1. Assume that $d \geq 1$ and that $\sigma \in \mathcal{B}^{\infty}_{\mathrm{loc}}(\mathbb{R})$ is such that the closure of the set of discontuities of σ has Lebesgue measure 0 and σ is not (almost everywhere equal to) a polynomial. Show that $S_d(\sigma)$ is dense in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ where $1 \leq p < \infty$.

Solution: Under the given assumptions the set $S_d(\sigma)$ is dense in $\mathcal{C}(\mathbb{R}^d)$ and since $\mathcal{C}(\mathbb{R}^d)$ is dense in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ the claim follows.

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2. Define the operator Δ_h by $(\Delta_h f)(t) = f(t+h) - f(t)$ where $h > 0$ and assume that f is an integrable function on every bounded interval. Show that f is a polynomial of degree at most m if and only if $\Delta_h^{m+1} f = 0$ for all $h > 0$.

Solution: Let $f(t) = \sum_{j=0}^m a_j t^j$. Then

$$(\Delta_h f)(t) = \sum_{j=0}^m a_j((t+h)^j - t^j),$$

and because

$$(t+h)^j - t^j = \sum_{k=1}^j \binom{j}{k} t^{j-k} h^k,$$

which is a polynomial of degree $j-1$ we conclude that $(\Delta_h f)$ is a polynomial of degree at most $m-1$. Thus $(\Delta_h^n f)$ is a polynomial of degree at most $m-n$ and we get one part of the claim by taking $n = m+1$.

It is clear that if $\Delta_h f = 0$ for all $h > 0$, then f is a constant (and for this part no integrability assumpitions are needed).

Assume for the moment that f is continuous. Suppose that we have already shown that if $\Delta_h^m f = 0$ for all $h > 0$ then f is a polynomial of degree at most $m-1$. Suppose now that $\Delta_h^{m+1} f = 0$ for all $h > 0$. Denote be E_h the operator $(E_h f)(t) = f(t+h)$. It follows that $\Delta_h = E_h - I$. Since $\Delta_h^{m+1} f(t) = 0$ for all t and h we have $E_h \Delta_h^m f(t) = \Delta_h^m f(t)$ and hence $E_{jh} \Delta_h^m f(t) = \Delta_h^m f(t)$ for all t, h, and j. Because $\Delta_{kh} = \left(\sum_{j=0}^{k-1} E_{jh}\right) \Delta_h$ we conclude that

$$\Delta_{kh}^m f(t) = k^m \Delta_h f(t), \quad t \in \mathbb{R}, \quad h > 0, \quad k \geq 1.$$

Now define

$$g(h,t) = |(\Delta_h^m f)(t)|^{\frac{1}{m}},$$

so that $g(kh,t) = kg(h,t)$ when $t \in \mathbb{R}$ and $h > 0$. It follows immediately that $g(rh,t) = rg(h,t)$ for all rational positive r, and by continuity for all $r > 0$. Thus we conclude that

$$|(\Delta_h^m f)(t)| = h^m \left|(\Delta_1^m f)(t)\right|,$$

and since $(\Delta_h^m f)(t+h) = (E_h \Delta_h^m f)(t) = (\Delta_h^m f)(t)$ we conclude that $|(\Delta_1^m f)(t+h)| = |(\Delta_1^m f)(t)|$. Since h is arbitrary we see that $|(\Delta_1^m f)(t)|$ is a constant, and then we can, again by continuity conclude that there is a constant c susch that $(\Delta_h^m f)(t) = ch^m$. But then it follows that $\Delta_h^m (f - \frac{ct^m}{m!}) = 0$ and the claim follows from the induction assumption.

If f is not apriori continuous we apply the argument above to the function $g(t) = \int_0^t f(s)\,\mathrm{d}s$ and we note that $\Delta_h^{j+1} g(t) = \int_t^{t+h} \Delta_h^j f(s)\,\mathrm{d}s$ when $j \geq 1$.

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3. Define the operator Δ_h by $(\Delta_h f)(t) = f(t+h) - f(t)$ where $h > 0$. Show that if $\varphi * \sigma$ is a polynomial of degree at most m for all infinitely many times differentiable functions that are 0 outside $[-1,1]$, then $\varphi * (\Delta_h^{m+1}\sigma) = 0$ for all such functions φ.

Solution: Since

$$\begin{aligned}(\varphi * (\Delta_h\sigma))(t) &= \int_{\mathbb{R}} \varphi(t-s)(\sigma(s+h) - \sigma(s))\,\mathrm{d}s \\ &= \int_{\mathbb{R}} \varphi(t+h-(s+h))\sigma(s+h)\,\mathrm{d}s - \int_{\mathbb{R}} \varphi(t-s)\sigma(s)\,\mathrm{d}s \\ &= \int_{\mathbb{R}} \sigma(t+h-s)\sigma(s)\,\mathrm{d}s - \int_{\mathbb{R}} \varphi(t-s)\sigma(s)\,\mathrm{d}s = (\Delta_h(\varphi * \sigma))(t),\end{aligned}$$

we see that we also have

$$\Delta_h^{m+1}(\varphi * \sigma) = \varphi * (\Delta_h^{m+1}\sigma).$$

If now $\varphi * \sigma$ is a polynomial of degree at most m we know that $\Delta_h^{m+1}(\varphi * \sigma) = 0$ and the claim follows immediately.

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4. Show that if $\varphi * \sigma$ is a polynomial of degree at most m for all infinitely many times differentiable functions that are 0 outside $[-1,1]$ then σ is (almost everywhere equal to) a polynomial of degree at most m.

Hint: One can use distribution theory for this or one can choose as the function φ the function $\psi_\lambda(\underline{t}) = \lambda\psi(\lambda\underline{t})$ where $\lambda \geq 1$ and $\psi(t) = 0$ when $|t| \geq 1$, then let $\lambda \to \infty$ and use the exercises above.

Solution: If $\varphi * \sigma$ is a polynomial of degree at most m, then $\varphi * (\Delta_h^{m+1}\sigma) = 0$ and by choosing φ to be the function $\psi_\lambda(\underline{t}) = \lambda\psi(\lambda\underline{t})$ we conlcude when $\lambda \to \infty$ that $\Delta_h^{m+1}\sigma = 0$ almost everywhere. But since this is true for arbitrary h, we conlcude that σ must be (almost everywhere equal to) a polynomial of degree at most m.
