Mat-1.1332, KP3-II, tentti 19.12.2006
RATKAISUT / HA

1

$$A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 0 & -2 & 10 & -8 \end{bmatrix}$$

$$\sim \mathrm{ref}(A) = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ tukisarakkeet

Nul(A): $\vec{x} \in Nul(A) \iff A\vec{x} = \vec{0} \iff$

$(\mathrm{ref}(A))\vec{x} = \vec{0} \iff \begin{cases} x_1 - 3x_3 + 2x_4 = 0 \\ x_2 - 5x_3 + 4x_4 = 0 \end{cases}$

Vapaasti valittavat: $x_4 = s$, $x_3 = t$

$\Rightarrow \begin{cases} x_2 = 5t - 4s \\ x_1 = 3t - 2s \end{cases}$

$\vec{x} = \begin{bmatrix} 3t - 2s \\ 5t - 4s \\ t \\ s \end{bmatrix}$ Kanta: Valinta 1: $t=0$, $s=1$
— " — 2: $t=1$, $s=0$

$\vec{x}_1 = \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}$ $\{\vec{x}_1, \vec{x}_2\}$ on kanta.

Col(A): Poimitaan tukisarakkeiden ilmaisemat sarakkeet alkuperäisestä A:sta

Col(A):n kanta: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$

Row(A):n kanta: $\{[0, 1, -5, 4]^T, [1, 0, -3, 2]^T\}$

Rangi = dim(Col(A)) (= dim(Row(A))) = 2
Nulliteetti = dim(N(A)) = 2
Niiden summa = 2 + 2 = 4 = n (sarakk. lkm.)

2 (a) Jos A on diagonalisoituva, voidaan esittää: $A = VDV^{-1}$, missä V:n sarakkeet ovat A:n ominaisvektorit ja $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$,

λ_k on V:n sarakevektoris numero k vastaava ominaisarvo.

(Diagonalisoituvuus merkitsee sitä, että V:llä on käänteismatriisi, tätä ei ~~vaadita~~ mainittavaksi.)

(b)

$$A = VDV^{-1}$$

$$A^2 = AA = VD\underbrace{V^{-1}V}_{I}DV^{-1} = VD^2V^{-1}$$

$$\vdots$$

$$A^p = VD^pV^{-1}$$

(D^p lasketaan korottamalla kukin diagonaalialkio potenssiin p.)

(c) $A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{7}{6} \end{bmatrix}$ Lasketaan ominaisarvot:

$$\det(A - \lambda I) = \begin{vmatrix} \frac{1}{3} - \lambda & \frac{1}{3} \\ -\frac{1}{3} & \frac{7}{6} - \lambda \end{vmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2}$$

$$\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0 \iff \lambda = \left\langle \begin{matrix} 1 \\ \frac{1}{2} \end{matrix} \right.$$

Ominaisvektorit :

$\lambda_1 = 1$:

$$A - 1 \cdot I = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$$(A - I)\vec{x} = \vec{0} \iff -\frac{2}{3}x_1 + \frac{1}{3}x_2 = 0$$

(2. on ekvivalentti)

val. $x_2 = 2$

$\Longrightarrow x_1 = 1$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = \frac{1}{2}$

$$A - \frac{1}{2}I = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

(Rivit LRV, kuten pitääkin)

Otetaan 2. rivi kerrottuna 3:lla :

$-x_1 + 2x_2 = 0$; val. $x_2 = 1$

$\Longrightarrow x_1 = 2$

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$V = [\vec{v}_1 \; \vec{v}_2] = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$V^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

Siis

$$A = VDV^{-1} \Rightarrow A^{15} = VD^{15}V^{-1}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (\frac{1}{2})^{15} \end{bmatrix} \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} =$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.000030518 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.33329 & 0.66665 \\ -0.66665 & 1.3333 \end{bmatrix}.$$

[Laskutyö: Diagonaalimatriisilla kertominen on nopeaa, jäljelle jäävä kärsimys: kahden 2×2-matriisin kertominen, pikkuvirheistä ei rangaista.
(pikkuvirheistä ehkä)]

(3) $\vec{y}\,' = A\vec{y}$, $\vec{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$

A:n ominaisarvot:

$$\det(A - \lambda I) = (-\tfrac{3}{2} - \lambda)^2 - \tfrac{1}{4} = 0$$

$$\Leftrightarrow \lambda + \tfrac{3}{2} = \pm\tfrac{1}{2} \Leftrightarrow \lambda = \left\langle \begin{matrix} -1 \\ -2 \end{matrix}\right.$$

Ominaisvektorit:

$\lambda_1 = -1$; $(-\frac{3}{2} + 1)x_1 + \frac{1}{2}x_2 = 0$

$\Leftrightarrow -x_1 + x_2 = 0$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -2$; $(-\frac{3}{2} + 2)x_1 + \frac{1}{2}x_2 = 0$

