WEAKLY SINGULAR INTEGRAL EQUATIONS

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Contents

1. Introduction

This course is devoted to the smoothness/singularities of the solutions of weakly singular integral equations of the second kind, and to piecewise polynomial collocation type methods to solve such equations. In Section 5 we prove theorems which characterise the boundary singulatities of the derivatives of a solution and undertake a change of variables that kills these singularities. This enables to justify some new collocation type methods not considered in the literature. Since two of these methods are based on the spline interpolation or quasi-interpolation, we undertake also a study of this approximation tool, see Section 8; occuring here computations are due to Evely Leetma.

It is assumed that the reader has taken an elementary course of functional analysis. In Section 2 we remind all or almost all that we need in the sequel about the functional spaces and the operator theory.

In the main text we minimise the quoting to literure. Bibliographical remarks can be found in the end of the lecture notes. There can be found also further comments on the central results of the lectures.

Besides elementary training exercises, section Exercises and Problems contains some more serious problem settings for possible master and doctor theses.

Let us recall standard designations used during the present notes:

$\mathbb{R} = (-\infty, \infty)$ is the set of real numbers, $\mathbb{R}_+ = [0, \infty)$,

$\mathbb{C}$ is the set of complex numbers,

$\mathbb{N} = \{1, 2, 3, ...\}$ is the set of natural numbers,

$\mathbb{Z} = \{... -1, 0, 1, 2, ...\}$ is the set of integers, $\mathbb{Z}_+ = \mathbb{N}_0 = \{0, 1, 2, ...\}$,

$\varphi(t) \asymp \psi(t)$ as $t \to 0$ means that $\frac{\varphi(t)}{\psi(t}$ and $\frac{\psi(t)}{\varphi(t}$ are bounded as $t \to 0$,

$\varphi(t) \sim \psi(t)$ as $t \to 0$ means that $\frac{\varphi(t)}{\psi(t} \to 1$ as $t \to 0$.

Sometimes we use abbreviated designations of partial derivatives:

$\partial_x = \frac{\partial}{\partial x}$, $\partial_x^k = (\frac{\partial}{\partial x})^k$.

By c we denote a generic constant that may have different values by different occurrences.

2. Requisites

2.1. Spaces. Below $\mathbf{K}$ stand for $\mathbb{R}$ or $\mathbb{C}$; its elements are called *scalars*.

A *vector space* X is a non-empty set with two operations – addition ($u, v \in X \mapsto u+v \in X$) and multiplication to scalars ($u \in X,\ \alpha \in \mathbf{K} \ \mapsto\ \alpha u \in X$) such that that the following axioms are satisfied:

$u + v = v + u, \quad u + (v + w) = (u + v) + w,$

$\alpha(u + v) = \alpha u + \alpha v, \quad (\alpha + \beta)u = \alpha u + \beta u, \quad (\alpha\beta)u = \alpha(\beta u), \quad 1u = u;$

there is an element $\mathbf{0}$ in X such that $u + \mathbf{0} = u$, $0u = \mathbf{0}$ for all $u \in X$.

The elements (called also vectors) $u_1, ..., u_n$ of a vector space X are *linearly dependent* if there are scalars $\alpha_1, ...,\alpha_n$ not all of which are zero such that $\alpha_1 u_1 + ... + \alpha_n u_n = \mathbf{0}$; otherwise $u_1, ..., u_n$ are called *linearly independent*. The *dimension* of X is n $(\mathrm{dim}X = n)$ if there are n linearly independent elements in X and every set of $n + 1$ elements is linearly dependent; the dimension of X is infinite ($\mathrm{dim}X = \infty$) if for any natural number n, there are n linearly independent elements in X. A *subspace* X_0 of a vector space X is a non-empty subset of X which itself is a vector space with respect to the operations of X (thus $u, v \in X_0 \Rightarrow\ u+v \in X_0$; $u \in X_0,\ \alpha \in \mathbf{K}\ \Rightarrow\ \alpha u \in X_0$). By $\mathrm{span}S$, the linear span of a subset $S \subset X$, is denoted the set of all linear combinations $\sum_{k=1}^{n} \alpha_k u_k$ with $\alpha_k \in \mathbf{K}$, $u_k \in S$, $n = 1, 2, ...$; clearly, $\mathrm{span}S$ is a subspace of X.

A *normed space* X is a vector space which is equipped with a norm $\| \cdot \| = \| \cdot \|_X$, a function from X into $\mathbb{R}_+$, such that

$\| u \| = 0$ if and only if $u = \mathbf{0}$;

$\| \alpha u \| = | \alpha | \| u \|\ \forall \alpha \in \mathbf{K},\ u \in X$;

$\| u + v \| \leq \| u \| + \| v \|\ \forall u, v \in X$.

A sequence $(u_n) \subset X$ *converges* to $u \in X$ (one writes $u_n \to u$ or $\lim u_n = u$) if $\| u_n - u \| \to 0$ as $n \to \infty$. A sequence $(u_n) \subset X$ is a *Cauchy sequence* if $\| u_m - u_n \| \to 0$ as $m, n \to \infty$. Every convergent sequence $(u_n) \subset X$ is Cauchy but the inverse is not true in general. A normed space X is called *complete* if every Cauchy sequence of its elemets converges to an element of X. A complete normed space is called *Banach space.*

For $u_0 \in X$ and $r > 0$, the set $B(u_0, r) := \{u \in X : \| u - u_0 \| \leq r\}$ is called (closed) *ball* of X with the centre u_0 and radius r. A set $S \subset X$ is called:

bounded if it is contained in a ball of X;

open if for any $u_0 \in S$ there is an $r > 0$ such that $B(u_0, r) \subset S$;

closed if $(u_n) \subset S$, $u_n \to u$ implies $u \in S$;

relatively compact if every sequence $(u_n) \subset S$ contains a convergent subsequence (with a limit in X not necessarily belonging to S);

compact if S is closed and relatively compact.

The *closure* $\overline{S}$ of a set $S \subset X$ is the smallest closed set containing S. A set $S \subset X$ is said to be dense in X if $\overline{S} = X$. A relatively compact set is bounded; in finite dimensional spaces, also the inverse is true.

The *Kolmogorov n-width* $d_n(S, X)$ of a set $S \subset X$ is defined by

$$d_n(S,X) = \inf_{X_n \subset X:\, \dim X_n = n} \sup_{u \in S} \inf_{u_n \in X_n} \| u - u_n \|_X$$

where the infimum is taken over all subspaces $X_n \subset X$ of dimension n.

Examples of Banach spaces:

$C[0,1]$ consists of all continuous functions $u : [0,1] \to \mathbf{K}$,

$$\| u \|_{C[0,1]} = \| u \|_\infty = \max_{0 \le x \le 1} | u(x) |;$$

$BC(0,1)$ consists of all bounded continuous functions $u : (0,1) \to \mathbf{K}$,

$$\| u \|_{BC[0,1]} = \| u \|_\infty = \sup_{0<x<1} | u(x) |;$$

$C^m[0,1]$ consists of all m times continuously differentiable functions $u : [0,1] \to \mathbf{K}$,

$$\| u \|_{C^m[0,1]} = \sum_{k=0}^{m} \| u^{(k)} \|_\infty;$$

$L^p(0,1)$, $1 \le p < \infty$, consists of all (equivalence classes of) measurable functions $u : (0,1) \to \mathbf{K}$ such that $\| u \|_p < \infty$,

$$\| u \|_{L^p(0,1)} = \| u \|_p = \left(\int_0^1 | u(x) |^p \right)^{1/p};$$

$L^\infty(0,1)$ consists of all measurable functions $u : (0,1) \to \mathbf{K}$ such that $\| u \|_\infty < \infty$,

$$\| u \|_{L^\infty(0,1)} = \| u \|_\infty = \sup_{0<x<1} | u(x) | .$$

All these spaces are infinite dimensional. The space $C[0,1]$ is a closed subspace of $BC(0,1)$; both are closed subspaces of $L^\infty(0,1)$.

Theorem 2.1 (Arzela). A set $S \subset C[0,1]$ is relatively compact in $C[0,1]$ if and only if the following two conditions are fulfilled:

(i) the functions $u \in S$ are uniformly bounded, i.e., there is a constant c such that $| u(x) | \le c$ for all $x \in [0,1]$, $u \in S$;

(ii) the functions $u \in S$ are equicontinuous, i.e., for every $\varepsilon > 0$ there is a $\delta > 0$ such that $x_1, x_2 \in [0,1]$, $| x_1 - x_2 | \le \delta$ implies $| u(x_1) - u(x_2) | \le \varepsilon$ for all $u \in S$.

2.2. **Linear operators.** Let X and Y be two vector spaces. *Operator* $A : X \to Y$ is a function defined on X and with values in Y; operator A is *called linear* if

$$A(u + v) = Au + Av, \quad A(\alpha u) = \alpha Au$$

for all $u, v \in X$ and $\alpha \in \mathbf{K}$.

Assume now that X and Y are normed spaces. An operator $A : X \to Y$ is said to be continuous if $\| u_n - u \|_X \to 0$ implies $\| Au_n - Au \|_Y \to 0$. A linear operator $A : X \to Y$ occurs to be continuous if and only if it is be bounded, i.e., if there is a constant c such that

$$\| Au \|_Y \leq c \| u \|_X$$

for all $u \in X$. The smallest constant c in this inequality is called the *norm* of A,

$$\| A \|_{X \to Y} = \sup\{\| Au \|_Y : \ u \in X, \ \| u \|_X = 1\}.$$

A sequence of linear bounded operators $A_n : X \to Y$ is said to be *pointwise convergent* (or strongly convergent) if the sequence $(A_n u)$ is convergent in Y for any $u \in X$.

Theorem 2.2 (Banach–Steinhaus). Let X and Y be Banach spaces. A sequence of linear bounded operators $A_n : X \to Y$ converges pointwise if and only if the following two conditions are fulfilled:

(i) there is a constant c such that $\| A_n \|_{X \to Y} \leq c$ for all n;

(ii) there is a dense set $S \subset X$ such that the sequence $(A_n u)$ is convergent in Y for every $u \in S$.

Under conditions (i) and (ii), the limit operator $A : X \to Y$, $Au = \lim A_n u$, is linear and bounded.

2.3. **Inverse operator.** Let X and Y be Banach spaces and $A : X \to Y$ a linear operator. Introduce the subspaces

$$\mathcal{N}(A) = \{u \in X : \ Au = \mathbf{0}\} \subset X \quad \text{(the null space of } A),$$

$$\mathcal{R}(A) = \{f \in Y : \ f = Au, \ x \in X\} \subset Y \quad \text{(the range of } A).$$

If $\mathcal{N}(A) = \{\mathbf{0}\}$ then the inverse operator $A^{-1} : \mathcal{R}(A) \subset Y \to X$ exists on $\mathcal{R}(A)$, i.e., $A^{-1}Au = u \ \ \forall u \in X$, $AA^{-1}f = f \ \ \forall f \in \mathcal{R}(A)$; clearly also A^{-1} is linear. If $\mathcal{N}(A) = \{\mathbf{0}\}$ and $\mathcal{R}(A) = Y$ then the inverse operator $A^{-1} : Y \to X$ is defined on whole Y; a nontrivial fact is that A^{-1} is bounded if A is. This is the essence of the following theorem.

Theorem 2.3 (Banach). Let X and Y be Banach spaces and let $A : X \to Y$ be a linear bounded operator with $\mathcal{N}(A) = \{\mathbf{0}\}$ and $\mathcal{R}(A) = Y$. Then the inverse operator $A^{-1} : Y \to X$ is linear and bounded.

Theorem 2.4 (Banach). Let X and Y be Banach spaces and $A : X \to Y$ a linear bounded operator having the inverse $A^{-1} : Y \to X$. Assume that the linear bounded operator $B : X \to Y$ satisfies the condition

$$\| B \|_{X \to Y} \| A^{-1} \|_{Y \to X} < 1.$$

Then $A+B: X \to Y$ has the inverse $(A+B)^{-1}: Y \to X$ (defined on whole Y) and

$$\| (A+B)^{-1} \|_{Y\to X} \le \frac{\| A^{-1} \|_{Y\to X}}{1- \| B \|_{X\to Y} \| A^{-1} \|_{Y\to X}}.$$

2.4. **Linear compact operators.** Let X , Y, U, V be Banach spaces. A linear operator $T: X \to Y$ is said to be *compact* if it maps bounded subsets of X into relatively compact subsets of Y. Equivalently, $T: X \to Y$ is compact if for every bounded sequence $(u_n) \subset X$, the sequence (Tu_n) contains a subsequence that converges in Y. Linear compact operators are bounded. A linear bounded finite dimensional operator (i.e., a linear bounded operator with finite dimensional range) is compact. For linear compact operators $T_1, T_2 : X \to Y$, $\alpha_1, \alpha_2 \in \mathbf{K}$, the operator $\alpha_1 T_1 + \alpha_2 T_2 : X \to Y$ is compact. For a linear compact operator $T: X \to Y$ and linear bounded operators $A: U \to X$ and $B: Y \to V$, the operators $TA: U \to Y$ and $BT: X \to V$ are compact.

Theorem 2.5. Let $T_n : X \to Y$, $n = 1, 2, ...,$ be linear compact operators, $T: X \to Y$ a linear bounded operator, and let $\| T_n - T \|_{X\to Y} \to 0$ as $n \to \infty$. Then $T: X \to Y$ is compact.

Theorem 2.6. Let $T: X \to Y$ be a linear compact operator and let the linear bounded operators $B_n : Y \to V$ converge pointwise to $B: Y \to V$ as $n \to \infty$. Then

$$\| B_n T - BT \|_{X\to V} \to 0 \quad \text{as} \quad n \to \infty.$$

(Similar claim about $\| TA_n - TA \|_{U\to Y}$ is wrong.)

Denote by $I = I_X$ the identity operator in X, i.e., $Iu = u$ for every $u \in X$.

Theorem 2.7 (Fredholm alternative)**.** Let $T: X \to X$ be a linear compact operator and let

$$\mathcal{N}(I-T) = \{\mathbf{0}\}.$$

Then $I - T$ has the bounded inverse $(I-T)^{-1}: X \to X$.

Theorem 2.8. Let X and Y be Banach spaces such that $Y \subset X$, Y is dense in X and $\| u \|_X \le c \| u \|_Y$ for every $u \in Y$. Let $T: X \to X$ be a linear compact operator that maps Y into Y, and let also $T: Y \to Y$ *be* compact. Assume that the equation $u = Tu + f$ with given $f \in Y$ has a solution $u \in X$. Then $u \in Y$.

The only claim $u \in Y$ of Theorem 2.8 will be trivial if we add the assumption that $\mathcal{N}(I - T) = \{\mathbf{0}\}$, since then by Theorem 2.7 equation $u = Tu + f$ is uniquely solvable in X as well as in Y. Actually this additional assumption is acceptable for our needs in the sequel so far as we do not treat eigenvalue problems.

Examples of linear compact integral operators. With the help of Theorem 2.1 it easy to see that the *Fredholm integral operator*

$$T: C[0,1] \to C[0,1], \;\; (Tu)(x) = \int_0^1 K(x,y)u(y)dy,$$

is compact provided that its *kernel* $K(x,y)$ is continuous on the square $\Box = [0,1] \times [0,1]$. Similarly, the *Volterra integral operator*

$$T : C[0,1] \to C[0,1], \quad (Tu)(x) = \int_0^x K(x,y)u(y)dy,$$

is compact provided that the kernel $K(x,y)$ is continuous on the triangle $\Delta = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le x \le 1\}$.

2.5. **Differentiation of composite functions. Theorem 2.9** (Faa di Bruno). Let u be an m times continuously differentiable function on an interval which contains the values of $\varphi \in C^m[0,1]$. Then the composite function $u(\varphi(x))$ is m times continuously differentiable on $[0,1]$ and the differentiation formula

$$\left(\frac{d}{dx}\right)^j u(\varphi(x)) = \sum_{k_1+2k_2+...+jk_j=j} \frac{j!}{k_1!...k_j!} u^{(k_1+...+k_j)}(\varphi(x)) \left(\frac{\varphi'(x)}{1!}\right)^{k_1} \cdots \left(\frac{\varphi^{(j)}(x)}{j!}\right)^{k_j}$$

holds for $j = 1, ..., m$; the sum is taken over all non-negative integers $k_1, ..., k_j$ such that $k_1 + 2k_2 + ... + jk_j = j$.

3. WEAKLY SINGULAR INTEGRAL OPERATORS

3.1. **Weakly singular kernels.** Consider the integral operator T defined by its kernel function $K(x,y)$ via the formula

$$(Tu)(x) = \int_0^1 K(x,y)u(y)dy, \quad 0 \le x \le 1,$$

where u is taken from some set of functions, for example, from $C[0,1]$. In the literature, the weak singularity of the kernel K and of the corresponding operator T may have different senses. A tight understanding is that K has the form

$$K(x,y) = a(x,y) \mid x-y \mid^{-\nu} \tag{3.1}$$

where a is a continuous function on $[0,1] \times [0,1]$ and $0 < \nu < 1$. This kernel has the property

$$\sup_{0 \le x \le 1} \int_0^1 \mid K(x,y) \mid dy < \infty \tag{3.2}$$

often used to define the weak singularity in the wide sense: a kernel K is weakly singular if it is absolutely integrable w.r.t. y and satisfies (3.2). The kernels we will consider in the sequel are somewhere in the middle of these two extremal understandings of the weak singularity: we assume that K is continuous on $([0,1] \times [0,1]) \setminus \mathrm{diag}$ and

$$\mid K(x,y) \mid \le c_K(1+ \mid x-y \mid^{-\nu}) \quad \text{for} \quad (x,y) \in ([0,1] \times [0,1]) \setminus \mathrm{diag} \tag{3.3}$$

where $\nu < 1$. Here diag means the diagonal of $\mathbb{R}^2$:

$$\mathrm{diag} = \mathrm{diag}(\mathbb{R}^2) = \{(x,y) \in \mathbb{R}^2 : x = y\}.$$

For instance, the kernels

$$K(x,y) = a(x,y)\log \mid x-y\mid, \;\; K(x,y) = a(x,y) \mid x-y\mid^{-\nu} \log^k \mid x-y\mid$$

with $a \in C([0,1]\times[0,1])$ and many others are weakly singular in this sense.

3.2. **The smoothness-singularity class $\mathcal{S}^{m,\nu}$ of kernels.** We are interested in kernels that are C^m-smooth outside the diagonal. Introduce the following smoothness-singularity class $\mathcal{S}^{m,\nu}$ of kernels. For given $m \in \mathbb{N}_0$ and $\nu \in \mathbb{R}$, denote by $\mathcal{S}^{m,\nu} = \mathcal{S}^{m,\nu}(([0,1]\times[0,1]) \setminus \text{diag})$ the set of m times continuously differentiable kernels K on $([0,1]\times[0,1]) \setminus \text{diag}$ that satisfy there for all $k,l \in \mathbb{N}_0$, $k+l \le m$, the inequality

$$\left|\left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^l K(x,y)\right| \le c_{K,m} \left\{ \begin{array}{cc} 1, & \nu+k<0 \\ 1+\mid \log\mid x-y\mid\mid, & \nu+k=0 \\ \mid x-y\mid^{-\nu-k}, & \nu+k>0 \end{array} \right\}. \tag{3.4}$$

Note that for $k=l=0$, $\nu>0$, condition (3.4) coincides with (3.3). A kernel $K \in \mathcal{S}^{m,\nu}$ is weakly singular if $\nu<1$. A kernel $K \in \mathcal{S}^{m,\nu}$ with $\nu<0$ is bounded but its derivatives may have singularities on the diagonal; $\nu=0$ corresponds to a logarithmically singular kernel. A consequene of (3.4) is that

$$\left|\left(\frac{\partial}{\partial y}\right)^k \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^l K(x,y)\right| \le c'_{K,m} \left\{ \begin{array}{cc} 1, & \nu+k<0 \\ 1+\mid \log\mid x-y\mid\mid, & \nu+k=0 \\ \mid x-y\mid^{-\nu-k}, & \nu+k>0 \end{array} \right\}. \tag{3.5}$$

Indeed, using the equality $\partial_y = (\partial_x+\partial_y)-\partial_x$, we can obtain (3.5) from (3.4) first for $k=1$, then for $k=2$ etc.

