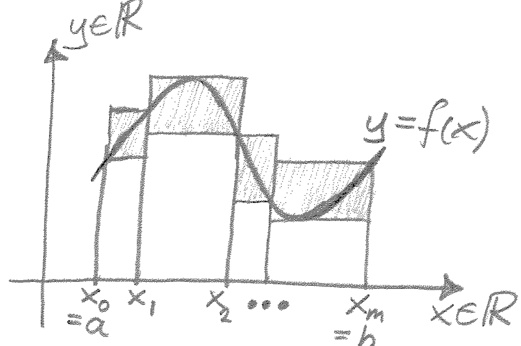
Riemann-integraali (kertaus)



Välin $[a,b] \subset \mathbb{R}$
ositus $\mathcal{P} = \{x_i\}_{i=0}^m$:

$$a = x_0 < x_1 < \dots < x_m = b.$$

Rajoitetun funktion $f : [a,b] \to \mathbb{R}$

"yläsumma" $U(f,\mathcal{P}) := \sum_{i=1}^m \left(\sup_{x \in [x_{i-1},x_i]} f(x) \right) \cdot (x_i - x_{i-1})$,

"alasumma" $L(f,\mathcal{P}) := \sum_{i=1}^m \left(\inf_{x \in [x_{i-1},x_i]} f(x) \right) \cdot (x_i - x_{i-1})$.

f on Riemann-integroituva, jos

$$\sup_{\mathcal{P}} L(f,\mathcal{P}) = \inf_{\mathcal{P}} U(f,\mathcal{P})$$

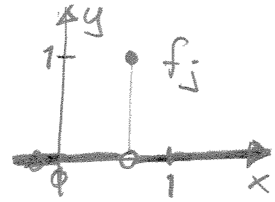
$$\left[=: \underset{\text{RIEMANN}}{\int_a^b} f(x)\,dx \quad \text{silloin.} \right]$$

[1854 (Georg Friedrich) Bernhard Riemann (1826-1866)]

Ongelma:
voi olla $\underset{\text{RIEMANN}}{\int_a^b} \sum_{j=1}^\infty f_j(x)\,dx \neq \sum_{j=1}^\infty \underset{\text{RIEMANN}}{\int_a^b} f_j(x)\,dx$,

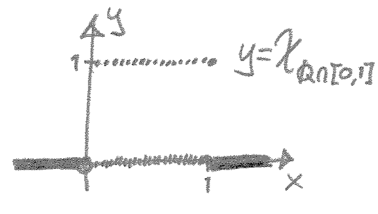
vaikka funktiot f_j "hyvin mukavia".

Esim.
Olkoon $\varphi : \mathbb{Z}^+ \to \mathbb{Q} \cap [0,1]$ bijektio,



$f_j := \chi_{\{\varphi(j)\}}$, jolloin

$\sum_{j=1}^\infty \underset{\text{RIEMANN}}{\int_0^1} f_j(x)\,dx = 0$, mutta



$\sum_{j=1}^\infty f_j(x) = \chi_{\mathbb{Q}\cap[0,1]}$ ei ole edes Riemann-integroituva!

Integraali (Lebesguen tapaan) [GZ 6]

Seuraavassa $(X, \mathcal{M}, \mu)$ on täydellinen mitta-avaruus,
esim. $(\mathbb{R}^n, \mathcal{M}(\lambda^*_{\mathbb{R}^n}), \lambda_{\mathbb{R}^n})$ [1901/1902 Henri Lebesgue (1875–1941)]

Alkeita [GZ 6.1]: Olkoon $f : X \to [-\infty, +\infty]$ $\mathcal{M}$-mitallinen.

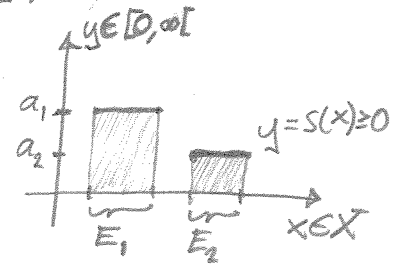
Määritellään $\int f \, d\mu$ vaiheittain:

Määr. 1/3 (yksinkertainen $\mathcal{M}$-mitallinen $s : X \to [0, +\infty[$)

Olkoon $\tilde{\tilde{\int}} 0 \, d\mu := 0$ ja

$$\tilde{\tilde{\int}} \underbrace{\sum_{j=1}^{k} a_j \chi_{E_j}}_{\substack{=: s \geq 0, \\ 0 < a_j < \infty}} d\mu := \sum_{j=1}^{k} a_j \, \mu(E_j)$$

