9 Polynomial approximations: Stone–Weierstrass

In this section we study densities of subalgebras in $C(X)$. These results will be applied in characterizing function algebras among Banach algebras. First we study continuous functions on $[a,b] \subset \mathbb{R}$:

Weierstrass Theorem (1885). *Polynomials are dense in* $C([a,b])$.

Proof. Evidently, it is enough to consider the case $[a,b]=[0,1]$. Let $f \in C([0,1])$, and let $g(x) = f(x) - (f(0) + (f(1)-f(0))x)$; then $g \in C(\mathbb{R})$ if we define $g(x)=0$ for $x \in \mathbb{R} \setminus [0,1]$. For $n \in \mathbb{N}$ let us define $k_n : \mathbb{R} \to [0,\infty[$ by

$$k_n(x) := \begin{cases} \frac{(1-x^2)^n}{\int_{-1}^1 (1-t^2)^n \ \mathrm{d}t}, & \text{when } |x|<1, \\ 0, & \text{when } |x| \geq 1. \end{cases}$$

Then define $P_n := g * k_n$ (*convolution* of g and k_n), that is

$$\begin{aligned} P_n(x) &= \int_{-\infty}^{\infty} g(x-t)\ k_n(t)\ \mathrm{d}t \ = \int_{-\infty}^{\infty} g(t)\ k_n(x-t)\ \mathrm{d}t \\ &= \int_0^1 g(t)\ k_n(x-t)\ \mathrm{d}t, \end{aligned}$$

and from this last formula we see that P_n is a polynomial on $[0,1]$. Notice that P_n is real-valued if f is real-valued. Take any $\varepsilon > 0$. Function g is uniformly continuous, so that there exists $\delta > 0$ such that

$$\forall x,y \in \mathbb{R}: \ |x-y|<\delta \Rightarrow |g(x)-g(y)|<\varepsilon.$$

Let $\|g\| = \max_{t \in [0,1]} |g(t)|$. Take $x \in [0,1]$. Then

$$\begin{aligned} |P_n(x)-g(x)| &= \left| \int_{-\infty}^{\infty} g(x-t)\ k_n(t)\ \mathrm{d}t - g(x) \ \int_{-\infty}^{\infty} k_n(t)\ \mathrm{d}t \right| \\ &= \left| \int_{-1}^{1} (g(x-t)-g(x))\ \ k_n(t)\ \mathrm{d}t \right| \\ &\leq \int_{-1}^{1} |g(x-t)-g(x)|\ \ k_n(t)\ \mathrm{d}t \\ &\leq \int_{-1}^{-\delta} 2\|g\|\ k_n(t)\ \mathrm{d}t + \int_{-\delta}^{\delta} \varepsilon\ k_n(t)\ \mathrm{d}t + \int_{\delta}^{1} 2\|g\|\ k_n(t)\ \mathrm{d}t \\ &\leq 4\|g\| \int_{\delta}^{1} k_n(t)\ \mathrm{d}t + \varepsilon. \end{aligned}$$

The reader may verify that $\int_\delta^1 k_n(t)\ \mathrm{d}t \ \to_{n\to\infty} 0$ for every $\delta > 0$. Hence $\|Q_n - f\| \to_{n\to\infty} 0$, where $Q_n(x) = P_n(x) + f(0) + (f(1)-f(0))x$ □

Exercise. Show that the last claim in the proof of Weierstrass Theorem is true.

For $f : X \to \mathbb{C}$ let us define $f^* : X \to \mathbb{C}$ by $f^*(x) := \overline{f(x)}$, and define $|f| : X \to \mathbb{C}$ by $|f|(x) := |f(x)|$. A subalgebra $\mathcal{A} \subset \mathcal{F}(X)$ is called *involutive* if $f^* \in \mathcal{A}$ whenever $f \in \mathcal{A}$.

Stone–Weierstrass Theorem (1937). *Let X be a compact space. Let $\mathcal{A} \subset C(X)$ be an involutive subalgebra separating the points of X. Then $\mathcal{A}$ is dense in $C(X)$.*

Proof. If $f \in \mathcal{A}$ then $f^* \in \mathcal{A}$, so that the real part $\Re f = \dfrac{f + f^*}{2}$ belongs to $\mathcal{A}$. Let us define

$$\mathcal{A}_\mathbb{R} := \{\Re f \mid f \in \mathcal{A}\};$$

this is a $\mathbb{R}$-subalgebra of the $\mathbb{R}$-algebra $C(X, \mathbb{R})$ of continuous real-valued functions on X. Then

$$\mathcal{A} = \{f + ig \mid f, g \in \mathcal{A}_\mathbb{R}\},$$

so that $\mathcal{A}_\mathbb{R}$ separates the points of X. If we can show that $\mathcal{A}_\mathbb{R}$ is dense in $C(X, \mathbb{R})$ then $\mathcal{A}$ would be dense in $C(X)$.