Observe also that the differentiation $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^l$ does not influence on the r.h.s. of (3.4). This tells us that (3.4) is somehow related to kernels that depend on the difference $x-y$ of arguments. For example, kernel (3.1) belongs to $\mathcal{S}^{m,\nu}$ if $a \in C^m([0,1]\times[0,1])$; actually the condition on a can be weakened, see Exercise 4. A further important example is given by $K(x,y) = a(x,y)\log\mid x-y\mid$ with an $a \in C^m([0,1]\times[0,1])$ – this kernel K belongs to $\mathcal{S}^{m,0}$.

Lemma 3.1. (i) If $K \in \mathcal{S}^{m,\nu}$ with an $m \ge 1$ then $\partial_x K(x,y)$ and $\partial_y K(x,y)$ belong to $\mathcal{S}^{m-1,\nu+1}$whereas $(\partial_x+\partial_y)K(x,y)$ belongs to $\mathcal{S}^{m-1,\nu}$.

(ii) If $K \in \mathcal{S}^{m,\nu}$ then $(x-y)K(x,y)$ belongs to $\mathcal{S}^{m,\nu-1}$.

Proof. These claims are elementary consequences of the definition of $\mathcal{S}^{m,\nu}$. □

3.3. **Compactness of a weakly singular integral operator in $C[0,1]$.** A weak singularity of the kernel implies that the corresponding integral operator is compact in the space $C[0,1]$. More precisely, the following statement holds true.

Lemma 3.2. A kernel $K \in \mathcal{S}^{m,\nu}$ *with* $m \ge 0$, $\nu<1$ defines a compact operator $T : L^\infty(0,1) \to C[0,1]$, hence also a compact operator $T : C[0,1] \to C[0,1]$ and a compact operator $T : L^\infty(0,1) \to L^\infty(0,1)$.

Proof. Take a smooth "cutting" function $e : [0, \infty) \to \mathbb{R}$ satisfying the conditions $e(r) = 0$ for $0 \le r \le \frac{1}{2}$, $e(r) = 1$ for $r \ge 1$ and $0 \le e(r) \le 1$ for all $r \ge 0$. Define

$$K_n(x,y) = e(n \mid x-y \mid)K(x,y), \quad (x,y) \in [0,1] \times [0,1],$$

and

$$(T_n u)(x) = \int_0^1 K_n(x,y)u(y)dy, \quad n \in \mathbb{N}.$$

The kernels $K_n(x,y)$ are continuous on $[0,1]\times[0,1]$ – the possible diagonal singularity is "cut" off by the factor $e(n \mid x-y \mid)$, $K_n(x,y) = 0$ in a neighborhood of the diagonal. Hence the operators $T_n : L^\infty(0,1) \to C[0,1]$ are compact. Further, for $u \in L^\infty(0,1)$, $0 \le x \le 1$, we have

$$(Tu - T_n u)(x) = \int_0^1 [K(x,y) - K_n(x,y)]u(y)dy = \int_0^1 K(x,y)[1 - e(n \mid x-y \mid)]u(y)dy,$$

$$\mid (Tu - T_n u)(x) \mid \le c_K \int_0^1 \mid x-y \mid^{-\nu} [1 - e(n \mid x-y \mid)]dy \parallel u \parallel_\infty$$

$$\le c_K \int_{|x-y|\le 1/n} \mid x-y \mid^{-\nu} dy \parallel u \parallel_\infty = 2c_K \int_0^{1/n} z^{-\nu}dz \parallel u \parallel_\infty = 2c_K \frac{(1/n)^{1-\nu}}{1-\nu} \parallel u \parallel_\infty$$

that implies $Tu \in C[0,1]$ as a uniform limit of $T_n u \in C[0,1]$, and

$$\parallel T - T_n \parallel_{L^\infty(0,1)\to C[0,1]} \le 2c_K \frac{(1/n)^{1-\nu}}{1-\nu} \to 0 \text{ as } n \to \infty.$$

Thus T maps $L^\infty(0,1)$ into $C[0,1]$ and $T : L^\infty(0,1) \to C[0,1]$ is compact as a norm limit of compact operators $T_n : L^\infty(0,1) \to C[0,1]$, see Theorem 2.5. $\square$

4. Differentiation of weakly singular integrals

First we recall a well known result about the closedness of the graph of the differentiation operator; the proof is left as an exercise.

Lemma 4.1. *Let* $v_n \in C^1(0,1)$ and $v_n \to v$, $v_n' \to w$ *uniformly on every closed subinterval* $[\delta, 1-\delta]$, $\delta > 0$. *Then* $v \in C^1(0,1)$ *and* $v' = w$.

We are ready to establish a differentiation formulae for weakly singular integrals with respect to a parameter.

Theorem 4.2. *Let* $g(x,y)$ *be a continuously differentiable function on* $((0,1) \times [0,1]) \backslash \mathrm{diag}$ *satisfiying there the inequalities*

$$\mid g(x,y) \mid \le c \mid x-y \mid^{-\nu}, \quad \left| \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) g(x,y) \right| \le c \mid x-y \mid^{-\nu}, \quad \nu < 1. \tag{4.1}$$

Then the function $x \mapsto \int_0^1 g(x,y)dy$ *is continuously differentiable in* $(0,1)$ *and*

$$\frac{d}{dx} \int_0^1 g(x,y)dy = \int_0^1 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) g(x,y)dy + g(x,0) - g(x,1), \quad 0 < x < 1. \tag{4.2}$$

Proof. For functions g that are continuously differentiable on $(0,1)\times[0,1]$ including the diagonal, formula (4.2) is obvious. Let g satisfy the conditons of the Lemma. Take a cutting function $e\in C^1[0,\infty)$ satisfying $e(r)=0$ for $0\le r\le \frac{1}{2}$, $e(r)=1$ for $r\ge 1$ and $0\le e(r)\le 1$ for all $r\ge 0$; we already used this cutting function in the proof of Lemma 3.2. Define $g_n(x,y)=e(n\mid x-y\mid)g(x,y)$, $n=1,2,\ldots.$ The functions g_n are continuously differentiable on $(0,1)\times[0,1]$ and (4.2) holds for them: denoting $v_n(x)=\int_0^1 e(n\mid x-y\mid)g(x,y)dy$, we have

$$v_n'(x)=\frac{d}{dx}\int_0^1 e(n\mid x-y\mid)g(x,y)dy=\int_0^1 e(n\mid x-y\mid)(\partial_x+\partial_y)g(x,y)dy$$
$$+e(nx)g(x,0)-e(n(1-x))g(x,1),\ \ 0<x<1.$$

We took into account that $(\partial_x+\partial_y)e(n\mid x-y\mid)=0$. With the help of (4.1) we find that

$$v_n(x)\to\int_0^1 g(x,y)dy,\ \ v_n'(x)\to\int_0^1(\partial_x+\partial_y)g(x,y)dy+g(x,0)-g(x,1)\ \text{ as }\ n\to\infty$$

uniformly on every closed subinterval $[\delta,1-\delta]$, $\delta>0$. By Lemma 4.1, the function $\int_0^1 g(x,y)dy$ is continuously differentiable on $(0,1)$ and (4.2) holds true for it. □

Theorem 4.3. *Let $g(x,y)$ be a continuously differentiable function for $0\le y<x<1$ satisfiying there the inequalities*

$$|\,g(x,y)\,|\le c(x-y)^{-\nu},\quad \left|\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)g(x,y)\right|\le c(x-y)^{-\nu},\ \ \nu<1. \tag{4.3}$$

Then the function $\int_0^x g(x,y)dy$ is continuously differentiable in $(0,1)$ and

$$\frac{d}{dx}\int_0^x g(x,y)dy=\int_0^x\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)g(x,y)dy+g(x,0),\ \ 0<x<1. \tag{4.4}$$

Proof. This can be proved by the same idea as Theorem 4.2. Alternatively, we can derive (4.4) from (4.2) extending g by the zero values to $((0,1)\times[0,1])$ \diag and noticing that (4.3) implies (4.1) for the extended g. The details of the argument are proposed as an exercise. □

5. Boundary singularities of the solution

5.1. **Boundary singularities of a solution to w.s.i.e. is a usual phenominon.** Consider the integral equation

$$u(x)=\int_0^1 K(x,y)u(y)dy+f(x),\quad 0\le x\le 1, \tag{5.1}$$

where $K\in\mathcal{S}^{m,\nu}$ with $m\ge 1$, $\nu<1$, $f\in C^m[0,1]$. Let us demonstrate that in general $u\notin C^1[0,1]$. Indeed, supposing that $u\in C^1[0,1]$, we can differentiate (5.1) as an equality and we obtain on the basis of Theorem 4.2

$$u'(x)=\int_0^1[(\partial_x+\partial_y)K(x,y)]u(y)]dy+\int_0^1 K(x,y)u'(y)dy$$
$$+K(x,0)u(0)-K(x,1)u(1)+f'(x).$$

Since the integral operators with the kernels $K(x,y)$ and $(\partial_x+\partial_y)K(x,y)$ are weakly singular and $u, u' \in C[0,1]$, the first two terms on the r.h.s. are on the basis of Theorem 3.2 continuous on $[0,1]$; the same is true for the term $f'(x)$. On the other hand, the term $K(x,0)u(0)$ has a singularity at $x=0$ provided that $u(0)\neq 0$ and $K(x,0)$ really has a singularity allowed by inequality (3.3), and similarly the term $K(x,1)u(1)$ has a singularity at $x=1$ if $u(1)\neq 0$ and $K(x,1)$ has a singularity. Thus the assumption $u\in C^1[0,1]$ leads to a contradiction if $K(x,0)$ or $K(x,1)$ is singular and $u(0)\neq 0$, $u(1)\neq 0$; these inequalities hold for most of $f\in C^m[0,1]$.

5.2. **Weighted space $C^{m,\nu}(0,1)$.** For $m\geq 1$, $\nu<1$, denote by $C^{m,\nu}(0,1)$ the space of functions $f\in C^m(0,1)$ that satisfy the inequalities

$$\mid f^{(j)}(x)\mid\leq c_f\left\{\begin{array}{cc} 1 & j+\nu-1<0\\ 1+\mid\log\rho(x)\mid, & j+\nu-1=0\\ \rho(x)^{-j-\nu+1}, & j+\nu-1>0\end{array}\right\},\quad 0<x<1,\ \ j=0,...,m, \tag{5.2}$$

where

$$\rho(x)=\min\{x,1-x\}$$

is the distance from $x\in(0,1)$ to the boundary of the interval $(0,1)$. Introduce the weight functions

$$w_\lambda(x)=\left\{\begin{array}{cc} 1, & \lambda<0\\ 1/(1+\mid\log\rho(x)\mid), & \lambda=0\\ \rho(x)^\lambda & \lambda>0\end{array}\right\},\quad 0<x<1,\ \ \lambda\in\mathbb{R}.$$

Equipped with the norm

$$\parallel f\parallel_{C^{m,\nu}(0,1)}=\sum_{j=0}^{m}\sup_{0<x<1} w_{j+\nu-1}(x)\mid f^{(j)}(x)\mid,$$

$C^{m,\nu}(0,1)$ becomes a Banach space.

For $j=0$ (5.2) yields $\mid f(x)\mid\leq c_f$ telling us that a function $f\in C^{m,\nu}(0,1)$ is bounded on $(0,1)$. For $j=1$ (5.2) yields

$$\mid f'(x)\mid\leq c_f\rho(x)^{-\nu},\ \ 0<x<1,$$

if $0<\nu<1$; for $\nu\leq 0$ we have a more strong inequality. This implies $f'\in L^q(0,1)$ for a $q>1$ such that $q\nu<1$. Hence, for any $x_1,x_2\in(0,1)$, we have

$$\mid f(x_1)-f(x_2)\mid=\left|\int_{x_1}^{x_2} f'(x)dx\right|\leq\left(\int_{x_1}^{x_2}\mid f'(x)\mid^q dx\right)^{\frac{1}{q}}\left(\int_{x_1}^{x_2} dx\right)^{\frac{1}{q'}}=\parallel f'\parallel_{L^q}\mid x_1-x_2\mid^{\frac{1}{q'}}$$

where $\frac{1}{q}+\frac{1}{q'}=1$. We see that f is uniformly continuous on $(0,1)$. A uniformly continuous function f on $(0,1)$ has the boundary limits

$$f(0):=\lim_{x\to 0}f(x),\quad f(1):=\lim_{x\to 1}f(x),$$

hence f has a continuous extension to $[0,1]$. So we established a natural imbedding

$$C^{m,\nu}(0,1) \subset C[0,1], \quad m \geq 1, \ \nu < 1. \tag{5.3}$$

Moreover, with the help of Arzela theorem (Theorem 2.1) we obtain that the imbedding operator is compact.

If $\nu < 0$ we can apply the same argument for f' and so on. We obtain the following generalization of imbedding (5.3):

$$C^{m,\nu}(0,1) \subset C^l[0,1], \quad m \geq 1, \ \nu < 1, \ l = \min\{m-1, \mid \mathrm{int}\nu \mid\} \tag{5.4}$$

where intν, the integer part of ν, is the greatest integer not exceeding ν. Imbedding (5.4) is compact.

5.3. **Compactness of integral operators in weighted spaces.** The following theorem is crucial in the smoothness considerations for the solutions of (5.1). It has a simple formulation but not so simple proof.

Theorem 5.1. Let $K \in \mathcal{S}^{m,\nu}$, $m \geq 1$, $\nu < 1$. Then the Fredholm integral operator T defined by $(Tu)(x) = \int_0^1 K(x,y)u(y)dy$ maps $C^{m,\nu}(0,1)$ into itself and $T : C^{m,\nu}(0,1) \to C^{m,\nu}(0,1)$ is compact.

Proof. (i) *A technical formulation of what we have to prove.* First of all, taking a function $u \in C^{m,\nu}(0,1)$, we have to ensure that $Tu \in C^{m,\nu}(0,1)$, or equivalently, $Tu \in C^m(0,1)$ and $w_{i+\nu-1}D^iTu \in BC(0,1)$, $i = 0,...,m$, where $D = \frac{d}{dx}$ is the differentiation operator and $w_{i+\nu-1}$ are the weight functions introduced in Section 5.2. Second, we have to prove that the operators $w_{i+\nu-1}D^iT : C^{m,\nu}(0,1) \to BC(0,1)$, $i = 0,...,m$, are compact. Then for a given bounded sequence $(u_n)_{n\in\mathbb{N}} \subset C^{m,\nu}(0,1)$, the sequences $(w_{i+\nu-1}D^iTu_n)$, $i = 0,...,m$, are relatively compact in $BC(0,1)$, and repeatedly extracting convergent subsequences from the preceding subsequences, first for $j = 0$, after that for $j = 1$ etc., we can arrive to a subsequence determined by an infinite set $N' \subset \mathbb{N}$ such that all $(w_{i+\nu-1}D^iTu_n)_{n\in N'}$, $i = 0,...,m$, converge uniformly in $(0,1)$, or equivalently, the sequence $(Tu_n)_{n\in N'}$ converges in $C^{m,\nu}(0,1)$ that means the compactness of $T : C^{m,\nu}(0,1) \to C^{m,\nu}(0,1)$. (A fastidious reader uses Lemma 4.1 to ensure that the limits of $(D^iTu_n)_{n\in N'}$, $i = 0,...,m$, are consistent in $(0,1)$.)

For $i = 0$, we have $w_{i+\nu-1}(x) \equiv 1$, and $w_{i+\nu-1}D^iT = T : C^{m,\nu}(0,1) \subset C[o,1] \to C[0,1]$ is compact by Lemma 3.2. Thus *we have to prove the compactness of* $w_{i+\nu-1}D^iT : C^{m,\nu}(0,1) \to BC(0,1)$ *for* $i = 1,...,m$.

(ii) *Differentiation of* Tu. Take an arbitrary $u \in C^{m,\nu}(0,1)$ and a "cutting" function $e \in C^m[0,\infty)$ that satisfies

$$0 \leq e(r) \leq 1 \text{ for } r \geq 0, \ \ e(r) = 0 \text{ for } 0 \leq r \leq \frac{1}{2}, \ \ e(r) = 1 \text{ for } r \geq 1.$$

Fix an arbitrary point $x' \in (0,1)$ and denote $r' = \frac{1}{2}\rho(x') = \frac{1}{2}\min\{x', 1-x'\}$. For x satisfying $| x - x' |\le \frac{1}{2}r'$, we split

$$\int_0^1 K(x,y)u(y)dy$$

$$= \int_0^1 e\left(\frac{| x-y |}{r'}\right) K(x,y)u(y)dy + \int_0^1 \left\{1 - e\left(\frac{| x-y |}{r'}\right)\right\} K(x,y)u(y)dy.$$

In the first integral on r.h.s., the diagonal singularity is cut off by the factor $e\left(\frac{|x-y|}{r'}\right)$; we may differentiate this integral m times under the integral sign. In the second integral on r.h.s., the coefficient function $1 - e(| x - y | /r')$ vanishes for $y = 0$ and $y = 1$. Due to estimate (3.4) and Theorem 4.2, this integral is also differentiable; differentiation formula (4.2) yields

$$\frac{\partial}{\partial x}\int_0^1 \left\{1 - e\left(\frac{| x-y |}{r'}\right)\right\} K(x,y)u(y)dy$$

$$= \int_0^1 \left\{1 - e\left(\frac{| x-y |}{r'}\right)\right\} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\{K(x,y)u(y)\}\, dy, \quad | x - x' |\le r'/2$$

(the boundary terms of (4.2) vanish in our case; $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) e(| x-y | /r')= 0$). In its turn, the last integral can be differentiated in a similar manner. By repeated differentiation we obtain

$$(D^iTu)(x) = \int_0^1 \left(\frac{\partial}{\partial x}\right)^i \left\{e\left(\frac{| x-y |}{r'}\right) K(x,y)\right\} u(y)dy$$

$$+ \int_0^1 \left\{1 - e\left(\frac{| x-y |}{r'}\right)\right\} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^i \{K(x,y)u(y)\}\, dy, \quad | x - x' |\le \frac{1}{2}r', \quad 1 \le i \le m.$$

Differentiating the product of functions under the integrals by the Leibnitz rule, setting after that $x = x'$ but writing again x instead of x', we arrive at the formula

$$(5.5) \qquad (D^iTu)(x) = \sum_{j=0}^{i} \binom{i}{j} \int_0^1 e_j(x,y) \left(\frac{\partial}{\partial x}\right)^{i-j} K(x,y)u(y)dy$$

$$+ \sum_{j=0}^{i} \binom{i}{j} \int_0^1 \left\{1 - e\left(\frac{2 | x-y |}{\rho(x)}\right)\right\} \left\{\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{i-j} K(x,y)\right\} u^{(j)}(y)dy, \quad 0 < x < 1,$$

$$e_j(x,y) := \left[\left(\frac{\partial}{\partial x}\right)^j e\left(\frac{| x-y |}{r}\right)\right]_{r=\rho(x)/2}, \quad 0 \le j \le i, \quad 1 \le i \le m.$$