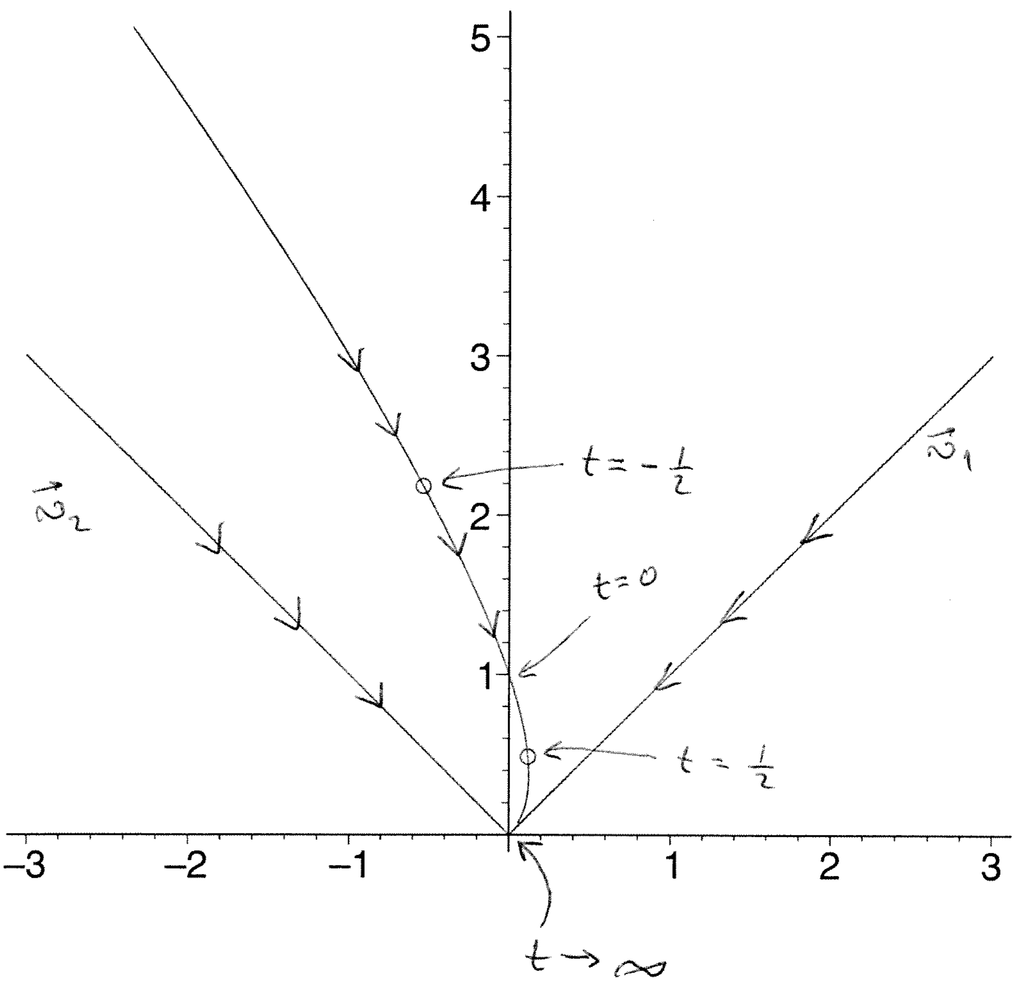
$\Leftrightarrow x_1 + x_2 = 0$; $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Yl. ratk.: $\vec{y}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(AE): $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{y}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\Longleftrightarrow \begin{cases} c_1 - c_2 = 0 \\ c_1 + c_2 = 1 \end{cases} \Longrightarrow c_1 = c_2 = \frac{1}{2}$$

Kysytty ratkaisu: $\vec{y}(t) = \frac{1}{2} e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



Trajektorit ovat ominaisvektorikoordinaatistossa paraabeleja: $z_2 = K z_1^2$, koska $\lambda_2 = 2 \lambda_1$

4 $\quad \theta'' + c\theta' + k\sin\theta = 0$

(a) Merk. $y_1 = \theta$, $y_2 = \theta' = y_1'$;

$$\begin{cases} y_1' = y_2 & = f_1(y_1, y_2) \\ y_2' = -k\sin y_1 - c\,y_2 & = f_2(y_1, y_2) \end{cases}$$

KRP:t
$$\begin{cases} y_2 = 0 \\ -k\sin y_1 - c\,y_2 = 0 \end{cases}$$

$\Leftrightarrow \quad y_2 = 0$ ja ($\sin y_1 = 0 \Leftrightarrow y_1 = n\pi$, $n \in \mathbb{Z}$)

Ts. : $\ldots(-2\pi, 0), (-\pi, 0), (0,0), (\pi, 0), (2\pi, 0), \ldots$

(b) Linearisointi $\vec{0}$:ssa :

$$J_{\vec{f}}(y_1, y_2) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k\cos y_1 & -c \end{bmatrix}$$

$$J_{\vec{f}}(0,0) = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}$$

$$D(\lambda) = +\lambda\left(\lambda + \frac{1}{2}\right) + 1 = \lambda^2 + \frac{1}{2}\lambda + 1$$

$$\lambda = -\frac{1}{4} \pm \sqrt{\frac{1}{16} - 1} = -\frac{1}{4} \pm i\,\frac{\sqrt{15}}{4}$$

Kompleksiset ominaisarvot, $\mathrm{Re}\,\lambda < 0$

$\Rightarrow$ nieluspiraali, stabiili (nahvasti).
Sopusoinnussa (epälineaarisen) kuvan kanssa.