First we have to show that $\overline{\mathcal{A}_\mathbb{R}}$ is closed under taking maximums and minimums. For $f, g \in C(X, \mathbb{R})$ we define

$$\max(f, g)(x) := \max(f(x), g(x)), \quad \min(f, g)(x) := \min(f(x), g(x)).$$

Notice that $\overline{\mathcal{A}_\mathbb{R}}$ is an algebra over the field $\mathbb{R}$. Since

$$\max(f, g) = \frac{f + g}{2} + \frac{|f - g|}{2}, \quad \min(f, g) = \frac{f + g}{2} - \frac{|f - g|}{2},$$

it is enough to prove that $|h| \in \overline{\mathcal{A}_\mathbb{R}}$ whenever $h \in \overline{\mathcal{A}_\mathbb{R}}$. Let $h \in \overline{\mathcal{A}_\mathbb{R}}$. By the Weierstrass Theorem there is a sequence of polynomials $P_n : \mathbb{R} \to \mathbb{R}$ such that

$$P_n(x) \to_{n \to \infty} |x|$$

uniformly on the interval $[-\|h\|, \|h\|]$. Thereby

$$\| |h| - P_n(h) \| \to_{n \to \infty} 0,$$

where $P_n(h)(x) := P_n(h(x))$. Since $P_n(h) \in \overline{\mathcal{A}_\mathbb{R}}$ for every n, this implies that $|h| \in \overline{\mathcal{A}_\mathbb{R}}$. Now we know that $\max(f, g), \min(f, g) \in \overline{\mathcal{A}_\mathbb{R}}$ whenever $f, g \in \overline{\mathcal{A}_\mathbb{R}}$.

Now we are ready to prove that $f \in C(X, \mathbb{R})$ can be approximated by elements of $\mathcal{A}_{\mathbb{R}}$. Take $\varepsilon > 0$ and $x, y \in X$, $x \neq y$. Since $\mathcal{A}_{\mathbb{R}}$ separates the points of X, we may pick $h \in \mathcal{A}_{\mathbb{R}}$ such that $h(x) \neq h(y)$. Let $g_{xx} = f(x)\mathbb{I}$, and let

$$g_{xy}(z) := \frac{h(z) - h(y)}{h(x) - h(y)}\, f(x) + \frac{h(z) - h(x)}{h(y) - h(x)}\, f(y).$$

Here $g_{xx}, g_{xy} \in \mathcal{A}_{\mathbb{R}}$, since $\mathcal{A}_{\mathbb{R}}$ is an algebra. Furthermore,

$$g_{xy}(x) = f(x), \quad g_{xy}(y) = f(y).$$

Due to the continuity of g_{xy}, there is an open set $V_{xy} \in \mathcal{V}(y)$ such that

$$z \in V_{xy} \;\Rightarrow\; f(z) - \varepsilon < g_{xy}(z).$$

Now $\{V_{xy} \mid y \in X\}$ is an open cover of the compact space X, so that there is a finite subcover $\{V_{xy_j} \mid 1 \leq j \leq n\}$. Define

$$g_x := \max_{1 \leq j \leq n} g_{xy_j};$$

$g_x \in \overline{\mathcal{A}_{\mathbb{R}}}$, because $\overline{\mathcal{A}_{\mathbb{R}}}$ is closed under taking maximums. Moreover,

$$\forall z \in X: \; f(z) - \varepsilon < g_x(z).$$

Due to the continuity of g_x (and since $g_x(x) = f(x)$), there is an open set $U_x \in \mathcal{V}(x)$ such that

$$z \in U_x \;\Rightarrow\; g_x(z) < f(z) + \varepsilon.$$

Now $\{U_x \mid x \in X\}$ is an open cover of the compact space X, so that there is a finite subcover $\{U_{x_i} \mid 1 \leq i \leq m\}$. Define

$$g := \min_{1 \leq i \leq m} g_{x_i};$$

$g \in \overline{\mathcal{A}_{\mathbb{R}}}$, because $\overline{\mathcal{A}_{\mathbb{R}}}$ is closed under taking minimums. Moreover,

$$\forall z \in X: \; g(z) < f(z) + \varepsilon.$$

Thus

$$f(z) - \varepsilon < \min_{1 \leq i \leq m} g_{x_i}(z) = g(z) < f(z) + \varepsilon,$$

that is $|g(z) - f(z)| < \varepsilon$ for every $z \in X$, i.e. $\|g - f\| < \varepsilon$. Hence $\mathcal{A}_{\mathbb{R}}$ is dense in $C(X, \mathbb{R})$ implying that $\mathcal{A}$ is dense in $C(X)$ □

Remark. Notice that under the assumptions of the Stone–Weierstrass Theorem, the compact space is actually a compact Hausdorff space, since continuous functions separate the points.

Exercise*. Let K be a compact Hausdorff space, $\emptyset \neq S \subset K$, and $\mathcal{J} \subset C(K)$ be an ideal. Let us define

$$\begin{aligned} \mathcal{I}(S) &:= \{f \in C(K) \mid \forall x \in S : \ f(x) = 0\}, \\ V(\mathcal{J}) &:= \{x \in K \mid \forall f \in \mathcal{J} : \ f(x) = 0\}. \end{aligned}$$

Prove that
(a) $\mathcal{I}(S) \subset C(K)$ a closed ideal,
(b) $V(\mathcal{J}) \subset K$ is a closed non-empty subset,
(c) $V(\mathcal{I}(S)) = \overline{S}$ (hint: Urysohn), and
(d) $\mathcal{I}(V(\mathcal{J})) = \overline{\mathcal{J}}$ (hint: Stone–Weierstrass).
Lesson to be learned:
topology of K goes hand in hand with the (closed) ideal structure of $C(K)$.