Computational Methods in Inverse Problems, Mat–1.3626

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1. (a) Let $A \in \mathbb{R}^{m\times n}$ be a matrix with singular value decomposition $A = UDV^{\mathrm{T}}$. Show that the eigen values of the symmetric matrix $A^{\mathrm{T}}A$ are the squared diagonal elements of D. What are the corresponding eigenvectors?

   (b) Let $U \in \mathbb{R}^{n\times n}$ be an orthogonal matrix, i.e., $U^{\mathrm{T}}U = I$, where I is the unit matrix. Deduce that also U^{T} must be orthogonal, i.e., $UU^{\mathrm{T}} = I$.

   (c) Starting from the fact that vector norm is invariant under orthogonal matrix transformations, show that $\|U^{\mathrm{T}}AV\| = \|A\|$ for all A and orthogonal U and V such that the matrix product makes sense.

2. An orthogonal projection $P : \mathbb{R}^n \to H$ to a subspace $H \subset \mathbb{R}^n$ is defined as a matrix $P \in \mathbb{R}^{n\times n}$ with the following properties:

   - $P^2 = P$,
   - $(I - P)x \perp Px$ for all $x \in \mathbb{R}^n$.

   Given an arbitrary matrix $A \in \mathbb{R}^{m\times n}$, express the orthogonal projections $P_1 : \mathbb{R}^n \to N(A)$ and $P_2 : \mathbb{R}^m \to R(A)$ in terms of the singular value decomposition of A.Verify also the identity $R(A)^{\perp} = N(A^{\mathrm{T}})$ with the singular value decomposition.

3. Design a 2×2 matrix that maps the unit circle to an ellipse whose longer semiaxis has length 2 and points to direction $(x, y) = (1, 2)$ and the shorter semiaxis is of length 1/2. How many degrees of freedom do you have, i.e., characterize the degree of non-uniqueness of such matrices.

4. Consider the matrix $A \in \mathbb{R}^{n\times n}$,

   $$A = I - 2\widehat{u}\widehat{u}^{\mathrm{T}},$$

   where $\widehat{u} \in \mathbb{R}^n$ is an arbitrary unit vector, i.e., $\|\widehat{u}\| = 1$. Show that A is unitary. Find the eigenvalues and (real) eigenvectors of A. Give a geometric interpretation of A.

   Given a vector $x \in \mathbb{R}^n$, $x \neq 0$, set $u = x - \|x\|e_1$, where $e_1 = [1\ 0\ \cdots\ 0]^{\mathrm{T}}$, and $\widehat{u} = u/\|u\|$. Compute Ax and interpret geometrically the result.