Mat-1.174 Computational Partial Differential Equations
1^{st} exercise, Jan 21 –

1. Using the 5-point discretization of the Laplacian, find the general form of the matrix A for numerically approximating the solution of $\nabla^2 u = 0$.

   Set up your matrix using $n \times n$ block matrices, call them B, mI, O (to be explained at the lecture) . Write your session in a text file and cut/paste into a MATLAB-session for instance as follows:

   ```
   n=3; B=4*diag(ones(1,n))-diag(ones(1,n-1),1)-diag(ones(1,n-1),-1)
   O=zeros(...)
   A=[B,mI,O;...]
   help diag
   ```

2. To practice the assembly of the RHS also do the following example. Assume the boundaries of the unit square (thin insulated metal plate) have the temperatures indicated in the figure below. Find an approximation for the steady-state temperature in the 16 interior nodes.

   ```
          50   20   10   0
      |----|----|----|----|---|
      |                       |
   80 |  T13  T14  T15  T16   | 80
      |                       |
   60 |   T9  T10  T11  T12   | 60
      |                       |
   40 |   T5   T6   T7   T8   | 40
      |                       |
   20 |   T1   T2   T3   T4   | 20
      |                       |
      |----|----|----|----|---|
           0    0    0    0
   ```

   Using the matrix A in exercise 1 you can now solve the linear system. Reshape the solution vector as a 4×4 temperature matrix. (`help reshape`). Append the boundary vectors also into your solution matrix. (Take the corners to be averages of the nearest neighbours.)

   Plot your solution using one or several of functions like `mesh, surf (or surfc, surfl, etc.)`

   To get some entertainment you can make a simple movie just by doing something like

   ```
   for j=1:10;view(-20,10*j),pause,end;
   ```

3. Now you are ready to program it. Write an m-file `lapsolve1.m` for solving numerically the Laplace equation on a square. Your function could start as follows:

   ```
   function u=lapsolve1(bot,right,top,left)
   % Solve Laplace equation on a square.
   % Parameters:
   %
   %    bot,right,top,left are vectors of boundary values,
   %    each has length n+1
   % The solution is returned as an (n+1)x(n+1) matrix (the boundary
   % values are appended)
   n=length(bot)-1;  % nr. of subintervals.
   % You may include a test to see that the lengths agree.
   ```

   Of course you may want to design your function differently. Please go ahead and do !

   Now you can take a finer grid in the previous example. Take the boundary values to be the values of an interpolation polynomial. (`help polyfit` and `help polyval`. Repeat problem 2 on finer grids.

   Later on this will be further developed and generalized. Some important points will be *sparse matrices* (`help sparse`), non-homogeneous (*Poisson*), etc.

4. The faces of a thin square plate with side length $a = 12$ are perfectly insulated. The upper side is kept at temperature $100°C$ and the other sides are kept at $0°C$. Find the steady state temperature $u(x, y)$. Make use of the worksheet ..., especially plot the solution.

   Having the formula, design a suitable way to plot the solution in MATLAB.

5. Motivation: Recall some ODE stuff and learn the basic use of MATLAB-ODE-solvers. In addition write a direction field program for autonomous 2×2-systems.

   The equation of a damped pendulum is

   $$\Theta'' + c\Theta' + k \sin \Theta = 0.$$

   a) Write it as a 2×2-system and find its critical points and linearizing matrix at the critical points..
   b) Find eigenvalues (and -vectors) and decide the type and stability of each critical point.
   c) Let $k = 1$. Plot the phase plane for various values of the damping constant, at least for $c = 1/2$ and $c = 1/4$. Display at least two stable and two unstable critical points with trajectories.

   (Take c as `global`).

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d) Look at the direction field code at `http://www.math.hut.fi/teaching/osdylask/L99/L1.html`. Modify it to plot the direction field of trajectories of an autonomous 2×2-system.

Perhaps some of you have studied this problem in MAPLE some time ago. Make comparisons.