**Closure and boundary.** Let $(X, \tau)$ be a topological space. Let $S \subset X$; its *closure* $\mathrm{cl}_\tau(S) = \overline{S}$ is the smallest closed set containing $S$. The set $S$ is *dense* in $X$ if $\overline{S} = X$. The *boundary* of $S$ is $\partial_\tau S = \partial S := \overline{S} \cap \overline{X \setminus S}$.

**Exercise.** Let $(X, \tau)$ be a topological space. Let $S, S_1, S_2 \subset X$. Show that
(a) $\overline{\emptyset} = \emptyset$,
(b) $S \subset \overline{S}$,
(c) $\overline{\overline{S}} = \overline{S}$,
(d) $\overline{S_1 \cup S_2} = \overline{S_1} \cup \overline{S_2}$.

**Exercise.** Let $X$ be a set, $S, S_1, S_2 \subset X$. Let $c : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfy Kuratowski's closure axioms (a-d):
(a) $c(\emptyset) = \emptyset$,
(b) $S \subset c(S)$,
(c) $c(c(S)) = c(S)$,
(d) $c(S_1 \cup S_2) = c(S_1) \cup c(S_2)$.
Show that $\tau := \{U \subset X \mid c(X \setminus U) = X \setminus U\}$ is a topology of $X$, and that $\mathrm{cl}_\tau(S) = c(S)$ for every $S \subset X$.

**Exercise.** Let $(X, \tau)$ be a topological space. Prove that
(a) $x \in \overline{S} \;\Leftrightarrow\; \forall U \in \mathcal{V}(x) : \; U \cap S \neq \emptyset$.
(b) $\overline{S} = S \cup \partial S$.