})$ can be projected to $\mathcal{E}$. Indeed, differentiating the last equation and combining it with the other two we obtain

$$(5) \qquad \begin{pmatrix} 1 & 0 & y^1 \\ 0 & 1 & y^2 \\ y^1 & y^2 & 0 \end{pmatrix} \begin{pmatrix} y^1_1 \\ y^2_1 \\ \lambda \end{pmatrix} = \begin{pmatrix} (x-1/2)e^x + 1 \\ (x^2 - 1/4)e^x + 2x \\ 2x^3 + x/2 - 1/2 \end{pmatrix}$$

Now the distribution is given by the nullspace of

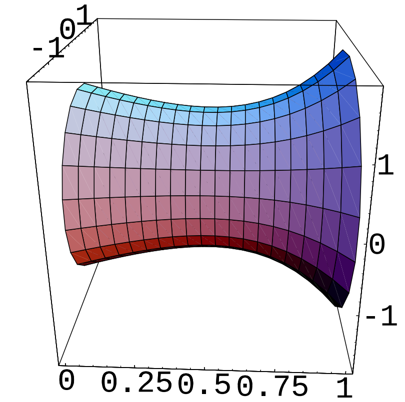
$$A = \begin{pmatrix} -y_1 & 1 & 0 \\ -y_2 & 0 & 1 \end{pmatrix}$$

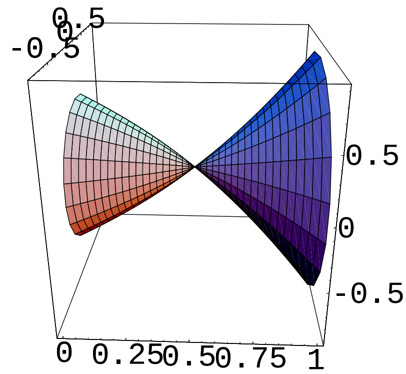
The linear system (5) can be solved, except on the x–axis, and computing further the nullspace of A gives a one-dimensional distribution $\mathcal{D}$ on $\mathcal{E} \setminus \{x\text{–axis}\}$. In this simple case one could compute $\mathcal{D}$ symbolically; however, this is not necessary. Let us define

$$f(x,y) = (y^1)^2 + (y^2)^2 - (x-1/2)^2 - (x^2 - 1/4)^2$$

and let $M = f^{-1}(0) \subset \mathcal{E}$. Evidently M is smooth except at the vertex $p = (1/2, 0, 0)$. The distribution $\mathcal{D}$ obtained above restricts further to $M \setminus \{p\}$.

Now in [3] it was observed that there is at least one smooth solution going through p. To compute this solution numerically it would be nice to have a smooth problem whose solutions would give the original solutions. To this end we introduce the map $\varphi : (x,y) \mapsto \big(x, \frac{\sqrt{|y|-1}}{|y|}y\big)$ and define $M^* = \varphi^{-1}(M)$. In the picture M^* is shown above and M below. Evidently φ is bijective, except at the vertex, where the fiber is the unit circle. This procedure of replacing M by M^* is similar (at least in spirit) to blowing up in algebraic geometry, see for example [6].





Now we can define a distribution $\mathcal{D}^*$ by transporting the distribution $\mathcal{D}$ from M to M^* by $(d\varphi)^t$. In this case we can work directly with distributions and do not need forms. Hence we have obtained a regular problem. The original solutions can be recovered by the map

$$\psi(x,y) = \begin{cases} \varphi(x,y), & x \le 1/2 \\ \varphi(x,-y), & x \ge 1/2 \end{cases}$$

The appearance of this 'discontinuous projection' can be understood by rotating M^* in different directions for $x > 1/2$ and $x < 1/2$, and at the same time shrinking the exceptional fiber to a point. Solution curves of the two systems correspond to each other after a half turn.

In both examples the map φ was needed analytically. However, the choice of the form of the map is not very critical. Also φ is required only locally near the singular set, which further facilitates the choice of the map.

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