<u>Euler</u> : $\vec{y}^{(m+1)} = \vec{y}^{(m)} + h\,F(\vec{y}^{(m)})$, ts.

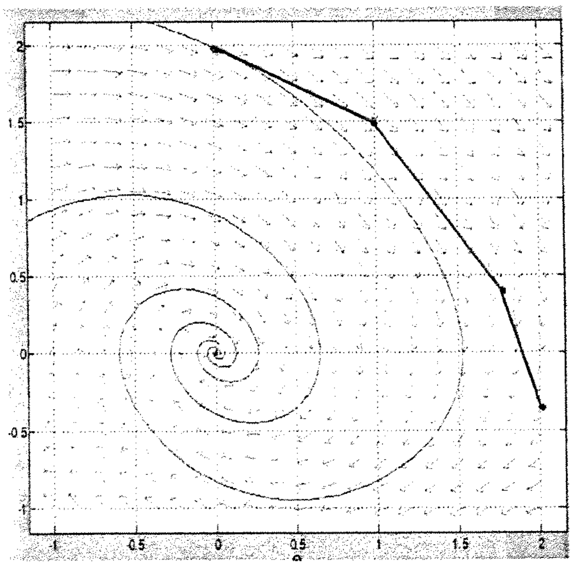
$$\vec{y}^{(m+1)} = \vec{y}^{(m)} + h \begin{bmatrix} y_2^{(m)} \\ -y_1^{(m)} - 0.5\,y_2^{(m)} \end{bmatrix}$$

$$\vec{y}^{(0)} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\vec{y}^{(1)} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \underbrace{h}_{0.5} \begin{bmatrix} 2 \\ \underbrace{0 - 0.5 \cdot 2}_{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$$

$$\vec{y}^{(2)} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} + 0.5 \begin{bmatrix} 1.5 \\ \underbrace{-1 - 0.5 \cdot 1.5}_{-1.75} \end{bmatrix} = \begin{bmatrix} 1.75 \\ 0.625 \end{bmatrix}$$

$$\vec{y}^{(3)} = \begin{bmatrix} 1.75 \\ 0.625 \end{bmatrix} + 0.5 \begin{bmatrix} 0.625 \\ -1.75 - 0.5 \cdot 0.625 \end{bmatrix} = \begin{bmatrix} 2.063 \\ -0.406 \end{bmatrix}$$



(5) Ratkaisun muoto on annettu:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t},$$

$$\lambda_n = \frac{cn\pi}{L} = \frac{n\pi}{5}.$$

Alkuehto: $f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{5}$.

Toteutuu valitsemalla $B_n = b_n$ = f:n sinisarjan kerroin.

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx =$$

$$\frac{2}{5}\int_0^5 20\sin\frac{n\pi x}{5}\,dx = -\frac{40}{n\pi}\Big/_0^5 \cos\frac{n\pi x}{5}$$

$$= \frac{40}{n\pi}\Big(1 - \underbrace{\cos n\pi}_{(-1)^n}\Big) = \frac{80}{n\pi}, \text{ n pariton}$$
$$= 0, \text{ n parill.}$$

$$\Rightarrow \quad u(x,t) = \frac{80}{\pi}\sum_{n=1,3,5,\dots} \frac{1}{n}\sin\frac{n\pi x}{5} e^{-\left(\frac{n\pi}{5}\right)^2 t}$$

1. termi: $u_1(x,t) = \frac{80}{\pi}\sin\frac{\pi x}{5} e^{-\left(\frac{\pi}{5}\right)^2 t}$

$x = 2.5$; $u_1(2.5, t) = \frac{80}{\pi} e^{-\left(\frac{\pi}{5}\right)^2 t}$.

$$5 = u_1(2.5, t_1) = \frac{80}{\pi} e^{-\left(\frac{\pi}{5}\right)^2 t_1} \Longrightarrow$$

$$-\frac{\pi^2}{25} t_1 = \ln\frac{5\pi}{80} \Longrightarrow t_1 = \frac{25}{\pi^2}\ln\frac{80}{5\pi}$$

$$= 4.123 \text{ s.}$$

u(x,t):n approksimaatio 25:n ensimmäisen 0:sta poikkeavan termin avulla.