Multiplying both sides of (5.5) to the weight function $w_{i+\nu-1}(x)$, the result can be rewritten in the form

$$(5.6) \qquad w_{i+\nu-1}D^iTu = \sum_{j=0}^{i} \binom{i}{j} \left(T_{i,j}u + S_{i,j}(w_{j+\nu-1}D^ju)\right), \quad 1 \le i \le m,$$

where for $j = 0, ..., i$, $i = 1, ..., m$,

$$(T_{i,j}u)(x) = \int_0^1 w_{i+\nu-1}(x)e_j(x,y)\left(\frac{\partial}{\partial x}\right)^{i-j} K(x,y)u(y)dy, \tag{5.7}$$

$$(S_{i,j}v)(x) = \int_0^1 \frac{w_{i+\nu-1}(x)}{w_{j+\nu-1}(y)}\left\{1 - e\left(\frac{2\mid x-y\mid}{\rho(x)}\right)\right\}\left\{\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{i-j} K(x,y)\right\} v(y)dy. \tag{5.8}$$

Now the proof of the compactness of the operators $w_{i+\nu-1}D^iT : C^{m,\nu}(0,1) \to BC(0,1)$, $i = 1, ..., m$, can be reduced to the study of the mapping properties of $T_{i,j}$ and $S_{i,j}$. In (5.6),

$$\sup_{0<y<1} w_{j+\nu-1}(y) \mid (D^j u)(y) \mid \leq \| u \|_{C^{m,\nu}(0,1)}, \quad j = 0, ..., i.$$

To prove compactness of the operators $w_{i+\nu-1}D^iT : C^{m,\nu}(0,1) \to BC(0,1)$, $i = 1, ..., m$, it is sufficient to establish that

$$T_{i,j} : C^{m,\nu}(0,1) \to BC(0,1), \ \ i = 1, ..., m, \ \ j = 0, ..., i, \ \ \text{are compact}, \tag{5.9}$$

$$S_{i,j} : BC(0,1) \to BC(0,1), \ \ i = 1, ..., m, \ \ j = 0, ..., i, \ \ \text{are compact}. \tag{5.10}$$

(iii) *Proof of* (5.10). Denote by $H_{i,j}$ the kernel of the integral operator $S_{i,j}$, $1 \leq i \leq m$, $0 \leq j \leq i$,

$$H_{i,j}(x,y) = \frac{w_{i+\nu-1}(x)}{w_{j+\nu-1}(y)}\left\{1 - e\left(\frac{2\mid x-y\mid}{\rho(x)}\right)\right\}\left\{\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{i-j} K(x,y)\right\}.$$

Observe that $1 - e\left(\frac{2|x-y|}{\rho(x)}\right) = 0$ for $\mid x - y \mid \geq \frac{\rho(x)}{2}$, hence

$$\operatorname{supp} H_{i,j} \subset \{(x,y) \in [0,1]\times[0,1] : \mid x-y \mid \leq \frac{\rho(x)}{2}\}$$

and $H_{i,j}$ is continuous on $([0,1]\times[0,1])/\text{diag}$. For $(x,y) \in \operatorname{supp} H_{i,j}$, the quantities $\rho(x)$ and $\rho(y)$ are of the same order:

$$\frac{\rho(x)}{2} \leq \rho(y) \leq \frac{3}{2}\rho(x) \ \text{ for } \ y \in \left(x - \frac{\rho(x)}{2}, x + \frac{\rho(x)}{2}\right).$$

Hence similar relations hold for the weight functions: with some positive constants c_1 and c_2,

$$c_1 w_{j+\nu-1}(x) \leq w_{j+\nu-1}(y) \leq c_2 w_{j+\nu-1}(x), \quad j = 0, ..., m. \tag{5.11}$$

Thus

$$\mid H_{i,j}(x,y) \mid \leq c\frac{w_{i+\nu-1}(x)}{w_{j+\nu-1}(x)} \mid \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{i-j} K(x,y) \mid \leq c \mid \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{i-j} K(x,y) \mid .$$

Now (3.4) tells us that the kernels $H_{i,j}$, $i = 1, ..., m$, $j = 0, ..., i$, are weakly singular, and (5.10) holds due to Lemma 3.2.

(iv) *Proof of* (5.9): *case* $0 < \nu < 1$. The following argument holds also for $\nu = 0$, except for $j = i$. Denote by $K_{i,j}(x,y)$ the kernel of the integral operator $T_{i,j}$,

$$K_{i,j}(x,y) = w_{i+\nu-1}(x) e_j(x,y) \partial_x^{i-j} K(x,y), \;\; 1 \le i \le m,$$

Observe that

$$\operatorname{supp} e_0 \subset \{(x,y) \in [0,1] \times [0,1] : |\, x - y \,| \ge \rho(x)/4\},$$

whereas for $j > 0$ the support of e_j is smaller,

$$\operatorname{supp} e_j \subset \{(x,y) \in [0,1] \times [0,1] : \; \rho(x)/4 \le |\, x - y \,| \le \rho(x)/2\}, \;\; 0 < j \le m. \tag{5.12}$$

Further,

$$|\, e_j(x,y) \,| \le c_j (\rho(x)/2)^{-j}, \;\; c_j := \max_{r \ge 0} |\, e^{(j)}(r) \,|, \;\; 0 \le j \le m.$$

Hence

$$|\, e_j(x,y) \,| \le c \,|\, x - y \,|^{-j}, \;\; 0 \le j \le m, \tag{5.13}$$

(for $j = 0$ this inequality is trivially true). For $0 < \nu < 1$, $0 \le j \le i$, and for $\nu = 0$, $0 \le j < i$, (3.4) yields $|\, \partial_x^{i-j} K(x,y) \,| \le c \,|\, x - y \,|^{-(i-j)-\nu}$, and we obtain

$$|\, K_{i,j}(x,y) \,| \le c w_{i+\nu-1}(x) \,|\, x - y \,|^{-i-\nu},$$

$$\int_0^1 |\, K_{i,j}(x,y) \,|\, dy \le c w_{i+\nu-1}(x) \int_{\{y: \, |x-y| \ge \frac{\rho(x)}{4}\}} |\, x - y \,|^{-i-\nu} \, dy \tag{5.14}$$

$$\le 2c w_{i+\nu-1}(x) \int_{\frac{\rho(x)}{4}}^1 z^{-i-\nu} dz \le c' w_{i+\nu-1}(x) \left\{ \begin{array}{ll} \rho(x)^{-i-\nu+1}, & i + \nu > 1 \\ 1 + |\, \log \rho(x) \,|, & i + \nu = 1 \end{array} \right\} = c', \;\; 0 < x < 1.$$

This means that for $0 < \nu < 1$, $0 \le j \le i$, and for $\nu = 0$, $0 \le j < i$, the operators $T_{i,j} : C[0,1] \to BC(0,1)$ are bounded that together with the compact imbedding $C^{m,\nu}(0,1) \subset C[0,1]$ implies the compactness of $T_{i,j} : C^{m,\nu}(0,1) \to BC(0,1)$. Thus (5.9) holds true for $0 < \nu < 1$, whereas for $\nu = 0$ we yet have to prove that also $T_{i,i} : C^{m,\nu}(0,1) \to BC(0,1)$ is compact.

(v) *Proof of* (5.9): *case* $\nu = 0$. To obtain (5.9) for $\nu = 0$, it remains to establish that the operators $T_{i,i} : C^{m,\nu}(0,1) \to BC(0,1)$, $1 \le i \le m$, are compact. Let us try to follow the estimation idea of the proof part (iv): now (for $j = i$) we have by (3.4) $|\, \partial_x^{i-j} K(x,y) \,| \le c(1 + \log |\, x - y \,|)$,

$$|\, K_{i,i}(x,y) \,| \le c w_{i-1}(x) \,|\, x - y \,|^{-i} \, (1 + \log |\, x - y \,|),$$

and instead of (5.14) we obtain

$$\int_0^1 |\, K_{i,i}(x,y) \,|\, dy \le c w_{i-1}(x) \int_{\{y: \, |x-y| \ge \frac{\rho(x)}{4}\}} |\, x - y \,|^{-i} \, (1 + \log |\, x - y \,|) dy$$

$$\leq c' \left\{ \begin{array}{cc} 1/(1+ \mid \log\rho(x) \mid), & i=1 \\ \rho(x)^{i-1}, & i>1 \end{array} \right\} \rho(x)^{-i+1}(1+ \mid \log\rho(x) \mid) = c' \left\{ \begin{array}{cc} 1, & i=1 \\ 1+ \mid \log\rho(x) \mid, & i>1 \end{array} \right\}.$$

We see that $T_{i,i} : BC(0,1) \to BC(0,1)$ is till bounded (and hence $T_{i,i} : C^{m,\nu}(0,1) \to BC(0,1)$ is compact) for $i = 1$ but not for $i > 1$. To prove the compactness of $T_{i,i} : C^{m,\nu}(0,1) \to BC(0,1)$ for $i \geq 2$ we need new ideas. Observe that

$$e_i(x,y) := -\partial_y e_{i-1}(x,y)$$

and integrate in (5.7) by parts. Clearly $e_{i-1}(x,0) = e_{i-1}(x,1) = 0$, so we obtain

$$(T_{i,i}u)(x) = \int_0^1 w_{i-1}(x)\left[-\partial_y e_{i-1}(x,y)\right] K(x,y)u(y)dy$$

$$= \int_0^1 w_{i-1}(x)e_{i-1}(x,y)[\partial_y K(x,y)]u(y)dy + \int_0^1 w_{i-1}(x)e_{i-1}(x,y)K(x,y)u'(y)dy,$$

$$T_{i,i} = T'_{i,i} + T''_{i,i}.$$

Due to (5.13) and (3.5), the kernel of the operator $T'_{i,i}$ has the estimate

$$w_{i-1}(x) \mid e_{i-1}(x,y)[\partial_y K(x,y)] \mid \leq c w_{i-1}(x) \mid x-y \mid^{-i+1} \mid x-y \mid^{-1} = c\rho(x)^{i-1} \mid x-y \mid^{-i},$$

and similarly as in (iv) we obtain that $T'_{i,i} : BC(0,1) \to BC(0,1)$ is bounded, hence $T'_{i,i} : C^{m,\nu}(0,1) \to BC(0,1)$ is compact. To prove the compactness of $T''_{i,i} : C^{m,\nu}(0,1) \to BC(0,1)$, we present it in the form

$$(T''_{i,i}u)(x) = \int_0^1 \frac{w_{i-1}(x)}{w_0(y)} e_{i-1}(x,y)K(x,y)[w_0(y)u'(y)]dy.$$

Here $\| w_0 u' \|_\infty \leq \| u \|_{C^{m,\nu}(0,1)}$, so it suffices to observe that

$$T'''_{i,i} : BC(0,1) \to BC(0,1) \text{ is compact for } (T'''_{i,i}v)(x) := \int_0^1 \frac{w_{i-1}(x)}{w_0(y)} e_{i-1}(x,y)K(x,y)v(y)dy$$

as for an integral operator with a weakly singular kernel. Indeed, taking into account (5.11) and (5.12) we can estimate the kernel of $T'''_{i,i}$ as follows:

$$\frac{w_{i-1}(x)}{w_0(y)} \mid e_{i-1}(x,y)K(x,y) \mid \leq c\rho(x)^{i-1}(1+ \mid \log\rho(x) \mid) \mid x-y \mid^{-i+1} (1+ \mid \log \mid x-y \mid\mid)$$

$$\leq c'(1+ \mid \log \mid x-y \mid\mid)^2, \quad (x,y) \in \operatorname{supp} e_{i-1}.$$

This completes the proof of the Theorem in the most important case $0 \leq \nu \leq 1$. In the case of $\nu < 0$, (5.9) can be established by same ideas; more terms and more times must be integrated by parts in (5.7). We do not go into details. Instead we demonstrate another idea, how the proof of the Theorem for $\nu < 0$ can be obtained from the case $0 \leq \nu < 1$.

(vi) *Extending the proof for negative ν.* Let $\nu \in [-1,0)$. Then $u \in C^{m,\nu}(0,1)$ is continuously differentiable on $[0,1]$ and Theorem 4.2 yields

$$\frac{d}{dx}\int_0^1 K(x,y)u(y)dy = \int_0^1 (\partial_x + \partial_y)[K(x,y)u(y)]dy + u(0)K(x,0) - u(1)K(x,1)$$

$$= \int_0^1 [(\partial_x + \partial_y)K(x,y)]u(y)]dy + \int_0^1 K(x,y)u'(y)dy + u(0)K(x,0) - u(1)K(x,1),$$

or

$$DTu = T^{(1)}u + TDu + R_1u,$$

$$w_{i+\nu-1}D^iTu = w_{i+\nu-1}D^{i-1}T^{(1)}u + w_{i+\nu-1}D^{i-1}TDu + w_{i+\nu-1}D^{i-1}R_1u, \ \ 1 \le i \le m,$$

where $T^{(1)}$ is the integral operator with the kernel $K^{(1)}(x,y) = (\partial_x+\partial_y)K(x,y)$ and $(R_1u)(x) = u(0)K(x,0) - u(1)K(x,1)$ is a finite dimensional (a two dimensional) operator. For $u \in C^{m,\nu}(0,1)$ it holds $Du \in C^{m-1,\nu+1}(0,1)$ with $\nu+1 \in [0,1)$,

$$\| Du \|_{C^{m-1,\nu+1}(0,1)} \le \| u \|_{C^{m,\nu}(0,1)} .$$

The operator $w_{i+\nu-1}D^{i-1}T = w_{(i-1)+(\nu+1)-1}D^{i-1}T : C^{m-1,\nu+1}(0,1) \to BC(0,1)$ is compact on the basis of (i)–(v), hence $w_{i+\nu-1}D^{i-1}TD : C^{m,\nu}(0,1) \to BC(0,1)$ is compact. The same is true for $w_{i+\nu-1}D^{i-1}T^{(1)} : C^{m,\nu}(0,1) \to BC(0,1)$ since $K^{(1)}(x,y)$ satisfies same inequalities as $K(x,y)$. Finally, the compactness of $w_{i+\nu-1}D^{i-1}R_1 : C^{m,\nu}(0,1) \to BC(0,1)$ is a consequence of the boundedness of this finite dimensional operator. As a summary, we obtain that $w_{i+\nu-1}D^iT : C^{m,\nu}(0,1) \to BC(0,1)$ is compact for $i = 1,...,m$. As explained in (i) this implies the claim of the Theorem.

Having stablished the compactness of $w_{i+\nu-1}D^iT : C^{m,\nu}(0,1) \to BC(0,1)$ for $\nu \in [-1,0)$, we in similar way extend the claim for $\nu \in [-2,-1)$ etc. □

5.4. **Smoothness and singularities of the solutions.** We are ready to present the basic result about the smoothness and singularities of the solutions to weakly singular integral equations.

Theorem 5.2. Let $K \in \mathcal{S}^{m,\nu}$, $f \in C^{m,\nu}(0,1)$, $m \ge 1$, $\nu < 1$, and let $u \in C[0,1]$ be a solution of equation (5.1). Then $u \in C^{m,\nu}(0,1)$.

Proof. By Lemma 3.2, the integral operator $T : C[0,1] \to C[0,1]$ is compact. By Theorem 5.1 T maps $C^{m,\nu}(0,1)$ into itself and $T : C^{m,\nu}(0,1) \to C^{m,\nu}(0,1)$ is compact. With $X = C[0,1]$ and $Y = C^{m,\nu}(0,1)$, Theorem 2.8 yields that $u \in C^{m,\nu}(0,1)$ for the solutions $u \in C[0,1]$ of (5.1). □

5.5. **A smoothing change of variables.** The derivatives of a solution $u \in C^{m,\nu}(0,1)$ to equation (5.1) may have boundary singularities. Now we undertake a change of variables that kills the singularities, i.e., the solution of the transformed equation will be C^m-smooth on $[0,1]$ including the boundary points.

Let $\varphi : [0,1] \to [0,1]$ be a smooth strictly increasing function such that $\varphi(0) = 0$, $\varphi(1) = 1$. Introducing the change of variables

$$x = \varphi(t), \ \ y = \varphi(s),$$

equation (5.1) takes the form

$$v(t) = \int_0^1 K_\varphi(t,s)v(s)ds + f_\varphi(t), \ \ 0 \le t \le 1, \tag{5.15}$$

where

$$f_\varphi(t) := f(\varphi(t)), \ \ K_\varphi(t,s) := K(\varphi(t), \varphi(s))\varphi'(s);$$

the solutions of equations (5.15) and (5.1) are in the relations

$$v(t) = u(\varphi(t)), \ \ u(x) = v(\varphi^{-1}(x)).$$

Under conditions we set on $\varphi : [0,1] \to [0,1]$, the inverse function $\varphi^{-1} : [0,1] \to [0,1]$ exists and is continuous.

Theorem 5.3. Given $m \ge 1$, $\nu < 1$, let $p \in \mathbb{N}$ satisfy

$$p > \left\{ \begin{array}{ll} m, & \nu \le 0 \\ \frac{m}{1-\nu}, & 0 < \nu < 1 \end{array} \right\}. \tag{5.16}$$

Let $\varphi \in C^p[0,1]$ satisfy the conditions $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(t) > 0$ for $0 < t < 1$ and

$$\varphi^{(j)}(0) = \varphi^{(j)}(1) = 0, \ \ j = 1, ..., p-1, \ \ \varphi^{(p)}(0) \ne 0, \ \ \varphi^{(p)}(1) \ne 0. \tag{5.17}$$

Then the following claims hold true.

(i) For $f \in C^{m,\nu}(0,1)$, the function $f_\varphi(t) = f(\varphi(t))$ belongs to $C^m[0,1]$ and

$$f_\varphi^{(j)}(0) = f_\varphi^{(j)}(1) = 0, \quad j = 1, ..., m. \tag{5.18}$$

(ii) For $K \in \mathcal{S}^{0,\nu}$, the kernel $K_\varphi(t,s) = K(\varphi(t), \varphi(s))\varphi'(s)$ belongs to $\mathcal{S}^{0,\nu}$ and hence defines a compact integral operator

$$T_\varphi : C[0,1] \to C[0,1], \ \ (T_\varphi v)(t) = \int_0^1 K_\varphi(t,s)v(s)ds.$$

Proof. (i) Clearly $f_\varphi \in C^m(0,1)$, thus claim (i) concerns the boundary behaviour of f_φ. Due to the imbedding (5.3), after the continuation by continuity, $f_\varphi \in C[0,1]$. So we have to show that

$$f_\varphi^{(j)}(0) := \lim_{t\to 0} f_\varphi^{(j)}(t) = 0, \ \ f_\varphi^{(j)}(1) := \lim_{t\to 1} f_\varphi^{(j)}(t) = 0, \quad j = 1, ..., m.$$

We establish these relations for $t \to 0$; for $t \to 1$ the argument is similar. By the formula of Faa di Bruno (see Theorem 2.9),

$$f_\varphi^{(j)}(t) = \sum_{k_1+2k_2...+jk_j=j} c_{k_1,...,k_j} f^{(k_1+...+k_j)}(\varphi(t))\, \varphi'(t)^{k_1}...\varphi^{(j)}(t)^{k_j}, \quad 0<t<1,$$

with some constants $c_{k_1,...,k_j}$. In a vicinity of 0, the inclusion $f \in C^{m,\nu}(0,1)$ yields

$$| f^{(k)}(\varphi(t)) | \leq c \left\{ \begin{array}{cc} 1, & k<1-\nu \\ 1+ | \log \varphi(t) |, & k=1-\nu \\ \varphi(t)^{1-\nu-k}, & k>1-\nu \end{array} \right\}.$$

Due to (5.17),

$$\varphi(t) \asymp t^p, \;\; \varphi^{(i)}(t) \asymp t^{p-i} \text{ as } t \to 0, \;\; i=0,...,p,$$

hence

$$| f_\varphi^{(j)}(t) | \leq c \sum_{k_1+2k_2...+jk_j=j} \left\{ \begin{array}{cc} 1, & k_1+...+k_j<1-\nu \\ 1+ | \log t |, & k_1+...+k_j=1-\nu \\ t^{p(1-\nu-k_1-...-k_j)}, & k_1+...+k_j>1-\nu \end{array} \right\} t^{(p-1)k_1} t^{(p-2)k_2} ... t^{(p-j)k_j}$$

$$= c \sum_{k_1+2k_2...+jk_j=j} \left\{ \begin{array}{cc} t^{p(k_1+...+k_j)-j}, & k_1+...+k_j<1-\nu \\ (1+ | \log t |) t^{p(k_1+...+k_j)-j}, & k_1+...+k_j=1-\nu \\ t^{p(1-\nu)-j}, & k_1+...+k_j>1-\nu \end{array} \right\}, \;\; 1 \leq j \leq m.$$

For $\nu > 0$, we have $k_1+...+k_j > 1-\nu$ and $| f_\varphi^{(j)}(t) | \leq ct^{p(1-\nu)-j}$ in accordance to lower line. For $\nu = 0$, there is one combination of $k_1,...,k_j$ such that $k_1+2k_2...+jk_j = j$ and $k_1+...+k_j = 1-\nu$, namely $k_1 = ... = k_{j-1} = 0$, $k_j = 1$, yielding $| f_\varphi^{(j)}(t) | \leq ct^{p-j}(1+ | \log t |)$. For $\nu < 0$, the smallest exponent $p(k_1+...+k_j)-j$ with restrictins $k_1+2k_2...+jk_j = j$ and $k_1+...+k_j < 1-\nu$ again corresponds to the combination $k_1 = ... = k_{j-1} = 0$, $k_j = 1$, yielding $| f_\varphi^{(j)}(t) | \leq t^{p-j}$ from the upper line which dominates over terms in in the lower and central lines. As a summary, in a neighborhood of 0, it holds

$$| f_\varphi^{(j)}(t) | \leq c \left\{ \begin{array}{cc} t^{p-j}, & \nu<0 \\ t^{p-j}(1+ | \log t |), & \nu=0 \\ t^{p(1-\nu)-j} & \nu>0 \end{array} \right\}, \;\; j=1,...,m.$$

Now condition (5.16) implies that $\lim_{t\to 0} f_\varphi^{(j)}(t) = 0$ for $j = 1,...,m$.

