Home Assignment 3

This exercise deals with the dynamic inverse problems and Kalman filtering. In this assignment, not all the details are fixed, so there is a bit space for experimentation.

Consider a heat conducting rod of unit length, modelled as an interval $[0, 1]$. The temperature of the rod is denoted by $u = u(x, t)$, $0 \leq x \leq 1$, $t = \text{time} \geq 0$. The initial temperature distribution of the rod is denoted by $u(x, 0) = u_0(x)$. The forward model for the time evolution is described by the heat equation,

$$\frac{\partial u}{\partial t}(u, t) = \frac{\partial^2 u}{\partial x^2}(x, t),$$

with the initial-boundary conditions,

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t) = 0, \quad u(x, 0) = u_0(x).$$

The inverse problem is described as follows: the temperature is measured at discrete times $t_k = k\Delta t$ at one end of the rod. The problem is to estimate the temperature distribution along the rod.

We discretize the forward problem. Divide the interval $[0, 1]$ in n equal subintervals, and denote $x^j = j/n$, $0 \leq j \leq n$. Further, let $u^j(t) = u(x^j, t)$. We write

$$\frac{\partial^2 u}{\partial x^2}(x^j, t) \approx n^2(u^{j-1}(t) - 2u^j(t) + u^{j+1}(t)), \quad 1 \leq j \leq n - 1.$$

The boundary conditions are approximated by

$$u^1(t) = u^0(t), \quad u^{n-1}(t) = u^n(t).$$

Hence, we can eliminate the temperatures at the end points and write a semidiscretized evolution equation for the temperature interior points,

$$\frac{d\mathbf{u}}{dt} = L\mathbf{u}(t), \tag{1}$$

where $\mathbf{u}(t9 \in \mathbb{R}^{n-1}$ and $L \in \mathbb{R}^{(n-1)\times(n-1)}$ are

$$\mathbf{u}(t) = \begin{bmatrix} u^1(t) \\ u^2(t) \\ \vdots \\ u^{n-1}(t) \end{bmatrix}, \quad L = n^2 \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}.$$

1. Define an initial temperature distribution, e.g., by assuming that the rod is initially heated up locally somewhere in the middle of the interval. Then select a time interval Δt, and propagate the initial temperature distribution by the equation (1) to $t = t_j = j\Delta t$, $j = 1, 2, \ldots$.

   having propagated the temperature distribution, read off the boundary values,

   $$u^0(t_j) = u^1(t_j), \quad j = 1, 2 \ldots$$

   Add noise to the boundary values to obtain the data

   $$y_j = u^0(t_j) + e_j, \quad e_j \sim \mathcal{N}(0, \sigma^2). \tag{2}$$

2. Kalman filtering. Suppose that we know the thermal properties of the rod plus the boundary conditions, that is, we know that the temperature distribution follows the equation (1). Write an evolution model using, e.g., the forward Euler discretization scheme,

   $$\mathbf{u}(t_{j+1}) = \mathbf{u}(t_j) + \Delta t L\mathbf{u}(t_j) + \mathbf{v}_{j+1}, \tag{3}$$

   where $\mathbf{v}_{j+1}$ is a term that account for modelling errors, e.g., due to the discretization of the time evolution equation.

   Using the observation equation (2), write a Kalman filtering algorithm to track the temperature distribution. Assume that you do not know the nitial temperature exactly, but only an approximation.

3. Try the Kalman filtering algorithm assuming that instead of the relatively good propagation model (3), you use a random walk model,

   $$\mathbf{u}(t_{j+1}) = \mathbf{u}(t_j) + \mathbf{w}_{j+1},$$

   where $\mathbf{w}_{j+1}$ is a Gaussian innovation term.