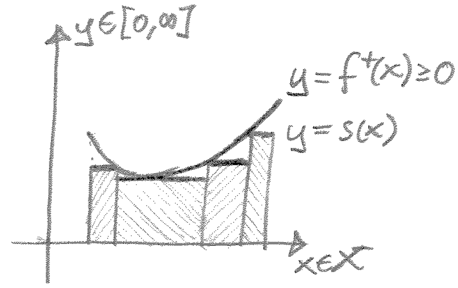
$E_j \in \mathcal{M}$, $i \neq j \Rightarrow E_i \cap E_j = \emptyset$.



Esim. $\tilde{\tilde{\int}} \chi_E \, d\mu = \mu(E)$.

Määr. 2/3 ($\mathcal{M}$-mitallinen $f^+ : X \to [0, \infty]$)

$$\tilde{\int} f^+ \, d\mu := \sup_{s \leq f^+} \tilde{\tilde{\int}} s \, d\mu.$$



Määr. 3/3

$\mathcal{M}$-mitallisen $f : X \to [-\infty, +\infty]$ μ-integraali on

$$\int f \, d\mu := \tilde{\int} f^+ d\mu - \tilde{\int} f^- d\mu, \quad \circledast$$

jos $\tilde{\int} f^+ d\mu < \infty$ tai $\tilde{\int} f^- d\mu < \infty$.

Jos $\tilde{\int} f^+ d\mu < \infty$ ja $\tilde{\int} f^- d\mu < \infty$,

sanotaan, että f on μ-integroituva.

[Usein merkitään $\int f(x) \, dx := \int f \, d\mu$, jos μ selvä.]

$$\circledast \begin{cases} f^+ = +f \cdot \chi_{\{f > 0\}}, \\ f^- = -f \cdot \chi_{\{f < 0\}}. \end{cases} \quad f = f^+ - f^-.$$

Huom. $\mu(E) = \int \chi_E \, d\mu$,

$\tilde{\tilde{\int}} s \, d\mu = \int s \, d\mu$,

$\tilde{\int} f^+ d\mu = \int f^+ d\mu$.

Lause 6.6

Olkoot $f, g : X \to [-\infty, +\infty]$ μ-integroituvia. Silloin:

(1) $f(x) \in \mathbb{R}$ μ-m.k. $x \in X$.

(2) $\int af \, d\mu = a \int f \, d\mu$ (kun $a \in \mathbb{R}$).

(3) $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$.

(4) $f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$.

(5) $E \in \mathcal{M} \Rightarrow f \cdot \chi_E$ μ-integroituva;

merkitään $\int_E f \, d\mu := \int f \cdot \chi_E \, d\mu$.

(6) Jos h $\mathcal{M}$-mitallinen, niin

h μ-integroituva $\Leftrightarrow$ $|h|$ μ-integroituva.

(7) $\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$.

Tod.

(1) $\{f^+ = +\infty\} = \bigcap_{k=1}^{\infty} \{f^+ > k\} \overset{f^+ \ \mathcal{M}\text{-mit.}}{\in} \mathcal{M}$

$$\mu(\{f^+ = +\infty\}) = \frac{1}{k} \cdot \left(k \cdot \mu(\{f^+ = +\infty\}) \right)$$

$$= \frac{1}{k} \tilde{\tilde{\int}} k \cdot \chi_{\{f^+ = +\infty\}} \, d\mu \qquad \Bigg\| \ 0 \leq \underbrace{k \cdot \chi_{\{f^+ = +\infty\}}}_{\mathcal{M}\text{-mit. yksinkert.}} \leq f^+$$

$$\leq \frac{1}{k} \underbrace{\tilde{\int} f^+ \, d\mu}_{< \infty} \xrightarrow[k \to \infty]{} 0$$

$\Rightarrow \mu(\{f^+ = +\infty\}) = 0$, vastaavasti $\mu(\{f^- = +\infty\}) = 0$. $\boxtimes_{(1)}$

(2) Harjoitustehtävä! $\boxtimes_{(2)}$

...

...

84.

(3) f, g $\mathcal{M}$-mitallisia $\Rightarrow$ $f+g$ $\mathcal{M}$-mitallinen
(voi yrittää integroida!)

f, g μ-integroituvia $\Rightarrow$ f^+, g^+ μ-integroituvia.