(ii) Claim (ii) is trivial for $\nu < 0$ since then $K_\varphi(t,s)$ is bounded together with $K(x,y)$. To prove claim (ii) for $0 \leq \nu < 1$, we first examine the properties of the function

$$\psi(t,s) := \left\{ \begin{array}{cc} \frac{\varphi(t)-\varphi(s)}{t-s}, & t \neq s \\ \varphi'(t), & t=s \end{array} \right\}, \;\; 0 \leq t,s \leq 1.$$

Due to the conditions set on φ, we have $\psi \in C^{p-1}([0,1] \times [0,1])$, $\psi(t,s) > 0$ for $(t,s) \in ([0,1] \times [0,1]) \setminus \{(0,0),\ (1,1)\}$; we show that there exists a positive constant c_0 such that

$$\psi(t,s) \geq c_0 \min\{(t+s)^{p-1},\ [(1-t)+(1-s)]^{p-1}\},\ \ 0 \leq t,s \leq 1. \tag{5.19}$$

It suffices to establish estimate (5.19) in a neighborhood of the points $(0,0)$; for a neighborhood of the point (1.1) the estimate follows by the symmetry; on the rest part of $[0,1]\times[0,1]$ function ψ is greater than a positive constant implying (5.19) also there, possibly with a smaller but still positive constant c_0. We choose a neighborhood $U_\delta \subset [0,1]\times[0,1]$ of $(0,0)$ of a sufficiently small radius $\delta > 0$ such that $\varphi^{(p)}(t) \neq 0$ for $0 \leq t \leq \delta$, see (5.17). Then $\varphi^{(p)}(t) > 0$ for $0 \leq t \leq \delta$, since $\varphi^{(p)}(t) < 0$ for $0 \leq t \leq \delta$ together with the conditions $\varphi'(0) = ... = \varphi^{(p-1)}(0) = 0$ should imply $\varphi'(t) < 0$ for $0 \leq t \leq \delta$ (use the Taylor formula!). Denote $d_0 := \min_{0\leq t\leq\delta} \varphi^{(p)}(t) > 0$. Let $0 < s < t \leq \delta$. Due to (5.17), the Taylor formula with the integral form of the rest term yields

$$\varphi(t) - \varphi(s) = \frac{1}{(p-1)!}\int_0^t (t-\tau)^{p-1}\varphi^{(p)}(\tau)d\tau - \frac{1}{(p-1)!}\int_0^s (s-\tau)^{p-1}\varphi^{(p)}(\tau)d\tau$$
$$= \frac{1}{(p-1)!}\int_0^s [(t-\tau)^{p-1} - (s-\tau)^{p-1}]\varphi^{(p)}(\tau)d\tau + \frac{1}{(p-1)!}\int_s^t (t-\tau)^{p-1}\varphi^{(p)}(\tau)d\tau.$$

The functions $(t-\tau)^{p-1} - (s-\tau)^{p-1}$ and $(t-\tau)^{p-1}$ under last two integrals are positive. Estimating $\varphi^{(p)}(\tau) \geq d_0 > 0$ we obtain

$$\varphi(t) - \varphi(s) \geq \frac{d_0}{(p-1)!}\left(\int_0^s [(t-\tau)^{p-1} - (s-\tau)^{p-1}]d\tau + \int_s^t (t-\tau)^{p-1}d\tau\right)$$
$$= \frac{d_0}{(p-1)!}\left(\int_0^t (t-\tau)^{p-1}d\tau - \int_0^s (s-\tau)^{p-1}d\tau\right) = \frac{d_0}{p!}(t^p - s^p),$$

and (5.19) follows for $0 < s < t \leq \delta$:

$$\frac{\varphi(t)-\varphi(s)}{t-s} \geq \frac{d_0}{p!}\frac{t^p - s^p}{t-s} = \frac{d_0}{p!}\sum_{j=0}^{p-1} t^j s^{p-1-j} \geq c_0 \sum_{j=0}^{p-1}\binom{p-1}{j} t^j s^{p-1-j} = c_0(t+s)^{p-1}.$$

The case $0 < t < s \leq \delta$ is symmetrical to the treated case $0 < s < t \leq \delta$. For $0 < s = t \leq \delta$, (5.19) follows by a limit argument. This completes the proof of (5.19).

Let us return to claim (ii) of the Theorem for $0 \leq \nu < 1$. Consider the case $0 < \nu < 1$. Due to (3.4) and (5.19),

$$\mid K_\varphi(t,s) \mid \leq c_K \mid \varphi(t) - \varphi(s) \mid^{-\nu} \varphi'(s) = c_K \left(\frac{\varphi(t)-\varphi(s)}{t-s}\right)^{-\nu} \mid t-s \mid^{-\nu} \varphi'(s)$$
$$\leq c_K c_0^{-\nu} \mid t-s \mid^{-\nu} \frac{\varphi'(s)}{[\min\{(t+s)^{p-1},\ ((1-t)+(1-s))^{p-1}\}]^\nu} \leq c \mid t-s \mid^{-\nu};$$

on the last step we took into account that $\varphi'(s) \asymp s^{p-1}$ as $s \to 0$, $\varphi'(s) \asymp (1-s)^{p-1}$ as $s \to 1$. Thus $K_\varphi \in \mathcal{S}^{0,\nu}$. In the case $\nu = 0$,

$$\begin{aligned} \mid K_\varphi(t,s) \mid &\leq c_K(1+ \mid \log \mid \varphi(t) - \varphi(s) \mid\mid \varphi'(s) \\ &= c_K(1+ \mid \log \frac{\varphi(t)-\varphi(s)}{t-s} \mid + \mid \log \mid t-s \mid\mid)\varphi'(s) \\ &\leq c(1+ \mid \log \min\{(t+s,\ [(1-t)+(1-s)]\} \mid + \mid \log \mid t-s \mid\mid)\varphi'(s) \\ &\leq c_1 + c_2(1+ \mid \log \mid t-s \mid\mid), \text{ i.e., } K_\varphi \in \mathcal{S}^{0,0}. \end{aligned}$$

Having established that $K_\varphi \in \mathcal{S}^{0,\nu}$ for $\nu < 1$, the compactness of the operator $T_\varphi : C[0,1] \to C[0,1]$ follows by Lemma 3.2. □

Corollary 5.4. Assume the conditions of Theorems 5.2 and 5.3. Then $v \in C^m[0,1]$,

$$v^{(j)}(0) = v^{(j)}(1) = 0, \quad j = 1, ..., m, \tag{5.20}$$

for the solutions of equation (5.15).

Remark 5.5. One can conjecture that, under condition of Theorem 5.3, $K \in \mathcal{S}^{m,\nu}$ implies $K_\varphi \in \mathcal{S}^{m,\nu}$. The argument becomes rather technical to justify this. For us the relation $K_\varphi \in \mathcal{S}^{0,\nu}$ established in Theorem 5.3 is sufficient in the sequel.

Example 5.6. Let us present an example of function φ that satisfies the conditions of Theorem 5.3:

$$\varphi(t) = c_p \int_0^t \tau^{p-1}(1-\tau)^{p-1} d\tau, \ \ c_p = \frac{1}{\int_0^1 \tau^{p-1}(1-\tau)^{p-1} d\tau}, \ \ p \in \mathbb{N}. \tag{5.21}$$

Clearly, $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(t) = c_p t^{p-1}(1-t)^{p-1} > 0$ for $0 < t < 1$, $\varphi^{(j)}(0) = \varphi^{(j)}(1) = 0$ for $j = 1, ..., p-1$, $\varphi^{(p)}(0) = c_p$, $\varphi^{(p)}(1) = (-1)^{p-1} c_p$. In this example, φ is a polynomial of degree $2p - 1$.

6. Specification for Volterra integral equations

The Volterra integral equation

$$u(x) = \int_0^x K(x,y)u(y)dy + f(x), \quad 0 \leq x \leq 1, \tag{6.1}$$

can be considered as a special case of the Fredholm integral equation (5.1) in which $K(x,y) = 0$ for $0 \leq x < y \leq 1$. The class $\mathcal{S}^{m,\nu}(([0,1] \times [0,1]) \setminus \text{diag})$ is well defined for such kernels, hence the results of Section 5 hold for equation (6.1). Nevertheless, it is worth to revisit the results of Section 5 since normally the derivatives of a solution $u(x)$ to (6.1) may have singularities only at $x = 0$. We "project" the formulations of the main concepts and results of Section 5 to the needs of Volterra equation (6.1). The proofs are omitted since they contain no new ideas, conversely, they are some simplifications of the arguments in Section 5.

Denote

$$\triangle = \{(x,y) : 0 \leq y < x \leq 1\}$$

and introduce the class $\mathcal{S}^{m,\nu}(\triangle)$ of kernels $K(x,y)$ that are defined and m times continuously differentiable on $\triangle$ and satisfy for $(x,y)\in\triangle$ and for all $k,l\in\mathbb{N}_0$, $k+l\leq m$, the inequality

$$\left|\left(\frac{\partial}{\partial x}\right)^k\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^l K(x,y)\right|\leq c_{K,m}\left\{\begin{array}{cc} 1, & \nu+k<0 \\ 1+\mid\log(x-y)\mid, & \nu+k=0 \\ (x-y)^{-\nu-k}, & \nu+k>0 \end{array}\right\}.$$

After an extension of $K\in\mathcal{S}^{m,\nu}(\triangle)$ by the zero value outside $\triangle$ we obtain a kernel $K\in\mathcal{S}^{m,\nu}(([0,1]\times[0,1])\setminus\mathrm{diag})$.

For $m\geq 1$, $\nu<1$, denote by $C^{m,\nu}(0,1]$ the space of functions $f\in C^m(0,1]$ that satisfy the inequalities

$$\mid f^{(j)}(x)\mid\leq c_f\left\{\begin{array}{cc} 1 & j+\nu-1<0 \\ 1+\mid\log x\mid, & j+\nu-1=0 \\ x^{-j-\nu+1}, & j+\nu-1>0 \end{array}\right\},\ \ 0<x\leq 1,\ \ j=0,...,m.$$

Introduce the weight functions

$$w^0_\lambda(x)=\left\{\begin{array}{cc} 1, & \lambda<0 \\ 1/(1+\mid\log x\mid), & \lambda=0 \\ x^\lambda & \lambda>0 \end{array}\right\},\ \ 0<x\leq 1,\ \ \lambda\in\mathbb{R}.$$

The norm in $C^{m,\nu}(0,1]$ is given by

$$\parallel f\parallel_{C^{m,\nu}(0,1]}=\sum_{j=0}^{m}\sup_{0<x\leq 1} w^0_{j+\nu-1}(x)\mid f^{(j)}(x)\mid .$$

There holds the compact imbeding

$$C^{m,\nu}(0,1]\subset C[0,1],\quad m\geq 1,\ \nu<1.$$

Theorem 6.1. Let $K\in\mathcal{S}^{m,\nu}(\Delta)$, $m\geq 1$, $\nu<1$. Then the Volterra integral operator T defined by $(Tu)(x)=\int_0^x K(x,y)u(y)dy$ maps $C^{m,\nu}(0,1]$ into itself and $T: C^{m,\nu}(0,1]\to C^{m,\nu}(0,1]$ is compact.

Theorem 6.2. Let $K\in\mathcal{S}^{m,\nu}(\Delta)$, $f\in C^{m,\nu}(0,1]$, $m\geq 1$, $\nu<1$, and let $u\in C[0,1]$ be the (unique) solution of equation (6.1). Then $u\in C^{m,\nu}(0,1]$.

See Exercise 14 about the existence and uniqueness of the solution.

An alternative proof of Theorem 6.2. Again, Theorem 6.2 is an elementary consequence ot Theorem 6.1 (similarly as Theorem 5.2 was an elementary consequence of Theorem 5.1). Alternatively, Theorem 6.2 can be derived from Theorem 5.2 by a prolongation argument, and we demonstrate how this can be done. First of all, we extend $K\in\mathcal{S}^{m,\nu}(\triangle)$ by the zero values from $\triangle$ to $([0,1]\times[0,1])\setminus$diag obtaining $K\in\mathcal{S}^{m,\nu}(([0,1]\times[0,1])\setminus\mathrm{diag})$. The Volterra equation (6.1) is equivalent to the Fredholm equation (5.1) with the extended kernel. By Theorem 5.2, we know about the solution of (5.1) that $u\in C^{m,\nu}(0,1)$. It remains to show that actually no singularities of the derivatives of $u(x)$, the solution of (6.1) and (5.1), occur at $x = 1$. To show this, we extend f from $[0, 1]$ to $[0, 1+\delta]$, $0 < \delta < 1/m$, using the reflection formula

$$f(x) = \sum_{j=0}^{m} d_j f(1 - j(x-1)), \;\; 1 < x \le 1+\delta, \tag{6.2}$$

where d_j are chosen so that the C^m-smooth joining takes place at $x = 1$. Namely, differentiating (6.2) k times we have

$$f^{(k)}(x) = \sum_{j=0}^{m} (-j)^k d_j f^{(k)}(1 - j(x-1)), \;\; 1 < x \le 1+\delta,$$

and the C^m-smooth joining at $x = 1$ happens if

$$\sum_{j=0}^{m} (-j)^k d_j = 1, \;\; k = 0, ..., m.$$

We obtained a uniquely solvable $(m+1)\times(m+1)$ Vandermonde system to determine $d_0, ..., d_m$. Using th reflection formula we extend also the kernel $K(x, y)$ along the lines $y = \gamma x$, $0 < \gamma < 1$, from the triangle $\triangle = \{(x, y) : 0 \le y < x \le 1\}$ onto the triangle $\triangle_\delta = \{(x, y) : 0 \le y < x \le 1+\delta\}$ with a $\delta > 0$. The extension procedure preserves f in $C^{m,\nu}(0, 1+\delta]$ and K in $\mathcal{S}^{m,\nu}(\triangle_\delta)$. Introduce the extended equation

$$u(x) = \int_0^x K(x, y)u(y)dy + f(x), \quad 0 \le x \le 1+\delta;$$

for $0 \le x \le 1$ this equation coincides with (6.1). By Theorem 5.2 applied to the extended equation, u is C^m-smooth for $0 < x < 1+\delta$, hence no singularities of the derivatives of u at $x = 1$ are possible. $\square$

Let $\varphi : [0, 1] \to [0, 1]$ be a smooth strictly increasing function such that $\varphi(0) = 0$, $\varphi(1) = 1$. Introducing the change of variables

$$x = \varphi(t), \;\; y = \varphi(s),$$

we rewrite the equation (6.1) in the form

$$v(t) = \int_0^t K_\varphi(t, s)v(s)ds + f_\varphi(t), \;\; 0 \le t \le 1, \tag{6.3}$$

where

$$f_\varphi(t) := f(\varphi(t)), \;\; K_\varphi(t, s) := K(\varphi(t), \varphi(s))\varphi'(s);$$

the solutions of equations (6.1) and (6.3) are in the relations

$$v(t) = u(\varphi(t)), \quad u(x) = v(\varphi^{-1}(x)).$$

An obtrusive mistake is to write “formally” $\int_0^{\varphi(t)} K(\varphi(t), \varphi(s))u(\varphi(s))\varphi'(s)ds$ as the result of the change of variables in the integral $\int_0^x K(x, y)u(y)dy$. We must be more careful! Actually the change of variables $y = \varphi(s)$ yields

$$\int_0^x K(x,y)u(y)dy = \int_0^{\varphi^{-1}(x)} K(x,\varphi(s))u(\varphi(s))\varphi'(s)ds,$$

and after that the change of variables $x = \varphi(t)$ completes the result as

$$\int_0^x K(x,y)u(y)dy \mid_{x=\varphi(t)} = \int_0^t K(\varphi(t),\varphi(s))u(\varphi(s))\varphi'(s)ds.$$

So we really obtain the transformed equation in the Volterra form (6.3).

Theorem 6.3. Given $m \geq 1$, $\nu < 1$, let

$$p > \left\{ \begin{array}{cc} m, & \nu \leq 0 \\ \frac{m}{1-\nu}, & 0 < \nu < 1 \end{array} \right\}.$$

Let $\varphi \in C^p[0,1]$ satisfy $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(t) > 0$ for $0 < t < 1$ and

$$\varphi^{(j)}(0) = 0, \;\; j = 1, ..., p-1, \;\; \varphi^{(p)}(0) \neq 0.$$

Then the following claims hold true.

(i) For $f \in C^{m,\nu}(0,1]$, the function $f_\varphi(t) := f(\varphi(t))$ belongs to $C^m[0,1]$ and

$$f_\varphi^{(j)}(0) = 0, \quad j = 1, ..., m.$$

(ii) For $K \in \mathcal{S}^{0,\nu}(\Delta)$, the kernel $K_\varphi(t,s) := K(\varphi(t),\varphi(s))\varphi'(s)$ belongs to $\mathcal{S}^{0,\nu}(\Delta)$ and defines a compact Volterra integral operator

$$T_\varphi : C[0,1] \to C[0,1], \;\; (T_\varphi v)(t) = \int_0^t K_\varphi(t,s)v(s)ds.$$

An example of function φ satisfying the conditions of Theorem 6.3 is given by $\varphi(t) = t^p$.

Corollary 6.4. Assume the conditions of Theorems 6.2 and 6.3. Then $v \in C^m[0,1]$,

$$v^{(j)}(0) = 0, \quad j = 1, ..., m,$$

for the solution of equation (6.3).

7. A collocation method for weakly singular integral equations

7.1. **Interpolation by polynomials on a uniform grid.** Denote by $\mathcal{P}_m$ the set of polynomials of degree not exceeding m.

Given an interval $[a,b]$ and $m \in \mathbb{N}$, introduce the uniform grid consisting of points

$$x_i = a + (i - \frac{1}{2})h, \quad i = 1, ..., m, \quad h = \frac{b-a}{m}. \tag{7.1}$$

Denote by Π_{m-1} the Lagrange interpolation projection operator assigning to any $u \in C[a,b]$ the polynomial $\Pi_{m-1}u \in \mathcal{P}_{m-1}$ that interpolates f at points (7.1).

Lemma 7.1. *For* $f \in C^m[a,b]$,

$$\max_{a\le x\le b} \mid f(x) - (\Pi_{m-1}f)(x) \mid \le \theta_m h^m \max_{a\le x\le b} \mid f^{(m)}(x) \mid, \tag{7.2}$$

$$\theta_m = \frac{1\cdot 3\cdot \ldots \cdot (2m-1)}{2\cdot 4\cdot \ldots \cdot 2m} \sim (\pi m)^{-\frac{1}{2}} \text{ as } m\to\infty.$$

Further, for $m = 2k$, $k \ge 1$,

$$\max_{x_k\le x\le x_{k+1}} \mid f(x) - (\Pi_{m-1}f)(x) \mid \le \vartheta_m h^m \max_{a\le x\le b} \mid f^{(m)}(x) \mid, \tag{7.3}$$

$$\vartheta_m = 2^{-2m}\frac{m!}{((m/2)!)^2} \sim \sqrt{2/\pi}\, m^{-\frac{1}{2}} 2^{-m},$$

whereas for $m = 2k+1$, $k \ge 1$,

$$\max_{x_k\le x\le x_{k+2}} \mid f(x) - (\Pi_{m-1}f)(x) \mid \le \vartheta_m h^m \max_{a\le x\le b} \mid f^{(m)}(x) \mid, \tag{7.4}$$

$$\vartheta_m = \frac{2\sqrt{3}}{9}\,\frac{(k!)^2}{(2k+1)!} \sim \frac{2\sqrt{3}}{9}\sqrt{2\pi}\, m^{-\frac{1}{2}} 2^{-m}.$$

These estimates are elementary consequences of the error formula

$$f(x) - (\Pi_{m-1}f)(x) = \frac{f^{(m)}(\xi)}{m!}(x-x_1)\ldots(x-x_m),\ \ x\in[a,b],\ \xi=\xi(x)\in(a,b).$$

Namely, for points (7.1), the maximum of $\mid (x-x_1)\ldots(x-x_m) \mid$ on $[a,b]$ is attained at the end points of the interval, whereas the maximum of $\mid (x-x_1)\ldots(x-x_{2k}) \mid$ on $[x_k, x_{k+1}]$ is attained at the centre of $[x_k, x_{k+1}]$ (which is also the centre of $[a,b]$). To establish (7.4), we take into account that the maximum of $\mid (x-x_k)(x-x_{k+1})(x-x_{k+2}) \mid$ on $[x_k, x_{k+2}]$ equals $\frac{2\sqrt{3}}{9}h^3$ and elementarily estimate the remaining product on $[x_{k-1}, x_{k+1}]$.

Comparing estimates (7.2)–(7.4) we observe that in the central parts of $[a,b]$, the estimates are approximately 2^m times preciser than on the whole interval. Surprisingly estimates (7.3) and (7.4) are for $m \ge 4$ preciser even than the error estimate of the Chebyshev interpolant on the same order on $[a,b]$, see estimate (7.6) below. In the central parts of $[a,b]$, the interpolation process on the uniform grid has also good stability properties: in contrast to an exponential growth of $\| \Pi_{m-1} \|_{C[a,b]\to C[a,b]}$ as $m\to\infty$, it holds

$$\| \Pi_{m-1} \|_{C[a,b]\to C[\frac{a+b}{2}-rh^{1/2},\, \frac{a+b}{2}+rh^{1/2}]} \le c_r(1+\log m) \tag{7.5}$$

with a constant c_r which depends only on $r>0$. This is the Runck's theorem (see [2], p. 170).

7.2. **Chebyshev interpolation.** Denote by Π'_{m-1} the Chebyshev interpolation projection operator assigning to any $f\in C[a,b]$ the polynomial $\Pi'_{m-1}f \in \mathcal{P}_{m-1}$ that interpolates f at the Chebyshev knots

$$x_i' = \frac{b-a}{2}\left(-\cos\frac{2i-1}{2m}\pi\right) + \frac{a+b}{2} \in (a,b), \quad i = 1,...,m.$$

Lemma 7.2 ([2], p.163). *For $f \in C^m[a,b]$, $m \geq 1$, it holds*

$$\max_{a\leq x\leq b} \mid f(x) - (\Pi_{m-1}' f)(x) \mid \leq \frac{(b-a)^m}{m!\, 2^{2m-1}} \max_{a\leq x\leq b} \mid f^{(m)}(x) \mid . \tag{7.6}$$

The Chebyshev interpolant $\Pi_{m-1}' f$ occurs to be the best approximation to the function $f(x) = x^m$ with respect to the uniform norm on $[a,b]$, and (7.6) turns into equality for this function. Computations with Chebyshev interpolant are numerically (relatively) stable, since (see [2], p. 164)

$$\| \Pi_{m-1}' \|_{C[a,b]\to C[a,b]} \leq 8 + \frac{4}{\pi}\log m, \quad m \in \mathbb{N}.$$

It is known that for any projection operator $P_{m-1} : C[a,b] \to \mathcal{P}_{m-1}$, i.e., for any operator $P_{m-1} : C[a,b] \to C[a,b]$ with $P_{m-1}^2 = P_{m-1}$ and the range $\mathcal{R}(P_{m-1}) = \mathcal{P}_{m-1}$, it holds

$$\| P_{m-1} \|_{C[a,b]\to C[a,b]} \geq c_0(1 + \log m), \quad m \in \mathbb{N},$$

where $c_0 > 0$. Thus $\| \Pi_{m-1}' \|_{C[a,b]\to C[a,b]}$ is of minimal possible growth order as $m \to \infty$.

7.3. **Piecewise polynomial interpolation.** Introduce in $\mathbb{R}$ the uniform grid $\mathbb{R}_h := \{jh : j \in \mathbb{Z}\}$, $h = 1/n$, $n \in \mathbb{N}$. Fix an $m \geq 2$. Given a function $f \in C[-\delta, 1+\delta]$, $\delta \geq \frac{mh}{2}$, we introduce a piecewise polynomial interpolant $\Pi_{n,m-1} f \in C[0,1]$ of degree $\leq m-1$ as follows. On every subinterval $[jh, (j+1)h]$, $0 \leq j \leq n-1$, the function $\Pi_{n,m-1} f$ is defined independently from other subintervals as a polynomial of degree $\leq m-1$ that interpolates f at m points lh neighboring jh from two sides:

$$(\Pi_{n,m-1} f)(lh) = f(lh), \quad l = j - \frac{m}{2} + 1, ..., j + \frac{m}{2} \text{ if } m \text{ is even,} \tag{7.7}$$

$$(\Pi_{n,m-1} f)(lh) = f(lh), \quad l = j - \frac{m-1}{2}, ..., j + \frac{m-1}{2} \text{ if } m \text{ is odd.} \tag{7.8}$$

To unify the writing form of conditions (7.7) and (7.8), introduce the designation

$$\mathbb{Z}_m = \{k \in \mathbb{Z} : -\frac{m}{2} < k \leq \frac{m}{2}\}.$$

Observe that $\mathbb{Z}_m$ contains m elements (integers),

$$\mathbb{Z}_m = \{-\frac{m}{2} + 1, -\frac{m}{2} + 2, ... \frac{m}{2}\} \text{ if } m \text{ is even,}$$

$$\mathbb{Z}_m = \{-\frac{m-1}{2}, -\frac{m-1}{2} + 1, ..., \frac{m-1}{2}\} \text{ if } m \text{ is odd.}$$

Conditions (7.7) and (7.8) determining $\Pi_{n,m-1} f$ on $[jh, (j+1)h]$ can be written in a unified form as

$$(\Pi_{n,m-1} f)(lh) = f(lh) \text{ for } l \text{ such that } l - j \in \mathbb{Z}_m. \tag{7.9}$$

For $1 \le j \le n-2$, conditions (7.9) contain the particular condition $(\Pi_{n,m-1}f)(jh) = f(jh)$ for $\Pi_{n,m-1}f$ as a function on $[jh,(j+1)h]$ as well as a function on $[(j-1)h, jh]$, hence $\Pi_{n,m-1}f$ is uniquely defined at these interior knots, $(\Pi_{n,m-1}f)(jh) = f(jh),\ \ j = 1, ..., n-2$, and this guarantees the continuity of $\Pi_{n,m-1}f$ on $[0,1]$.

To write a representation formula for $\Pi_{n,m-1}f$, introduce the Lagrange fundamental polynomials $L_k \in \mathcal{P}_{m-1}$, $k \in \mathbb{Z}_m$, satisfying $L_k(l) = \delta_{k,l}$, $k,l \in \mathbb{Z}_m$, where $\delta_{k,l}$ is the Kronecker symbol, $\delta_{k,l} = 0$ for $k \neq l$ and $\delta_{k,k} = 1$. An explicit formula for L_k is given by

$$L_k(t) = \prod_{l \in \mathbb{Z}_m \setminus \{k\}} \frac{t-l}{k-l}, \ \ k \in \mathbb{Z}_m. \tag{7.10}$$

Now it is easy to see that

$$(\Pi_{n,m-1}f)(t) = \sum_{k \in \mathbb{Z}_m} f((j+k)h) L_k(nt - j) \ \text{ for } t \in [jh,(j+1)h],\ j = 0,..,n-1. \tag{7.11}$$

Indeed, the restriction of $\Pi_{n,m-1}f$ onto $[jh,(j+1)h]$ given by (7.11) is really a polynomial of degree $\le m-1$ that satisfies interpolation conditions (7.9): for l with $l - j \in \mathbb{Z}_m$ we have

$$(\Pi_{n,m-1}f)(lh) = \sum_{k \in \mathbb{Z}_m} f((j+k)h) L_k(l-j) = \sum_{k \in \mathbb{Z}_m} f((j+k)h)\delta_{k,l-j} = f(lh).$$

The interpolant $\Pi_{n,m-1}f$ could be defined on $[0,1]$ also for $m = 1$ as a piecewise constant function with possible jumps at jh, $j = 1, ..., n-2$. We lose the continuity of the interpolant at the interior knots jh, $j = 1, ..., n-2$. But the real reason why we exclude the case $m = 1$ from our consideration is that the interpolation points jh are not properly located. A natural location of an interpolation point is the centre of the interval $[jh,(j+1)h)$ on which the interpolant is constant. The case $m = 1$ with interpolation points $(j+\frac{1}{2})h$ can be examined independently in an elementary way.

For $m = 2$, the interpolant $\Pi_{n,m-1}f$ is the usual piecewise linear function joining the points $(jh, f(jh)) \in \mathbb{R}^2$, $j = 0, ..., n$, by straight lines. For $m = 2$, $\Pi_{n,m-1}f$ does not need values of f outside $[0,1]$, and $\Pi_{n,m-1}$is a projection operator in $C[0,1]$, i.e. $\Pi^2_{n,m-1} = \Pi_{n,m-1}$.

For $m \ge 3$, $\Pi_{n,m-1}f$ uses values of f outside of $[0,1]$. For $f \in C[0,1]$, $\Pi_{n,m-1}f$ obtains a sense after an extension of f onto $[-\delta, 1+\delta]$. For instance, the reflection formulae of the type (6.2) can be exploited to extend f so that $f(kh)$ for $k < 0$ and $k > n$ is a linear combination of $f(jh)$, $j = 0, ..., n$, and the extended function maintains the smoothness of f. We are in a lucky situation if $f \in C^m[0,1]$ satisfies the boundary conditions $f^{(j)}(0) = f^{(j)}(1) = 0$, $j = 1, ..., m$, then the simplest extension operator

$$E_\delta : C[0,1] \to C[-\delta, 1+\delta], \ \ (E_\delta f)(t) = \left\{ \begin{array}{ll} f(0), & -\delta \le t \le 0 \\ f(t), & 0 \le t \le 1 \\ f(1), & 1 \le t \le 1+\delta \end{array} \right\}. \tag{7.12}$$

maintains the smoothness of f. The operator

$$P_{n,m-1} := \Pi_{n,m-1} E_\delta : \, C[0,1] \to C[0,1] \tag{7.13}$$

is well defined and $P^2_{n,m-1} = P_{n,m-1}$, i.e., $P_{n,m-1}$ is a projection operator in $C[0,1]$. The range $\mathcal{R}(P_{n,m-1})$ of $P_{n,m-1}$ is a subspace of dimension $n+1$ in $C[0,1]$. A function $f_n \in \mathcal{R}(P_{n,m-1})$ is uniquely determined by its $n+1$ values at the knots jh, $j = 0, ..., n$:

$$f_n(t) = \sum_{k \in \mathbb{Z}_m} (E_\delta f_n)((j+k)h) L_k(nt - j) \ \text{ for } t \in [jh, (j+1)h], \ j = 0, .., n-1; \tag{7.14}$$

here $(E_\delta f_n)(ih) = f_n(ih)$ for $i = 0, ..., n$, $(E_\delta f_n)(ih) = f_n(0)$ for $i < 0$ and $(E_\delta f_n)(ih) = f_n(1)$ for $i > n$. Clearly, $f_n = 0$ if and only if $f_n(ih) = 0$, $i = 0, ..., n$.

The definition of $\Pi_{n,m-1} f$ is closely related to the "central" part interpolation of f on the uniform grid treated in Section 7.1. On $[jh, (j+1)h]$, $\Pi_{n,m-1} f$ coincides with the polynomial interpolant $\Pi_{m-1} f$ for f on the interval $[a_j, b_j]$ with $a_j = (j - \frac{m-1}{2})h$, $b_j = (j + \frac{m+1}{2})h$ in the case of even m and $a_j = (j - \frac{m}{2})h$, $b_j = (i + \frac{m}{2})h$ in the case of odd m, moreover, $[jh, (j+1)h]$ is a "central" part of $[a_j, b_j]$; the interpolation error on $[jh, (j+1)h]$ can be estimated by (7.3) and (7.4).

Lemma 7.3. (i) For $f \in C^m[-\delta, 1+\delta]$,

$$\max_{0 \le t \le 1} \mid f(t) - (\Pi_{n,m-1} f)(t) \mid \le \vartheta_m h^m \max_{-\delta \le t \le 1+\delta} \mid f^{(m)}(t) \mid$$

with ϑ_m defined in (7.3) and (7.4) respectively for even and odd m.

(ii) For $f \in V^{(m)} = \{v \in C^m[0,1] : \, v^{(j)}(0) = v^{(j)}(1) = 0, \ j = 1, ..., m\}$, it holds

$$\max_{0 \le t \le 1} \mid f(t) - (P_{n,m-1} f)(t) \mid \le \vartheta_m h^m \max_{0 \le t \le 1} \mid f^{(m)}(t) \mid . \tag{7.15}$$

Proof. The claim (i) is a direct consequence of Lemma 7.1. Further, for $f \in V^{(m)}$, we have $E_\delta f \in C^m[-\delta, 1+\delta]$,

$$\max_{-\delta \le t \le 1+\delta} \mid (E_\delta f)^{(m)}(t) = \max_{0 \le t \le 1} \mid f^{(m)}(t) \mid, \quad (E_\delta f)(t) = f(t) \text{ for } 0 \le t \le 1,$$

and (7.15) follows from (i) applied to $E_\delta f$. $\square$

From (7.5) we obtain

$$\| P_{n,m-1} \|_{C[0,1 \to C[0,1]} \le \| \Pi_{n,m-1} \|_{C[-\delta, 1+\delta] \to C[0,1]} \le c(1 + \log m).$$

Thus the norms of projection operators are uniformly bounded with respect to n. Together with (7.15), noticing that $V^{(m)}$ is dense in $C[0,1\}$, the Banach–Steinhaus theorem (Theorem 2.2) yields the following result.

Corollary 7.4. For any $f \in C[0,1]$, $\max_{0 \le t \le 1} \mid f(t) - (P_{n,m-1} f)(t) \mid \to 0$ as $n \to \infty$.

7.4. **A piecewise polynomial collocation method: error estimate.** Having a weakly singular integral equation (5.1), $u = Tu + f$, with $K \in \mathcal{S}^{m,\nu}$, $f \in C^{m,\nu}(0,1)$, $m \ge 2$, $0 < \nu < 1$, we rewrite it with the help of a smoothing change of variables in the form (5.15), $v = T_\varphi v + f_\varphi$, and after that approximate (5.15) by the $n+1$ dimensional equation

$$v_n = P_{n,m-1}T_\varphi v_n + P_{n,m-1}f_\varphi. \tag{7.16}$$

This is the operator form of the *piecewise polynomial collocation method* corresponding to the interpolation projection operator $P_{n,m-1}$ which is defined by (7.10)–(7.13). (A *collocation method* is always related with a "projection" of a given equation with the help of an *interpolation* projection operator. In this sense, collocation methods can be treated as a subclass of Galerkin methods. Galerkin methods correspond to general class of projection operators, not necessarily interpolation ones.)

Theorem 7.5. Let $K \in \mathcal{S}^{m,\nu}$, $f \in C^{m,\nu}(0,1)$, $m \geq 2$, $\nu < 1$, and let $\varphi : [0,1] \to [0,1]$ satisfy the conditions of Theorem 5.3. Further, assume that $\mathcal{N}(I-T) = \{\mathbf{0}\}$ (or equivalently, $\mathcal{N}(I-T_\varphi) = \{\mathbf{0}\}$). Then there exists an n_0 such that for $n \geq n_0$, the collocation equation (7.16) has a unique solution v_n. The accuracy of v_n can be estimated by

$$\| v - v_n \|_\infty \leq ch^m \| v^{(m)} \|_\infty \tag{7.17}$$

where $v(t) = u_\varphi(t) = u(\varphi(t))$ is the solution of (5.15), $u(x)$ is the solution of (5.1); by Corollary 5.4, $v \in C^m[0,1]$. The constant c in (7.17) is independent of n and f (it depends on K, m and φ).

Proof. The following proof argument is typical for Galerkin (including collocation) methods.

By Theorems 5.3, $T_\varphi : C[0,1] \to C[0,1]$ is compact. By assumption, $\mathcal{N}(I-T_\varphi) = \{\mathbf{0}\}$, and the bounded inverse $(I-T_\varphi)^{-1} : C[0,1] \to C[0,1]$ exists due the Fredholm alternative (Theorem 2.7); denote

$$\kappa := \| (I-T_\varphi)^{-1} \|_{C[0,1]\to C[0,1]} .$$

Further, the compactness of $T_\varphi : C[0,1] \to C[0,1]$ and the pointwise convergence $P_{n,m-1}$ to I in $C[0,1]$ (see Corollary 7.4) imply by Theorem 2.6 the norm convergence

$$\varepsilon_n := \| P_{n,m-1}T_\varphi - T_\varphi \|_{C[0,1]\to C[0,1]} \to 0 \quad \text{as } n \to \infty.$$

Hence there is an n_0 such that $\kappa\varepsilon_n < 1$ for $n \geq n_0$. With the help of Theorem 2.4 we conclude that $I - P_{n,m-1}T_\varphi$ is invertible in $C[0,1]$ for $n \geq n_0$ and

$$\kappa_n := \| (I - P_{n,m-1}T_\varphi)^{-1} \|_{C[0,1]\to C[0,1]} \leq \frac{\kappa}{1-\kappa\varepsilon_n} \to \kappa \quad \text{as } n \to \infty. \tag{7.18}$$

This proves the unique solvability of the collocation equation (7.16) for $n \geq n_0$.