Pätee $\int (f^+ + g^+)\,d\mu = \int f^+ d\mu + \int g^+ d\mu$. ⊛

koska

$$\int (f^+ + g^+)\,d\mu = \sup_{t \le f^+ + g^+} \tilde{\tilde{\int}} \underbrace{t}_{\substack{\leq \\ = r+s,}}\, d\mu \qquad \begin{matrix} 0 \le r \le f^+ \\ 0 \le s \le g^+ \end{matrix} \Big\} \text{ } r, s \text{ yksinkertaisia } \mathcal{M}\text{-mitallisia}$$

$$\underbrace{}_{\tilde{\tilde{\int}} r\,d\mu + \tilde{\tilde{\int}} s\,d\mu}$$

$$\le \sup_{r \le f^+} \tilde{\tilde{\int}} r\,d\mu + \sup_{s \le g^+} \tilde{\tilde{\int}} s\,d\mu$$

$$= \int f^+ d\mu + \int g^+ d\mu \quad \Big\| \text{ Ota } \varepsilon > 0. \text{ Olkoot } 0 \le r_\varepsilon \le f^+ \text{ ja } 0 \le s_\varepsilon \le g^+, \text{ joille } \int f^+ d\mu < \tilde{\tilde{\int}} r_\varepsilon\, d\mu + \varepsilon \text{ jne.}$$

$$= \left(\tilde{\tilde{\int}} r_\varepsilon\, d\mu + \varepsilon\right) + \left(\tilde{\tilde{\int}} s_\varepsilon\, d\mu + \varepsilon\right)$$

$$= \tilde{\tilde{\int}} (r_\varepsilon + s_\varepsilon)\, d\mu + 2\varepsilon \quad \Big\| \quad r_\varepsilon + s_\varepsilon \le f^+ + g^+ \ldots$$

$$\le \int (f^+ + g^+)\, d\mu + 2\varepsilon \xrightarrow[\varepsilon \to 0^+]{} \int (f^+ + g^+)\, d\mu. \ \boxtimes ⊛$$

$$f + g = \begin{cases} (f+g)^+ - (f+g)^-, \\ (f^+ - f^-) + (g^+ - g^-). \end{cases}$$

$$\Rightarrow (f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

$$\overset{⊛}{\Rightarrow} \int (f+g)^+ d\mu + \int f^- d\mu + \int g^- d\mu = \int (f+g)^- d\mu + \int f^+ d\mu + \int g^+ d\mu \quad ⊛⊛$$

$$\Rightarrow \int (f+g)\, d\mu := \int (f+g)^+ d\mu - \int (f+g)^- d\mu$$

$$\overset{⊛⊛}{=} \int f^+ d\mu - \int f^- d\mu + \int g^+ d\mu - \int g^- d\mu$$

$$= \int f\, d\mu + \int g\, d\mu. \ \boxtimes (3)$$

(4) Nyt $f \leq g$, joten $f^+ \leq g^+$, $f^- \geq g^-$.

$$\int f^+ d\mu = \sup_{s \leq f^+} \tilde{\tilde{\int}} s\, d\mu \overset{f^+ \leq g^+}{\leq} \sup_{t \leq g^+} \tilde{\tilde{\int}} t\, d\mu = \int g^+ d\mu,$$

$$\int f\, d\mu = \underbrace{\int f^+ d\mu}_{\leq \int g^+ d\mu} - \underbrace{\int f^- d\mu}_{\geq \int g^- d\mu} \leq \int g^+ d\mu - \int g^- d\mu = \int g\, d\mu.$$

$\square_{(4)}$

(5) f $\mathcal{M}$-mitallinen, $E \in \mathcal{M}$ ($\Leftrightarrow \chi_E$ $\mathcal{M}$-mitallinen)
$\Rightarrow f \cdot \chi_E$ $\mathcal{M}$-mitallinen.

$$\int (f \cdot \chi_E)^+ d\mu = \int \underbrace{f^+ \cdot \chi_E}_{\leq f^+} d\mu$$

$$\overset{(4)}{\leq} \int f^+ d\mu \overset{f\ \mu\text{-int.}}{<} \infty.$$

Vastaavasti $\int (f \cdot \chi_E)^- d\mu \leq \int f^- d\mu < \infty$. $\square_{(5)}$

(6) Jos h on μ-integroituva, niin

$$|h| = \underbrace{h^+}_{h \cdot \chi_{\{h>0\}}} + \underbrace{h^-}_{-h \cdot \chi_{\{h<0\}}} \quad \text{on } \mu\text{-integroituva.}$$

Jos $|h|$ on μ-integroituva ja h $\mathcal{M}$-mitallinen, niin

$$h = \underbrace{|h| \cdot \chi_{\{h>0\}}}_{h^+} - \underbrace{|h| \cdot \chi_{\{h<0\}}}_{h^-} \quad \text{on } \mu\text{-integroituva.}$$

$\square_{(6)}$

(7) Jätetään $\left|\int f\, d\mu\right| \leq \int |f|\, d\mu$ harjoitustehtäväksi! $\blacksquare$

Huom. Jos h on $\mathcal{M}$-mitallinen, $|h| \le g$, g μ-integroituva, niin h on μ-integroituva.

Kolme pientä tulosta:

Olkoon $f : X \to [-\infty, +\infty]$ $\mathcal{M}$-mitallinen.

(a) Jos $E \in \mathcal{M}$, $\mu(E) = 0$, niin $\int_E f \, d\mu = 0$.

(b) Jos $f = 0$ μ-m.k., niin $\int f \, d\mu = 0$.

(c) Jos $\int |f| \, d\mu = 0$, niin $f = 0$ μ-m.k.

Tod.

(a) $$\int_E f^+ d\mu = \int f^+ \cdot \chi_E \, d\mu = \sup_{s \le f^+ \chi_E} \tilde{\int} \underbrace{s}_{= s \cdot \chi_E} d\mu \overset{\mu(E)=0}{=} 0$$

$$\Rightarrow \int_E f \, d\mu = \int_E f^+ d\mu - \int_E f^- d\mu = 0. \quad \boxtimes$$

(b) $$\int f \, d\mu = \int (f \cdot \chi_{\{f \ne 0\}} + f \cdot \chi_{\{f = 0\}}) \, d\mu$$
$$= \int_{\{f \ne 0\}} f \, d\mu \underset{(a)}{\overset{\mu(\{f \ne 0\}) = 0}{=}} 0. \quad \boxtimes$$

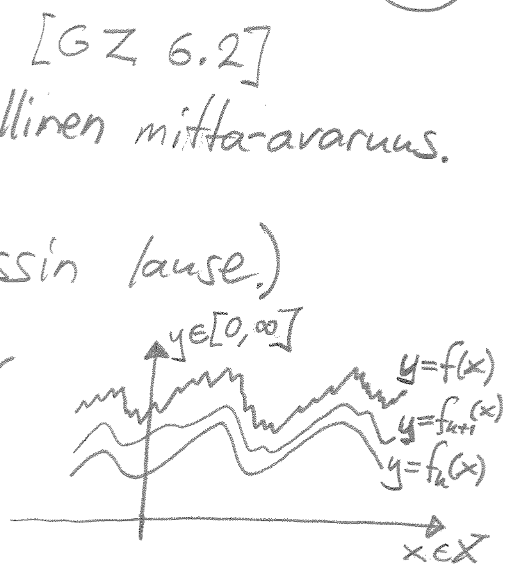
(c) $$0 \le \mu(\{f \ne 0\})$$
$$= \mu\Big(\bigcup_{k=1}^{\infty} \{|f| > 1/k\}\Big)$$
$$\le \sum_{k=1}^{\infty} \mu(\{|f| > 1/k\})$$
$$= \sum_{k=1}^{\infty} \int \chi_{\{|f| > 1/k\}} \, d\mu$$
$$\le \sum_{k=1}^{\infty} \int k \cdot |f| \, d\mu = \sum_{k=1}^{\infty} k \cdot \underbrace{\int |f| \, d\mu}_{=0} = 0. \quad \blacksquare$$

Integraali ja raja-arvo [GZ 6.2]

[Seuraavassa $(X, \mathcal{M}, \mu)$ on täydellinen mitta-avaruus.

Lause 6.11 (Monotonisen konvergenssin lause.)

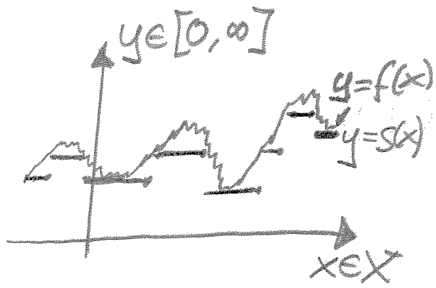
Olkoot $f_k \geq 0$ $\mathcal{M}$-mitallisia,
$f_k \leq f_{k+1}$ $(\forall k \in \mathbb{Z}^+)$.