Let v and v_n be the solutions of (5.15) and (7.16), respectively. Then

$$(I - P_{n,m-1}T_\varphi)(v - v_n) = v - P_{n,m-1}T_\varphi v - P_{n,m-1}f_\varphi = v - P_{n,m-1}v,$$

$$v - v_n = (I - P_{n,m-1}T_\varphi)^{-1}(v - P_{n,m-1}v)$$

and

$$\| v - v_n \|_\infty \leq \kappa_n \| v - P_{n,m-1}v \|_\infty, \quad n \geq n_0. \tag{7.19}$$

By Theorem 5.2, for the solution u of (5.1) we have $u \in C^{m,\nu}(0,1)$; by Corollary 5.4, for $v(t) = u_\varphi(t) = u(\varphi(t))$ we have $v \in C^m[0,1]$ and $v^{(j)}(0) = v^{(j)}(1) = 0$, $j = 1, ..., m$; by Lemma 7.3(ii),

$$\| v - P_{n,m-1}v \|_\infty \le \vartheta_m h^m \| v^{(m)} \|_\infty .$$

Now (7.19) yields

$$\| v - v_n \|_\infty \le \kappa_n \vartheta_m h^m \| v^{(m)} \|_\infty$$

that together with (7.18) implies (7.17). □

Proving the convergence of the method, without the convergence speed, the assumptions of Theorem 7.6 can be relaxed, see Exercise 16.

7.5. **The matrix form of the collocation method.** The solution v_n of equation (7.16) belongs to $\mathcal{R}(P_{n,m-1})$, so the knot values $v_n(ih)$, $i = 0, ..., n$, determine v_n uniquely. Equation (7.16) is equivalent to a system of linear algebraic equation with respect to $v_n(ih)$, $i = 0, ..., n$, and our task is to find this system out.

First of all, for $w_n \in \mathcal{R}(P_{n,m-1})$, we have $w_n = 0$ if and only if $w_n(ih) = 0$, $i = 0, ..., n$. Further, $(P_{n,m-1}w)(ih) = w(ih)$, $i = 0, ..., n$. Hence equation (7.15) is equivalent to the (so-called collocation) conditions

$$v_n(ih) = (T_\varphi v_n)(ih) + f(ih), \quad i = 0, ..., n,$$

i.e. $v_n \in \mathcal{R}(P_{n,m-1})$ satisfies equation (5.15) at the knots ih, $i = 0, ..., n$. (Actually collocation methods are usually a priori described by conditions of such type and after that an operator form of the method is derived; we follow an equivalent inverse way.) Using the representation (7.14) for v_n we obtain

$$(T_\varphi v_n)(ih) = \int_0^1 K_\varphi(ih,s)v_n(s)ds = \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} K_\varphi(ih,s)v_n(s)ds$$

$$= \sum_{j=0}^{n-1} \sum_{k\in\mathbb{Z}_m} \int_{jh}^{(j+1)h} K_\varphi(ih,s)L_k(ns-j)ds\, (E_\delta v_n)((j+k)h)$$

$$= \sum_{j=0}^{n-1} \sum_{k\in\mathbb{Z}_m} \alpha_{i,j,k} \cdot \left\{ \begin{array}{cc} v_n(0), & j+k \le 0 \\ v_n((j+k)h), & 1 \le j+k \le n-1 \\ v_n(1), & j+k \ge n \end{array} \right\} = \sum_{l=0}^{n} b_{i,l} v_n(lh), \quad i = 0, ..., n,$$

where

$$\alpha_{i,j,k} = \int_{jh}^{(j+1)h} K_\varphi(ih,s)L_k(ns-j)ds, \quad i = 0, ..., n, \; j = 0, ..., n-1, \; k \in \mathbb{Z}_m, \tag{7.20}$$

$$b_{i,l} = \left\{ \begin{array}{cc} \sum_{k\in\mathbb{Z}_m} \sum_{\{j:\, 0\le j\le n-1,\, j+k\le 0\}} \alpha_{i,j,k}\,, & l = 0 \\ \sum_{k\in\mathbb{Z}_m} \sum_{\{j:\, 0\le j\le n-1,\, j+k=l\}} \alpha_{i,j,k}\,, & l = 1, ...n-1 \\ \sum_{k\in\mathbb{Z}_m} \sum_{\{j:\, 0\le j\le n-1,\, j+k\ge n\}} \alpha_{i,j,k}\,, & l = n \end{array} \right\}, \quad i,l = 0, ..., n. \tag{7.21}$$

Hence the matrix form of the collocation method (7.16) is given by

$$v_n(ih) = \sum_{l=0}^{n} b_{i,l} v_n(lh) + f(ih), \quad i = 0, ..., n, \tag{7.22}$$

with $b_{i,l}$ defined by (7.20), (7.21). Having determined $v_n(ih)$, $i = 0, ..., n$, through solving the system (7.22), the collocation solution $v_n(t)$ at any intermediate point $t \in [jh, (j+1)h]$, $j = 0, .., n-1$, is given by

$$v_n(t) = \sum_{k \in \mathbb{Z}_m} \left\{ \begin{array}{cc} v_n(0), & j+k \le 0 \\ v_n((j+k)h), & 1 \le j+k \le n-1 \\ v_n(1), & j+k \ge n \end{array} \right\} \cdot L_k(nt - j) \tag{7.23}$$

where L_k, $k \in \mathbb{Z}_m$, are the Lagrange fundamental polynomials defined in (7.10).

8. Spline interpolation and quasi-interpolation

8.1. **Cardinal B-splines.** There are different equivalent definitions of the B-splines. We present the recursive definition of the *cardinal B-spline* B_m *of degree* m, $m \ge 0$, as follows:

$$B_0(x) = \left\{ \begin{array}{cc} 1, & 0 \le x < 1 \\ 0, & x < 0 \text{ or } x \ge 1 \end{array} \right\},$$

$$B_m(x) = \frac{1}{m} \left(x B_{m-1}(x) + (m+1-x) B_{m-1}(x-1) \right), \quad m = 1, 2, \dots .$$

Here are the explicit formulae of B_m for $m = 1, 2, 3$:

$$B_1(x) = \left\{ \begin{array}{cc} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{array} \right\},$$

$$B_2(x) = \left\{ \begin{array}{cc} \frac{1}{2}x^2, & 0 \le x \le 1 \\ \frac{1}{2}(-2x^2 + 6x - 3), & 1 \le x \le 2 \\ \frac{1}{2}(3-x)^2, & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{array} \right\},$$

$$B_3(x) = \left\{ \begin{array}{cc} \frac{1}{6}x^3, & 0 \le x \le 1 \\ \frac{1}{6}(-3x^3 + 12x^2 - 12x + 4), & 1 \le x \le 2 \\ B_3(4-x), & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{array} \right\}.$$

Let us list most important properties of B_m:

$$B_m \in C^{m-1}(\mathbb{R}) \text{ for } m \ge 1, \quad B_m \mid_{[j,j+1]} \in \mathcal{P}_m, \quad j \in \mathbb{Z},$$

$$\text{supp} B_m = [0, m+1], \quad B_m(x) > 0 \text{ for } 0 < x < m+1,$$

$$B_m(\frac{m+1}{2} - x) = B_m(\frac{m+1}{2} + x), \quad x \in \mathbb{R}, \quad B_m(\frac{m+1}{2}) = \max_{x \in \mathbb{R}} B_m(x),$$

$$\sum_{j\in\mathbb{Z}} B_m(x-j) = 1, \;\; x \in \mathbb{R}.$$

Some of the listed properties are obvious, some need proofs; some of proofs become easier when equivalent definitions of B_m are introduced. We quote [1], [4], [10], [11], [19] for equivalent definitions and proofs. In particular, $B_m(x) = \int_{x-1}^{x} B_{m-1}(y)dy$, that implies by recursion the smoothness claim $B_m \in C^{m-1}(\mathbb{R})$. Together with the property $B_m \mid_{[j,j+1]} \in \mathcal{P}_m$, $j \in \mathbb{Z}$ (which is elementary), this means that B_m is a spline of degree m (and defect 1) on the "cardinal" knot set $\mathbb{Z}$.

We complete the listed properties by the Marsden identity

$$\sum_{j\in\mathbb{Z}} (t-j-1)(t-j-2)...(t-j-m)B_m(x-j) = (t-x)^m, \;\; t,x \in \mathbb{R}, \;\; m \geq 1.$$

Equalizing the coefficients by t^{m-k} on the r.h.s and l.h.s., we obtain the *Marsden formulae*

$$x^k = \sum_{j\in\mathbb{Z}} \beta_{j,k,m} B_m(x-j), \quad k = 0,...,m, \;\; x \in \mathbb{R},$$

in which the coefficients $\beta_{j,k,m}$ satisfy $\mid \beta_{j,k,m} \mid \leq c_m \mid j \mid^k$ for $0 \neq j \in \mathbb{Z}$. Exact expressions for $\beta_{j,k,m}$ can be written down but they are somewhat complicated and we do not need them; for us it is sufficient to to be sure that $\mid \beta_{j,k,m} \mid$ do not grow exponentially as $\mid j \mid \to \infty$.

8.2. **Wiener interpolant.** Now we assume $m \geq 2$ to be fixed. Introduce in $\mathbb{R}$ the uniform grid $\mathbb{R}_h = \{jh : j \in \mathbb{Z}\}$, $h = 1/n$, $n \in \mathbb{N}$. Given a bounded function $f \in C(\mathbb{R})$, we look for its interpolant $Q_{n,m-1}f$ in the form

$$(Q_{n,m-1}f)(x) = \sum_{j\in\mathbb{Z}} d_j B_{m-1}(nx-j), \;\; x \in \mathbb{R}, \tag{8.1}$$

and determine the coefficients d_j from the interpolation conditions

$$(Q_{n,m-1}f)((k+\frac{m}{2})h) = f((k+\frac{m}{2})h), \quad k \in \mathbb{Z}. \tag{8.2}$$

This leads to the bi-infinite system of linear equations

$$\sum_{j\in\mathbb{Z}} B_{m-1}(k+\frac{m}{2}-j)d_j = f((k+\frac{m}{2})h), \quad k \in \mathbb{Z},$$

or

$$\sum_{j\in\mathbb{Z}} b_{k-j} d_j = f_k, \quad k \in \mathbb{Z}, \tag{8.3}$$

where

$$b_k = B_{m-1}(k+\frac{m}{2}), \;\; f_k = f((k+\frac{m}{2})h), \;\; k \in \mathbb{Z}. \tag{8.4}$$

For $m = 2$, system (8.3) reduces to the relations $d_k = f((k+1)h))$, $k \in \mathbb{Z}$, and $(Q_{n,1}f)(x) = \sum_{j\in\mathbb{Z}} f((j+1)h))B_1(nx-j)$ is the usual piecewise linear interpolant which can be determined on every subinterval $[ih,(i+1)h]$ independently from other subintervals. All is clair in the case $m=2$ and from now we focuse our attention to the case $m \geq 3$. A delicate problem is that, as we show below , the solution of system (8.3) is nonunique for $m \geq 3$ if we allow an exponential growth of $\mid d_j \mid$ as $\mid j \mid \to \infty$. Note that there are no problems with the convergence of the series in (8.1) since this series is locally finite: it follows from the relation $\mathrm{supp} B_{m-1} = [0,m]$ that

$$(Q_{n,m-1}f)(x) = \sum_{j=i-m+1}^{i} d_j B_{m-1}(nx-j) \ \text{ for } \ x \in [ih,(i+1)h], \ i \in \mathbb{Z}.$$

Consider the case of even m, then the interpolation points are ih, $i \in \mathbb{Z}$. We have

$$(Q_{n,m-1}f)(x) = \sum_{j=-m+1}^{0} d_j B_{m-1}(nx-j) \ \text{ for } \ x \in [0,h].$$

After the satisfying the interpolation conditions at $x=0$ and $x=h$, there remains $m-2$ dimensional manifold of coefficients $d_{-m+1},...,d_0$ undetermined. Arbitrarily fixing the values of $d_{-m+1},...,d_0$ from this manifold, we can uniquely determine d_{-m} and d_1 so that $(Q_{n,m-1}f)(x) = \sum_{j=-m}^{1} d_j B_{m-1}(nx-j)$ satisfies the interpolation conditions at $x=-h$ and at $x=2h$; due the support property of B_{m-1}, the values of $(Q_{n,m-1}f)(x)$ at $x=0$ and $x=h$ remain to be as they were. After that we can uniquely determine the next pair of coefficients d_{-m-1} and d_2 so that $(Q_{n,m-1}f)(x) = \sum_{j=-m-1}^{2} d_j B_{m-1}(nx-j)$ satisfies the interpolation conditions at $x=-2h$ and $x=3h$; the values of $(Q_{n,m-1}f)(x)$ at $x=-h,0,h,2h$ remain to be as they were. Continuing in this manner we determine all d_j, $j \in \mathbb{Z}$, so that $(Q_{n,m-1}f)(x) = \sum_{j\in\mathbb{Z}} d_j B_{m-1}(nx-j)$ satisfies all conditions (8.2). Hence there exists an $m-2$ dimensional manifold of interpolants (8.1)–(8.3). For odd $m \geq 3$, satisfying on the first step the interpolation conditions at $x=h/2$ and continuing as above, we easily see that there exists an $m-1$ imensional manifold of interpolants (8.1)–(8.3).

Only one of solutions of system (8.3) is suitable to obtain a bounded interpolant $Q_{n,m-1}f$ for a bounded function $f \in C(\mathbb{R})$. This solution is related to the Wiener theorem for Fourier series, or equivalently, for Laurent series on the unit circle $\mid z \mid= 1$ of the complex plane. Wiener theorem [20] can be formulated as follows: if the given complex numbers b_k, $k \in \mathbb{Z}$, satisfy the conditions $\sum_{k\in\mathbb{Z}} \mid b_k \mid< \infty$ and

$$b(z) := \sum_{k\in\mathbb{Z}} b_k z^k \neq 0 \ \text{ for all } \ z \in \mathbb{C} \text{ with } \ \mid z \mid= 1, \tag{8.5}$$

then $a(z) := 1/b(z)$ has the (Laurent) expansion $a(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ with $a_k \in \mathbb{C}$, $k \in \mathbb{Z}$, such that $\sum_{k\in\mathbb{Z}} \mid a_k \mid< \infty$. (Hence the expansion of a converges uniformly on the circle $\mid z \mid= 1$ of the complex plane, similarly as the expansion of b.) It is easy to understand (the argument is presented in Section 8.3) that $\mathfrak{A} = (a_{k-j})_{k,j\in\mathbb{Z}}$ is the inverse to $\mathfrak{B} = (b_{k-j})_{k,j\in\mathbb{Z}}$, i.e., $\mathfrak{B}\mathfrak{A} = \mathfrak{A}\mathfrak{B} = \mathfrak{I}$. We call $\mathfrak{A}$ the Wiener inverse of $\mathfrak{B}$. Condition (8.5) occurs to be fulfilled in our interpolation problem (8.1)–(8.4), so we can use the Wiener inverse $\mathfrak{A}$ of $\mathfrak{B}$ and define the *Wiener interpolant* $Q_{n,m-1}f$ by (8.1) with

$$d_k = \sum_{j\in\mathbb{Z}} a_{k-j} f_j, \quad k \in \mathbb{Z}. \tag{8.6}$$

Only a finite number of b_k do not vanish in the interpolation system (8.3). As we will see Section 8.3, this enables an elementary construction of the numbers a_k, $k \in \mathbb{Z}$; it occurs that a_k are real and decay exponentially as $\mid k \mid \to \infty$.

Due to the exponential decay of a_k, we may truncate the series in (8.6) to $O(\log n)$ terms maintaining the highest possible accuracy order $O(h^m)$ of $Q_{n,m-1}f$. Hence the error $f(x) - (Q_{n,m-1}f)(x)$ significantly depends on the values of f only in a neighborhood of x of a width $O(n^{-1}\log n)$; to a change of f outside this interval, the error $f(x) - (Q_{n,m-1}f)(x)$ responds by a change of an order $O(n^{-m})$.

8.3. **Construction of the Wiener interpolant.** Denote

$$\mu = \text{int } ((m-1)/2) \quad (\text{int} = \text{integer part}).$$

Due to the properties of B_{m-1}, for b_k defined in (8.4), it holds

$$b_k = b_{-k} > 0 \text{ for } \mid k \mid \le \mu, \;\; b_k = 0 \text{ for } \mid k \mid > \mu, \;\; \sum_{k=-\mu}^{\mu} b_k = 1.$$

Introduce the functions

$$b(z) := \sum_{|k|\le\mu} b_k z^k = b_0 + \sum_{k=1}^{\mu} b_k (z^k + z^{-k}), \;\; 0 \neq z \in \mathbb{C}, \;\; P_{2\mu}(z) = z^{\mu} b(z), \tag{8.7}$$

$$a(z) := 1/b(z) = z^{\mu}/P_{2\mu}(z), \;\; z \in \mathbb{C}, \;\; z \neq z_\nu, \;\; \nu = 1, ..., 2\mu, \tag{8.8}$$

where z_ν, $\nu = 1, ..., 2\mu$, are the roots of the $P_{2\mu} \in \mathcal{P}_{2\mu}$ (called the *characteristic roots*). From (8.7) we observe that together with z_ν also $1/z_\nu$ is a characteristic root . As stated in [11], all characteristic roots are simple and real; then clearly $z_\nu < 0$, $\nu = 1, ..., 2\mu$ and $z_\nu \neq -1$, $\nu = 1, ..., 2\mu$, thus there is exactly μ characteristic roots in the interval $(-1, 0)$. We omit the relatively long and complicated proof of this statement of [11]. It is possible to check the statement when the interpolant (8.1), (8.6) is constructed in the practice, since the algorithm needs the values of z_ν, $\nu = 1, ..., 2\mu$, so they must be computed in any case. Let us turn to examples:

$$m = 3: \;\; \mu = 1, \;\; b_{-1} = b_1 = \frac{1}{8}, \;\; b_0 = \frac{3}{4}, \;\; P_2(z) = \frac{1}{8}(z^2 + 6z + 1), \;\; z_{1,2} = -3 \pm \sqrt{8};$$

$$m = 4: \;\; \mu = 1, \;\; b_{-1} = b_1 = \frac{1}{6}, \;\; b_0 = \frac{2}{3}, \;\; P_2(z) = \frac{1}{6}(z^2 + 4z + 1), \;\; z_{1,2} = -2 \pm \sqrt{3};$$

$$m = 6: \;\; \mu = 2, \;\; P_4(z) = \frac{1}{5!}(z^4 + 26z^3 + 66z^2 + 26z + 1), \;\; w_{1,2} := -13 \pm \sqrt{105},$$

$$z_{1,2,3,4} = \frac{w_{1,2} \pm \sqrt{w_{1,2}^2 - 4}}{2}, \ z_1 \approx -0,043096, \ z_2 \approx -0,430575, \ z_3 = \frac{1}{z_1}, \ z_4 = \frac{1}{z_2};$$

$$m = 10: \ \mu = 4, \ P_8(z) = \frac{1}{9!}(z^8 + 502z^7 + 14608z^6 + 88234z^5 + 156190z^4$$
$$+88234z^3 + 14608z^2 + 502z + 1), \ z_5 = \frac{1}{z_1}, \ z_6 = \frac{1}{z_2}, \ z_7 = \frac{1}{z_3}, \ z_8 = \frac{1}{z_4},$$
$$z_1 = -2.121307 \cdot 10^{-3}, \ z_2 = -0,043223, \ z_3 = -0,201751, \ z_4 = -0,607997.$$

For two functions $a(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ and $b(z) = \sum_{k\in\mathbb{Z}} b_k z^k$ defined by the absolutely convergent Laurent series on the unit circle $| \ z \ |= 1$, it is easily seen that $a(z)b(z) = \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} a_{k-j} b_j\right) z^k, \quad k \in \mathbb{Z}$. For a and b defined in (8.7), (8.8), this yields $\sum_{j\in\mathbb{Z}} a_{k-j} b_j = \delta_{k,0}$, $k \in \mathbb{Z}$, where $\delta_{k,l}$ is the Kronecker symbol. Replacing here k by $k-l$ we rewrite it as $\sum_{j\in\mathbb{Z}} a_{k-l-j} b_j = \delta_{k-l,0}$, $k,l \in \mathbb{Z}$, or introducing the new summation index $j' = j + l$, as $\sum_{j'\in\mathbb{Z}} a_{k-j'} b_{j'-l} = \delta_{k,l}$. Finally, writing j instead j', the equality takes the form

$$\sum_{j\in\mathbb{Z}} a_{k-j} b_{j-l} = \delta_{k,l}, \quad k,l \in \mathbb{Z}.$$

Similarly (or simply by a symmetry argument),

$$\sum_{j\in\mathbb{Z}} b_{k-j} a_{j-l} = \delta_{k,l}, \quad k,l \in \mathbb{Z}.$$

The last two equalities mean that the matrix $\mathfrak{B} = (b_{k-j})_{k,j\in\mathbb{Z}}$ of system (8.3) has the inverse $\mathfrak{B}^{-1} = \mathfrak{A} = (a_{k-j})_{k,j\in\mathbb{Z}}$. Thus our task can be reformulared as follows: find the coefficients a_k of the Laurent series $a(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ for the function a defined by (8.7), (8.8).