Silloin
$$\lim_{k\to\infty} \int f_k \, d\mu = \int \underbrace{\lim_{k\to\infty} f_k}_{=:f} \, d\mu.$$

Tod.

Nyt $f := \lim_{k\to\infty} f_k$ on $\mathcal{M}$-mitallinen (joten sitä saa integroida).



Olkoon $0 < \varepsilon < 1$.

Olkoon $0 \leq s \leq f$,

$(*)$ $\int s \, d\mu \geq (1-\varepsilon) \cdot \int f \, d\mu$. (s yksinkertainen $\mathcal{M}$-mitallinen)

Saadaan
$$\int f \, d\mu \overset{f \geq f_k}{\geq} \int f_k \, d\mu \qquad \Big| \quad E_k := \{f_k > (1-\varepsilon)\cdot s\} \overset{f_k, s\ \mathcal{M}\text{-mit.}}{\in} \mathcal{M}$$

$$\geq \int_{E_k} (1-\varepsilon)\cdot s \, d\mu$$

$$= (1-\varepsilon) \cdot \int s \cdot \chi_{E_k} \, d\mu \qquad \Big| \quad s = \sum_{j=1}^n a_j \chi_{A_j}, \text{ missä } 0 \leq a_j < \infty,\ A_j \in \mathcal{M},\ i \neq j \Rightarrow A_i \cap A_j = \emptyset$$

$$= (1-\varepsilon) \cdot \int \sum_{j=1}^n a_j \chi_{A_j \cap E_k} \, d\mu$$

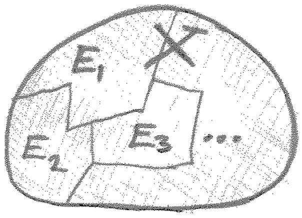
$$= (1-\varepsilon) \cdot \sum_{j=1}^n a_j \underbrace{\mu(A_j \cap E_k)}_{\xrightarrow[k\to\infty]{} \mu(A_j)}, \text{ koska } \underset{\in \mathcal{M}}{E_k} \subset E_{k+1}, \bigcup_{k=1}^\infty E_k = X$$

$$\xrightarrow[k\to\infty]{} (1-\varepsilon) \cdot \int s \, d\mu$$

$$\overset{(*)}{\geq} (1-\varepsilon)^2 \cdot \int f \, d\mu \xrightarrow[\varepsilon \to 0^+]{} \int f \, d\mu. \quad \blacksquare$$

Seuraus 6.12, 6.13

Jos $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ pistevieras, $X = \bigcup_{j=1}^{\infty} E_j$ ja
f μ-integroituva,



niin $\int f \, d\mu = \sum_{j=1}^{\infty} \int_{E_j} f \, d\mu.$

Tod.

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu,$$

$$\int f^+ \, d\mu = \int f^+ \cdot \sum_{j=1}^{\infty} \chi_{E_j} \, d\mu = \int \lim_{k\to\infty} \underbrace{\sum_{j=1}^{k} f^+ \cdot \chi_{E_j}}_{=: f_k} \, d\mu$$

f_k Monotonisen konvergenssin lauseen oletuksissa!

$$\overset{\text{mon. konv.}}{=} \lim_{k\to\infty} \int \sum_{j=1}^{k} f^+ \cdot \chi_{E_j} \, d\mu$$

$$= \lim_{k\to\infty} \sum_{j=1}^{k} \int f^+ \cdot \chi_{E_j} \, d\mu = \sum_{j=1}^{\infty} \int_{E_j} f^+ \, d\mu$$

ja vastaavasti $\int f^- \, d\mu = \sum_{j=1}^{\infty} \int_{E_j} f^- \, d\mu.$ ∎

Fatoun lemma

Olkoot $g_k \geq 0$ $\mathcal{M}$-mitallisia ($\forall k \in \mathbb{Z}^+$). Silloin

$$\int \liminf_{k\to\infty} g_k \, d\mu \leq \liminf_{k\to\infty} \int g_k \, d\mu.$$

Tod.

$$\int \liminf_{k\to\infty} g_k \, d\mu = \int \underbrace{\sup_{k\geq 1} \underbrace{\inf_{j\geq k} g_j}_{=: f_k}}_{= \lim_{k\to\infty} f_k} \, d\mu$$

f_k $\mathcal{M}$-mit., $f_k \leq f_{k+1}$

$$\overset{\text{mon. konv.}}{=} \lim_{k\to\infty} \int f_k \, d\mu$$

$$= \liminf_{k\to\infty} \int f_k \, d\mu$$

$$\overset{f_k \leq g_k}{\leq} \liminf_{k\to\infty} \int g_k \, d\mu.$$ ∎

Tehtävä

(a) Osoita, että

Fatoun lemma $\Rightarrow$ Monotonisen konvergenssin lause.

(b) Näytä esimerkillä, että Fatoun lemman epäyhtälö voi olla aito: joskus $\int \liminf_{k\to\infty} g_k \, d\mu < \liminf_{k\to\infty} \int g_k \, d\mu$.

Lause (Lebesguen dominoidun konvergenssin lause).

Olkoot $f_k : X \to [-\infty,+\infty]$ $\mathcal{M}$-mitallisia ($\forall k \in \mathbb{Z}^+$),

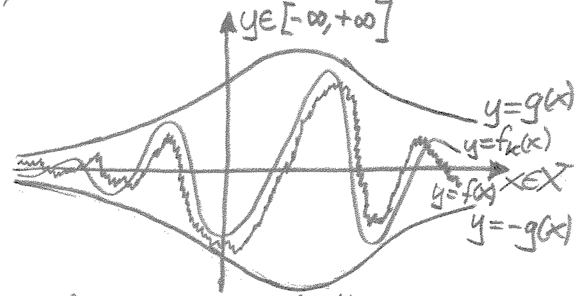
$f_k \to f$ μ-m.k.,

$|f_k| \leq g$ μ-m.k., g μ-integroituva.

Silloin

$$\int |f_k - f| \, d\mu \longrightarrow 0 \quad \text{ja}$$

$$\int f_k \, d\mu \longrightarrow \int f \, d\mu.$$



Tod.

Nyt f_k, f μ-integroituvia, koska $\mathcal{M}$-mitallisia ja koska $|f_k|, |f| \leq g$ μ-m.k. & g μ-integroituva.

$$\int 2g \, d\mu = \int \liminf_{k\to\infty} (2g - |f_k - f|) \, d\mu$$

Huom. Integroitavan funktion käytös nollamittaisessa joukossa ei muuta integraalia, ks. s. 86.

$$\overset{\text{Fatou}}{\leq} \liminf_{k\to\infty} \int (2g - |f_k - f|) \, d\mu$$

$$= \int 2g \, d\mu + \underbrace{\liminf_{k\to\infty} -\int |f_k - f| \, d\mu}_{= -\limsup_{k\to\infty} \int |f_k - f| \, d\mu \; \leq \; 0}$$

$$\overset{\int 2g\,d\mu \in \mathbb{R}}{\Longrightarrow} 0 \leq \limsup_{k\to\infty} \int |f_k - f| \, d\mu \leq 0 \Rightarrow \lim_{k\to\infty} \int |f_k - f| \, d\mu = 0. \quad (*)$$

$$\left| \int f_k \, d\mu - \int f \, d\mu \right| = \left| \int (f_k - f) \, d\mu \right|$$

$$\leq \int |f_k - f| \, d\mu \xrightarrow[k\to\infty]{(*)} 0. \quad \blacksquare$$

Lebesgue-integraali käyttäytyy siis raja-arvojen suhteen paremmin kuin Riemann-integraali, esim.

$$\int \chi_{\mathbb{Q}\cap[0,1]}\, d\lambda_{\mathbb{R}} = 0 \qquad (\text{ks. s. } 81).$$

Kuinka Riemann ja Lebesgue yleisesti suhtautuvat toisiinsa?

Lause 6.15–16.

Olkoon $f: [a,b] \to \mathbb{R}$ rajoitettu. Silloin

f Riemann-integroituva $\Leftrightarrow$

f jatkuva $\lambda_{\mathbb{R}}$-m.k. $x \in [a,b]$.

Tällöin f on myös Lebesgue-integroituva ja

$$\underset{\text{RIEMANN}}{\int_a^b f(x)\,dx} = \int_{[a,b]} f \, d\lambda_{\mathbb{R}}.$$

Tod.

Katso esim. [GZ 6.3, s. 142–144.] ▨

Lebesgue-integraali täten "laajentaa" Riemann-integraalin.

Lisätietoja mittateoriasta ja integroinnista esim. kurssilla Mat-1.3281 Analyysi I.

Kirja [GZ] Ziemerin versiona:

http://www.indiana.edu/~mathwz/PRbook.pdf