Let us order the characteristic roots $z_1, ..., z_{2\mu}$ so that $z_1, ..., z_\mu$ are in the interval $(-1, 0)$ and $z_{\mu+\nu} = 1/z_\nu$, $\nu = 1, ..., \mu$. Since all roots are simple, the function $a(z) := 1/b(z) = \frac{z^\mu}{P_{2\mu}(z)}$ has a representation

$$\frac{z^\mu}{P_{2\mu}(z)} = \sum_{\nu=1}^{2\mu} \frac{c_\nu}{z - z_\nu}.$$

Multiplying by $\prod_{\lambda=1}^{2\mu}(z - z_\lambda) = P_{2\mu}(z)/b_\mu$ we rewrite it as

$$\frac{z^\mu}{b_\mu} = \sum_{\nu=1}^{2\mu} c_\nu \prod_{\nu\neq\lambda=1}^{2\mu} (z - z_\lambda) = c_1 \prod_{\lambda=2}^{2\mu}(z - z_\lambda) + ... + c_{2\mu} \prod_{\lambda=1}^{2\mu-1} (z - z_\lambda).$$

Setting $z = z_\nu$ we determine the coefficients c_ν:

$$c_\nu = \frac{z_\nu^\mu}{b_\mu \prod_{\nu\neq\lambda=1}^{2\mu}(z_\nu - z_\lambda)} = \frac{z_\nu^\mu}{P'_{2\mu}(z_\nu)}, \quad \nu = 1, ..., 2\mu.$$

Thus

$$a(z) = \sum_{\nu=1}^{2\mu} \frac{z_\nu^\mu}{P'_{2\mu}(z_\nu)} \frac{1}{z - z_\nu} = \sum_{\nu=1}^{\mu} \left(\frac{z_\nu^\mu}{P'_{2\mu}(z_\nu)} \frac{1}{z - z_\nu} + \frac{z_\nu^{-\mu}}{P'_{2\mu}(z_\nu^{-1})} \frac{1}{z - z_\nu^{-1}} \right).$$

It follows from (8.7) that $P_{2\mu}(z^{-1}) = z^{-2\mu} P_{2\mu}(z)$. Differentiating this equality and setting then $z = z_\nu$ we find that $-P'_{2\mu}(z_\nu^{-1}) z_\nu^{-2} = z_\nu^{-2\mu} P'_{2\mu}(z_\nu)$, or $\frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} = -\frac{z_\nu^{-\mu+1}}{P'_{2\mu}(z_\nu^{-1})}$. Now we can rewrite

$$a(z) = \sum_{\nu=1}^{\mu} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \left(\frac{z_\nu}{z - z_\nu} - \frac{z_\nu^{-1}}{z - z_\nu^{-1}} \right) = \sum_{\nu=1}^{\mu} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \left(\frac{z_\nu z^{-1}}{1 - z_\nu z^{-1}} + \frac{1}{1 - z_\nu z} \right).$$

Expanding

$$\frac{z_\nu z^{-1}}{1 - z_\nu z^{-1}} = \sum_{k=1}^{\infty} z_\nu^k z^{-k} \quad \text{for} \quad |\, z \,|>|\, z_\nu \,|, \quad \nu = 1, ..., \mu,$$

$$\frac{1}{1 - z_\nu z} = \sum_{k=0}^{\infty} z_\nu^k z^k \quad \text{for} \quad |\, z \,|<|\, z_\nu \,|^{-1}, \quad \nu = 1, ..., \mu,$$

we arrive at the desired expansion

$$a(z) = \sum_{\nu=1}^{\mu} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \left(\sum_{k=1}^{\infty} z_\nu^k z^{-k} + \sum_{k=0}^{\infty} z_\nu^k z^k \right) = \sum_{k \in \mathbb{Z}} a_k z^k, \ \ \theta_m <|\, z \,|< \theta_m^{-1},$$

where $\theta_m = \max_{1 \le \nu \le \mu} |\, z_\nu \,|< 1$ and

$$a_k = \sum_{\nu=1}^{\mu} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} z_\nu^{|k|}, \ \ |\, a_k \,| \le c_m \theta_m^{|k|}, \ \ k \in \mathbb{Z}, \ \ c_m = \sum_{\nu=1}^{\mu} \frac{|\, z_\nu \,|^{\mu-1}}{|\, P'_{2\mu}(z_\nu) \,|}. \tag{8.9}$$

Thus a_k decays exponentially. Clearly a_k are real, $a_k = a_{-k}$, $k \in \mathbb{Z}$. In the following theorem we summarize the results and present formulae for $\sum_{k\in\mathbb{Z}} a_k$ and $\sum_{k\in\mathbb{Z}} |\, a_k \,|$.

Theorem 8.1. For a_k defined in (8.9), it holds

$$\sum_{k\in\mathbb{Z}} a_k = 1, \ \ \sum_{k\in\mathbb{Z}} |\, a_k \,| = \frac{(-1)^\mu}{P_{2\mu}(-1)}, \ \ a_k = (-1)^k \,|\, a_k \,| \neq 0, \ \ k \in \mathbb{Z}, \tag{8.10}$$

where $P_{2\mu}$ is the characteristic polynomial defined in (8.7). The Wiener interpolant $Q_{n,m-1}f$ of f is given by (8.1), (8.6), (8.9) where z_ν, $\nu = 1, ...\mu$, are the roots of the characteristic polynomial $P_{2\mu}(z)$ in the interval $(-1, 0)$ and $f_k = f((k + \frac{m}{2})h)$, $k \in \mathbb{Z}$. Moreover, $\|\, Q_{n,m-1} \,\|_{BC(\mathbb{R})\to BC(\mathbb{R})} \le \sum_{k\in\mathbb{Z}} |\, a_k \,|$.

Proof. The first one of relations (8.10) immediately follows from (8.7), (8.8):

$$\sum_{k\in\mathbb{Z}} a_k = a(1) = 1/b(1) = 1/\sum_{k=-\mu}^{\mu} b_k = 1.$$

Next we prove the third one of relations (8.10). We start from equalities

$$a(z) = 1/b(z) = \frac{z^\mu}{P_{2\mu}(z)} = \frac{z^\mu}{b_\mu \prod_{\lambda=1}^{2\mu}(z - z_\lambda)} = \frac{z^\mu}{b_\mu \prod_{\lambda=1}^{2\mu}(z + \mid z_\lambda \mid)}.$$

Setting $-z$ into the place of z, we have

$$\begin{aligned} a(-z) &= \frac{(-z)^\mu}{b_\mu \prod_{\lambda=1}^{2\mu}(-z + \mid z_\lambda \mid)} = \frac{(-z)^\mu}{b_\mu \prod_{\lambda=1}^{\mu}(-z + \mid z_\lambda \mid) \prod_{\lambda=1}^{\mu}(-z + \mid z_\lambda^{-1} \mid)} \\ &= \frac{z^\mu(-1)^\mu}{b_\mu \prod_{\lambda=1}^{\mu}(z - \mid z_\lambda \mid) \prod_{\lambda=1}^{\mu}(z - \mid z_\lambda^{-1} \mid)} \\ &= \frac{1}{b_\mu \prod_{\lambda=1}^{\mu}(1 - \mid z_\lambda \mid z^{-1}) \prod_{\lambda=1}^{\mu}(\mid z_\lambda^{-1} \mid - z)} \\ &= \frac{\prod_{\lambda=1}^{\mu} \mid z_\lambda \mid}{b_\mu \prod_{\lambda=1}^{\mu}(1 - \mid z_\lambda \mid z^{-1}) \prod_{\lambda=1}^{\mu}(1 - \mid z_\lambda \mid z)} \\ &= \frac{\prod_{\lambda=1}^{\mu} \mid z_\lambda \mid}{b_\mu} \prod_{\lambda=1}^{\mu} \left(\sum_{k=0}^{\infty} \mid z_\lambda \mid^k z^{-k} \right) \prod_{\lambda=1}^{\mu} \left(\sum_{k=0}^{\infty} \mid z_\lambda \mid^k z^k \right) = \sum_{k \in \mathbb{Z}} c_k z^k, \quad \mid z \mid = 1, \end{aligned}$$

with some $c_k > 0$, $k \in \mathbb{Z}$. Returning to z instead of $-z$, we obtain $a(z) = \sum_{k \in \mathbb{Z}} (-1)^k c_k z^k$. The Laurent expansion $a(z) = \sum_{k \in \mathbb{Z}} a_k z^k$ is unique, therefore $a_k = (-1)^k c_k = (-1)^k \mid a_k \mid$, $k \in \mathbb{Z}$, as asserted in (8.10). Equivalently, $\mid a_k \mid = (-1)^k a_k$, $k \in \mathbb{Z}$. Now the second one of relations (8.10) follows by the same argument as the first one:

$$\sum_{k \in \mathbb{Z}} \mid a_k \mid = \sum_{k \in \mathbb{Z}} a_k (-1)^k = a(-1) = 1/b(-1) = (-1)^\mu / P_{2\mu}(-1).$$

The proof of the inequality $\| Q_{n,m-1} \|_{BC(\mathbb{R}) \to BC(\mathbb{R})} \leq \sum_{k \in \mathbb{Z}} \mid a_k \mid$ is elementary and left as an exercise. Other claims have been already established. □

The values of $\sum_{k \in \mathbb{Z}} \mid a_k \mid = 1 / \mid P_{2\mu}(-1) \mid = 1 / \mid P_{2\mu,m}(-1) \mid$ for $m = 2, ..., 10$ are given in the following table:

m	$\frac{1}{\lvert P_{2\mu,m}(-1)\rvert}$	$\frac{\lvert P_{2\mu,m}(-1)\rvert}{\lvert P_{2\mu,m+1}(-1)\rvert}$
2	1	2
3	2	1.5
4	3	1.6
5	4.8	1.5625
6	7.5	1.5738
7	11.803279	1.5699
8	18.529412	1.5711
9	29.111913	1.5707
10	45.725806	1.5708

(one can guess that $\mid P_{2\mu,m}(-1) \mid / \mid P_{2\mu,m+1}(-1) \mid \to \pi/2 = 1.570796...$ as $m \to \infty$; for $m = 20$ this ratio is 1.570796327). We see that $\| Q_{n,m-1} \|_{BC(\mathbb{R}) \to BC(\mathbb{R})}$ has for $m \leq 10$ bounds that enable a numerically stable interpolation.

It is easily seen that the null space $\mathcal{N}(\mathfrak{B})$ of the matrix $\mathfrak{B} = (b_{k-j})_{k,j\in\mathbb{Z}}$ in the vector space X of all bi-infinite vectors $(d_j)_{j\in\mathbb{Z}}$ is of the dimension 2μ and is spanned by $(z_\nu^j)_{j\in\mathbb{Z}}$, $\nu = 1,...,2\mu$, where z_ν, $\nu = 1,...,2\mu$, are the characteristic roots. Thus any nontrivial element of $\mathcal{N}(\mathfrak{B})$ in X is exponentially growing either as $j \to \infty$ or as $j \to -\infty$. Together with the Marsden formula for x^k, $k = 0,...,m$-1 (see Section 8.1), we conclude that x^k coincides with its Wiener interpolant. Thus

$$Q_{n,m-1}g = g \text{ for } g \in \mathcal{P}_{m-1}.$$

8.4. **A difference representation of the Wiener solution** (a_k)**.** Denote by $\mathfrak{s}(\mathbb{Z})$ the vector space of (fast decaying) bisequences $\underline{a} = (a_j)_{j\in\mathbb{Z}}$ that satisfy the condition

$$\forall r \geq 0 \ \exists c_r < \infty \text{ such that } \mid a_j \mid\leq c_r \mid j \mid^{-r}, \ \ 0 \neq j \in \mathbb{Z}.$$

For instance, exponentially decaying sequences belong to $\mathfrak{s}(\mathbb{Z})$. Introduce also the subspaces

$$\mathfrak{s}_{\text{sym}}(\mathbb{Z}) = \{\underline{a} \in \mathfrak{s}(\mathbb{Z}) : \ a_{-j} = a_j, \ j \in \mathbb{Z}\},$$
$$\mathfrak{s}_0(\mathbb{Z}) = \{\underline{a} \in \mathfrak{s}(\mathbb{Z}) : \ \sum_{j\in\mathbb{Z}} a_j = 0\}$$

and denote by $\underline{e}$ the bisequence $(e_j)_{j\in\mathbb{Z}}$ with $e_j = \delta_{j,0}$ (the Kronecker symbol).

Introduce the difference operators

$$D^+ : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}), \ \ (D^+\underline{a})_j = a_{j+1} - a_j, \ j \in \mathbb{Z} \ \text{(forward difference)},$$

$$D^- : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}), \ \ (D^-\underline{a})_j = a_j - a_{j-1}, \ j \in \mathbb{Z} \ \text{(backward difference)}$$

and their one side inverses

$$J^+ : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}), \ \ (J^+\underline{a})_k = \left\{ \begin{array}{ll} \sum_{j=-\infty}^{k-1} a_j, & k \leq 0 \\ -\sum_{j=k}^{\infty} a_j, & k > 0 \end{array} \right\}, \ \ k \in \mathbb{Z},$$

$$J^- : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}), \ \ (J^-\underline{a})_k = \left\{ \begin{array}{ll} \sum_{j=-\infty}^{k} a_j, & k < 0 \\ -\sum_{j=k+1}^{\infty} a_j, & k \geq 0 \end{array} \right\}, \ \ k \in \mathbb{Z}.$$

Namely, a straightforward check shows that for any $\underline{a} \in \mathfrak{s}(\mathbb{Z})$, it holds

$$D^+J^+\underline{a} = \underline{a} - \left(\sum_{j\in\mathbb{Z}} a_j\right)\underline{e}, \ \ D^-J^-\underline{a} = \underline{a} - \left(\sum_{j\in\mathbb{Z}} a_j\right)\underline{e}, \tag{8.11}$$
$$J^+D^+\underline{a} = \underline{a}, \ \ J^-D^-\underline{a} = \underline{a}.$$

(Observe that $D^\pm : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z})$ does not have a two side inverse since $\underline{e}$ does not belong to the range of $D^\pm$;on the other side, $J^\pm\underline{e} = 0$.) Finally, introduce the second order central difference operator

$$D = D^+D^- = D^-D^+ : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}), \ \ (D\underline{a})_j = a_{j-1} - 2a_j + a_{j+1}, \ j \in \mathbb{Z},$$

and its one side inverse

$$J = J^+J^- : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}).$$

(Caution: $J^+J^- \neq J^-J^+$ although $D^+D^- = D^-D^+$; nevertheless, $J^+J^-\underline{a} = J^-J^+\underline{a}$ for $\underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$.) Equalities (8.11) imply

$$DJ\underline{a} = (D^+D^-)(J^+J^-)\underline{a} = D^-(D^+J^+)(J^-\underline{a}) = D^-\left(J^-\underline{a} - \left(\sum_{j\in\mathbb{Z}}(J^-\underline{a})_j\right)\underline{e}\right)$$

$$= \underline{a} - \left(\sum_{j\in\mathbb{Z}} a_j\right)\underline{e} - \left(\sum_{j\in\mathbb{Z}}(J^-\underline{a})_j\right)D^-\underline{e}, \ \ \underline{a} \in \mathfrak{s}(\mathbb{Z}),$$

$$JD\underline{a} = (J^+J^-)(D^+D^-)\underline{a} = J^+(J^-D^-)D^+\underline{a} = J^+D^+\underline{a} = \underline{a}, \ \ \underline{a} \in \mathfrak{s}(\mathbb{Z});$$

note that $J^\pm\underline{a} \in \mathfrak{s}_0(\mathbb{Z})$ for $\underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$, and the formula for $DJ\underline{a}$ simplifies to

$$DJ\underline{a} = \underline{a} - \left(\sum_{j\in\mathbb{Z}} a_j\right)\underline{e} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z}) \ \text{ for } \ \underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z}). \tag{8.12}$$

It is also easy to check that $J\underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$ for $\underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$.

Lemma 8.2. For $p \in \mathbb{N}$ and $\underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$, it holds

$$D^pJ^p\underline{a} = \underline{a} - \sum_{q=0}^{p-1}\left(\sum_{j\in\mathbb{Z}}(J^q\underline{a})_j\right)D^q\underline{e}. \tag{8.13}$$

Hence, $\underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$ has the representation

$$\underline{a} = \sum_{q=0}^{p-1}\left(\sum_{j\in\mathbb{Z}}(J^q\underline{a})_j\right)D^q\underline{e} + D^pJ^p\underline{a}, \quad p \in \mathbb{N}. \tag{8.14}$$

Proof. For $p = 1$, (8.13) is (8.12). Assume that, for a $p \in \mathbb{N}$, (8.13) holds for all $\underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$, and check that then this is true also for $p+1$. Indeed, by the induction assumption and (8.12),

$$D^{p+1}J^{p+1}\underline{a} = D(D^pJ^p)(J\underline{a}) = D\left(J\underline{a} - \sum_{q=0}^{p-1}\left(\sum_{j\in\mathbb{Z}}(J^qJ\underline{a})_j\right)D^q\underline{e}\right)$$

$$= \underline{a} - \left(\sum_{j\in\mathbb{Z}} a_j\right)\underline{e} - \sum_{q=0}^{p-1}\left(\sum_{j\in\mathbb{Z}}(J^{q+1}\underline{a})_j\right)D^{q+1}\underline{e} = \underline{a} - \sum_{q=0}^{p}\left(\sum_{j\in\mathbb{Z}}(J^q\underline{a})_j\right)D^q\underline{e}. \qquad \square$$

Lemma 8.3. For $\underline{a} = (a_k)_{k\in\mathbb{Z}}$ defined in (8.9), we have the representations

$$\underline{a} = \underline{e} + \sum_{q=1}^{p-1}\gamma_q D^q\underline{e} + D^pJ^p\underline{a}, \ \ p \in \mathbb{N}, \ \ \gamma_q = \sum_{\nu=1}^{\mu}\frac{(1+z_\nu)z_\nu^{\mu+q-1}}{(1-z_\nu)^{2q+1}P'_{2\mu}(z_\nu)}. \tag{8.15}$$

Proof. We have to compute the coefficients $\sum_{j\in\mathbb{Z}}(J^q\underline{a})_j$ in (8.14). By (8.9),

$$a_j=\sum_{\nu=1}^{\mu}\frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)}z_\nu^{|j|},\ \ j\in\mathbb{Z}.$$

For the sequence $\underline{z}^{(\nu)}=(z_\nu^{|j|})_{j\in\mathbb{Z}}$ we find

$$(J^-\underline{z}^{(\nu)})_k=\frac{1}{1-z_\nu}\left\{\begin{array}{ll} z_\nu^{-k}, & k<0\\ -z_\nu^{k+1}, & k\ge 0\end{array}\right\},$$

$$(J\underline{z}^{(\nu)})_k=(J^+J^-\underline{z}^{(\nu)})_k=\frac{1}{(1-z_\nu)^2}\left\{\begin{array}{ll} z_\nu^{-k+1}, & k\le 0\\ z_\nu^{k+1}, & k>0\end{array}\right\}=\frac{z_\nu}{(1-z_\nu)^2}z_\nu^{|k|},\ \ k\in\mathbb{Z}.$$

Hence

$$(J\underline{a})_k=\sum_{\nu=1}^{\mu}\frac{z_\nu}{(1-z_\nu)^2}\frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)}z_\nu^{|k|},\ \ k\in\mathbb{Z},$$

and by repeating this formula,

$$(J^q\underline{a})_k=\sum_{\nu=1}^{\mu}\frac{z_\nu^q}{(1-z_\nu)^{2q}}\frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)}z_\nu^{|k|},\ \ k\in\mathbb{Z},\ \ q\in\mathbb{N}. \tag{8.16}$$

Since $\sum_{k\in\mathbb{Z}}z_\nu^{|k|}=\frac{1+z_\nu}{1-z_\nu}$, we obtain

$$\sum_{k\in\mathbb{Z}}(J^q\underline{a})_k=\sum_{\nu=1}^{\mu}\frac{(1+z_\nu)z_\nu^{\mu+q-1}}{(1-z_\nu)^{2q+1}P'_{2\mu}(z_\nu)},\ \ q\in\mathbb{N}.$$

Recalling also that $\sum_{k\in\mathbb{Z}}a_k=1$, (8.14) takes the form (8.15). □

8.5. **Quasi-interpolants.** Formula (8.15) enables a new representation form for the coefficients d_k of the Wiener interpolant $(Q_{n,m-1}f)(x)=\sum_{k\in\mathbb{Z}}d_kB_{m-1}(nx-k)$.

Theorem 8.4. Let $f\in C(\mathbb{R})$ be bounded or of a polynomial growth. For the coefficients d_k of the Wiener interpolant (8.1), (8.6), it holds

$$d_k=\sum_{q=0}^{p-1}\gamma_qD^qf_k+\sum_{j\in\mathbb{Z}}(J^p\underline{a})_{k-j}D^pf_j,\ \ k\in\mathbb{Z},\ p\in\mathbb{N}, \tag{8.17}$$

where $\gamma_0=1$ and other coefficients γ_q are given in (8.15), $D^0f_k=f_k$, $D^1f_k=Df_k=D^+D^-f_k=f_{k-1}-2f_k+f_{k+1}$ is the central second difference, $D^2f_k=D(Df_k)$ is the central fourth difference etc., $f_k=f((k+\frac{m}{2})h)$ and $J^p\underline{a}\in\mathfrak{s}_{\rm sym}(\mathbb{Z})$ is presented in (8.16).

Proof. We need some formulae of summation by parts. For $\underline{a}\in\mathfrak{s}(\mathbb{Z})$ and a bounded or polynomially growing sequence $\underline{f}$, an elementary check confirms that

$$\sum_{j\in\mathbb{Z}}f_jD^+a_j=-\sum_{j\in\mathbb{Z}}(D^-f_j)a_j,\ \ \sum_{j\in\mathbb{Z}}f_jD^-a_j=-\sum_{j\in\mathbb{Z}}(D^+f_j)a_j.$$

For $D = D^+D^-$ these formulae imply

$$\sum_{j\in\mathbb{Z}} f_j D a_j = \sum_{j\in\mathbb{Z}} (Df_j)a_j, \quad \sum_{j\in\mathbb{Z}} f_j D^p a_j = \sum_{j\in\mathbb{Z}} (D^p f_j)a_j. \tag{8.18}$$

Recalling that $\underline{e} = (e_j) = (\delta_{j,0})$, we obtain (8.17) from (8.6) with the help of (8.15) and (8.18):

$$d_k = \sum_{j\in\mathbb{Z}} a_{k-j} f_j = \sum_{j\in\mathbb{Z}} f_{k-j} a_j = \sum_{j\in\mathbb{Z}} f_{k-j} (\sum_{q=0}^{p-1} \gamma_q D^q \underline{e} + D^p J^p \underline{a})_j$$

$$= \sum_{q=0}^{p-1} \gamma_q \sum_{j\in\mathbb{Z}} (D^q f_{k-j}) e_j + \sum_{j\in\mathbb{Z}} (D^p f_{k-j})(J^p \underline{a})_j = \sum_{q=0}^{p-1} \gamma_q D^q f_k + \sum_{j\in\mathbb{Z}} (J^p \underline{a})_{k-j} D^p f_j.$$

We took into account that $D_j f_{k-j} = D_k f_{k-j}$ where the designations $D_j f_{k-j}$ and $D_k f_{k-j}$ mean that the second central difference Df_{k-j} is taken with respect to j or k, respectively; due to the equality of these differences, we may omit the indexes j or k by D. □

Let us look at (8.17). Approximating the coefficients d_k by

$$d_k^{(p-1)} := \sum_{q=0}^{p-1} \gamma_q D^q f_k, \tag{8.19}$$

the interpolant $(Q_{n,m-1} f)(x) = \sum_{j\in\mathbb{Z}} d_j B_{m-1}(nx - j)$ will be approximated by the co-called *quasi-interpolant*

$$(Q_{n,m-1}^{(p-1)} f)(x) = \sum_{j\in\mathbb{Z}} d_j^{(p-1)} B_{m-1}(nx - j), \quad p \in \mathbb{N}. \tag{8.20}$$

Quasi-interpolants are determined locally from local information about f. For $x \in [ih, (i+1)h]$, (8.20) reduces to $(Q_{n,m-1}^{(p-1)} f)(x) = \sum_{j=i-m+1}^{i} d_j^{(p-1)} B_{m-1}(nx - j)$ and occuring here $d_j^{(p-1)}$ exploit the values $f_k = f((k + \frac{m}{2})h)$ for $k = i - m - p + 2, ..., i + p - 2$. Thus quasi-interpolants can be used for the approximation of functions given on an interval.

Theorem 8.5. If $f \in C^r(\mathbb{R})$, $0 \le r \le 2p$, and if $f^{(r)}(x)$ is bounded in $\mathbb{R}$ then

$$\sup_{x\in\mathbb{R}} | (Q_{n,m-1} f)(x) - (Q_{n,m-1}^{(p-1)} f)(x) | \le c'_m 2^{-r} h^r \sup_{x\in\mathbb{R}} | f^{(r)}(x) |, \tag{8.21}$$

$$c'_m = \sum_{\nu=1}^{\mu} \frac{| z_\nu |^{\mu-1}}{| P'_{2\mu}(z_\nu) |} \frac{1+ | z_\nu |}{1- | z_\nu |}.$$

(Recall that $\mu = \mathrm{int}((m-1)/2)$ and $-1 < z_\nu < 0$, $\nu = 1, ..., \mu$.)

Proof. We have $(Q_{n,m-1} f)(x) - (Q_{n,m-1}^{(p-1)} f)(x) = \sum_{k\in\mathbb{Z}} (d_k - d_k^{(p-1)}) B_{m-1}(nx - k)$. Since $B_{m-1}(nx - k) \ge 0$ and $\sum_{k\in\mathbb{Z}} B_{m-1}(nx - k) \equiv 1$, this implies

$$| (Q_{n,m-1} f)(x) - (Q_{n,m-1}^{(p-1)} f)(x) | \le \sup_{k\in\mathbb{Z}} | d_k - d_k^{(p-1)} | = \sup_{k\in\mathbb{Z}} | \sum_{j\in\mathbb{Z}} (J^p \underline{a})_{k-j} D^p f_j |$$

$$\le \sum_{k\in\mathbb{Z}} | (J^p \underline{a})_k | \sup_{j\in\mathbb{Z}} | D^p f_j | .$$

We used formulae (8.17) and (8.19) for d_k and $d_k^{(p-1)}$. Further, by (8.16),

$$\sum_{k\in\mathbb{Z}} \mid (J^p\underline{a})_k \mid \le \sum_{k\in\mathbb{Z}}\sum_{\nu=1}^{\mu} \frac{\mid z_\nu \mid^p}{(1+\mid z_\nu \mid)^{2p}} \frac{\mid z_\nu \mid^{\mu-1}}{\mid P'_{2\mu}(z_\nu) \mid} \mid z_\nu \mid^{|k|}$$

$$= \sum_{\nu=1}^{\mu} \frac{\mid z_\nu \mid^p}{(1+\mid z_\nu \mid)^{2p}} \frac{\mid z_\nu \mid^{\mu-1}}{\mid P'_{2\mu}(z_\nu) \mid} \frac{1+\mid z_\nu \mid}{1-\mid z_\nu \mid}$$

$$\le 2^{-2p} \sum_{\nu=1}^{\mu} \frac{\mid z_\nu \mid^{\mu-1}}{\mid P'_{2\mu}(z_\nu) \mid} \frac{1+\mid z_\nu \mid}{1-\mid z_\nu \mid} = 2^{-2p} c'_m;$$

we took into account that the function $x \mapsto \frac{x^p}{(1+x)^{2p}}$ grows on $[0,1]$ and hence its maximal value in this interval is attained at $x = 1$ and equals to 2^{-2p}. So we have

$$\mid (Q_{n,m-1}f)(x) - (Q^{(p-1)}_{n,m-1}f)(x) \mid \le 2^{-2p} c'_m \sup_{j\in\mathbb{Z}} \mid D^p f_j \mid .$$

It remains to estimate $\sup_{j\in\mathbb{Z}} \mid D^p f_j \mid$ under the assumption that $f^{(r)}(x)$ is bounded and $0 \le r \le 2p$. For even r, we estimate

$$\mid D^p f_j \mid = \mid D^{p-\frac{r}{2}} D^{\frac{r}{2}} f_j \mid \le \parallel D^+ \parallel^{p-\frac{r}{2}} \parallel D^- \parallel^{p-\frac{r}{2}} \parallel D^{\frac{r}{2}} f_j \parallel$$

where we use the supremum norm for the bi-infinite vectors $(f_j)_{j\in\mathbb{Z}}$ and the induced norm for the operators. In particular, $\parallel D^+ \parallel = \parallel D^- \parallel = 2$. The central difference $D^{\frac{r}{2}} f_j$ of the order r can be represented in the form

$$D^{\frac{r}{2}} f_j = D_j^{\frac{r}{2}} f((j + \frac{m}{2})h) = h^r f^{(r)}(\xi_j) \text{ with a } \xi_j \in \left((j + \frac{m}{2} - \frac{r}{2})h, (j + \frac{m}{2} + \frac{r}{2})h\right).$$

Hence $\sup_{j\in\mathbb{Z}} \mid D^p f_j \mid \le 2^{2p-r} h^r \sup_{x\in\mathbb{R}} \mid f^{(r)}(x) \mid$, and estimate (8.21) follows in the case of even r. For odd r we represent $D^p f_j = D^{p-\frac{r+1}{2}} D^+ \cdot D^- D^{\frac{r-1}{2}} f_j$ and estimating in a similar way as above we obtain again (8.21). □

For $r = 0$ estimate (8.21) implies

$$\parallel Q_{n,m-1} - Q^{(p-1)}_{n,m-1} \parallel_{BC(\mathbb{R})\to BC(\mathbb{R})} \le c'_m.$$

The values of c'_m for $m = 3, ..., 10$ are given in the following table:

m	3	4	5	6	7	8	9	10
c'_m	2	3	4.9554	8.0512	13.1157	21.2309	34.2665	55.1381

Together with the estimate of $\parallel Q_{n,m-1} \parallel_{BC(\mathbb{R})\to BC(\mathbb{R})}$ presented in the end of Section 8.3 we see that

$$\parallel Q^{(p-1)}_{n,m-1} \parallel_{BC(\mathbb{R})\to BC(\mathbb{R})} \le \parallel Q_{n,m-1} \parallel_{BC(\mathbb{R})\to BC(\mathbb{R})} + \parallel Q_{n,m-1} - Q^{(p-1)}_{n,m-1} \parallel_{BC(\mathbb{R})\to BC(\mathbb{R})}$$

remains for $m \le 10$ in acceptable bounds for numerics.

8.6. **Error bounds for quasi-interpolants and the Wiener interpolant.** We first establish an error estimate for the quasi-interpolants and after that we exploit Theorem 8.5 to extend the estimate for the Wiener interpolant. Introduce the notation

$$\mathcal{E}_{m,[a,b]}(f) = \inf_{g \in \mathcal{P}_m} \max_{a \le x \le b} \mid f(x) - g(x) \mid .$$

This is the error of the best uniform approximation to f on the interval $[a, b]$ by polynomials of degree $\le m$.

Theorem 8.6. For $p \in \mathbb{N}$, $x \in [ih, (i+1)h]$, $i \in \mathbb{Z}$, it holds

$$\mid f(x) - (Q^{(p-1)}_{n,m-1} f)(x) \mid \le c^{(p-1)}_m \mathcal{E}_{m_p, [(i-p+2-\frac{m}{2})h, (i+p-1+\frac{m}{2})h]}(f), \tag{8.22}$$

$$c^{(p-1)}_m = 2 + \sum_{q=1}^{p-1} \mid \gamma_q \mid 2^{2q} \le 2 + (\sqrt{2p-1} - 1) c_m$$

where $m_p = \min\{m-1, 2p-1\}$ and the constant c_m is defined in (8.9).

Proof. From (8.21) with $r = 2p$ we observe that $Q^{(p-1)}_{n,m-1} g = Q_{n,m-1} g$ for $g \in \mathcal{P}_{2p-1}$. As explained in the end of Section 8.3, $Q_{n,m-1} g = g$ for $g \in \mathcal{P}_{m-1}$. Thus $Q^{(p-1)}_{n,m-1} g = g$ for $g \in \mathcal{P}_{m_p}$, $m_p = \min\{m-1, 2p-1\}$. We obtain $f - Q^{(p-1)}_{n,m-1} f = (f-g) - Q^{(p-1)}_{n,m-1}(f-g)$ and for $x \in [ih, (i+1)h]$,

$$f(x) - (Q^{(p-1)}_{n,m-1} f)(x) = f(x) - g(x) - \sum_{k=i-m+1}^{i} d^{(p-1)}_k B_{m-1}(nx - k), \quad g \in \mathcal{P}_{m_p}, \tag{8.23}$$

where $d^{(p-1)}_k$ now corresponds to $f - g$ (cf.(8.19)),

$$d^{(p-1)}_k = \sum_{q=0}^{p-1} \gamma_q D^q (f_k - g_k), \quad f_k = f((k + \frac{m}{2})h), \quad g_k = g((k + \frac{m}{2})h).$$

Since $\mid D^q(f_k - g_k) \mid \le 2^{2q} \max_{|j-k| \le q} \mid f_j - g_j \mid$, $q \ge 1$, we obtain the estimate

$$\mid d^{(p-1)}_k \mid \le \sum_{q=0}^{p-1} \mid \gamma_q \mid\mid D^q(f_k - g_k) \mid \le [1 + \sum_{q=1}^{p-1} \mid \gamma_q \mid 2^{2q}] \max_{|j-k| \le p-1} \mid f_j - g_j \mid,$$

and (8.23) yields

$$\mid f(x) - (Q^{(p-1)}_{n,m-1} f)(x) \mid \le \mid f(x) - g(x) \mid + \max_{i-m+1 \le k \le i} \mid d^{(p-1)}_k \mid$$

$$\le \mid f(x) - g(x) \mid + [1 + \sum_{q=1}^{p-1} \mid \gamma_q \mid 2^{2q}] \max_{i-m+1 \le k \le i} \max_{|j-k| \le p-1} \mid f((j + \frac{m}{2})h) - g((j + \frac{m}{2})h) \mid$$

$$\le c^{(p-1)}_m \max_{(i-p+2-\frac{m}{2})h \le \xi \le (i+p-1+\frac{m}{2})h} \mid f(\xi) - g(\xi) \mid .$$

Since $g \in \mathcal{P}_{m_p}$ is in our argument arbitrary, this proves (8.22). It remains to estimate the constant $c^{(p-1)}_m = 2 + \sum_{q=1}^{p-1} \mid \gamma_q \mid 2^{2q}$. According to (8.15), (8.9)

$$\mid \gamma_q \mid \le \sum_{\nu=1}^{\mu} \frac{\mid z_\nu \mid^q (1 - \mid z_\nu \mid)}{(1 + \mid z_\nu \mid)^{2q+1}} \frac{\mid z_\nu \mid^{\mu-1}}{\mid P'_{2\mu}(z_\nu) \mid} \le \max_{0 \le x \le 1} \frac{x^q (1-x)}{(1+x)^{2q+1}} c_m, \quad q \ge 1.$$

The maximum of the function $x \mapsto \frac{x^q(1-x)}{(1+x)^{2q+1}}$ in $[0,1]$ is attained at $x = x_\star := 1+\frac{1}{q}-\sqrt{\frac{2}{q}+\frac{1}{q^2}}$. Since $x_\star^q(1+x_\star)^{-2q} \le 2^{-2q}$, $(1-x_\star)/(1+x_\star) = (2q+1)^{-\frac{1}{2}}$, we obtain $\mid \gamma_q \mid \le (2q+1)^{-\frac{1}{2}} 2^{-2q} c_m$ and

$$\sum_{q=1}^{p-1} \mid \gamma_q \mid 2^{2q} \le c_m \sum_{q=1}^{p-1} (2q+1)^{-\frac{1}{2}} \le c_m \int_0^{p-1} (2t+1)^{-\frac{1}{2}} dt = c_m(\sqrt{2p-1)}-1)$$

that completes the proof of the theorem. □

Remark 8.7. Exploiting the Chebyshev interpolant $\Pi'_{m-1} f \in \mathcal{P}_{m-1}$ in the role of g we obtain by Lemma 7.2 that for $f \in C^m[ah, bh]$, it holds

$$\mathcal{E}_{m-1,[ah,bh]}(f) \le \frac{(b-a)^m}{m! 2^{2m-1}} h^m \max_{ah \le x \le bh} \mid f^{(m)}(x) \mid . \tag{8.24}$$

The smallest value of p for which estimate (8.22) has the order $O(h^m)$ is $p = \mu + 1$ (still with $\mu = \mathrm{int}((m-1)/2)$):

(8.25)

$$\sup_{ih \le x \le (i+1)h} \mid f(x) - (Q^{(\mu)}_{n,m-1} f)(x) \mid \le c_m^{(\mu)} \frac{(m+2\mu-1)^m}{m! 2^{2m-1}} h^m \sup_{(i-\mu+1-\frac{m}{2})h \le x \le (i+\mu+\frac{m}{2})h} \mid f^{(m)}(x) \mid,$$

$$c_m^{(\mu)} = 2 + \sum_{q=1}^{\mu} \mid \gamma_q \mid 2^{2q} \le 2 + (\sqrt{2\mu+1} - 1) c_m.$$

Example 8.8. For $m = 3$ (quadratic splines), $p = 2$, $x \in [ih, (i+1)h]$, formulae (8.19), (8.20) and (8.22) yield after elementary calculation $\gamma_1 = -\frac{1}{8}$,

$$(Q^{(1)}_{n,2} f)(x) = \sum_{k=i-2}^{i} \left(-\frac{1}{8} f((k+\frac{1}{2}))h + \frac{5}{4} f((k+\frac{3}{2})h) - \frac{1}{8} f((k+\frac{5}{2})h \right) B_2(nx-k),$$

$$\mid f(x) - (Q^{(1)}_{n,2} f)(x) \mid \le \frac{5}{2} \mathcal{E}_{2,[(i-\frac{3}{2})h,(i+\frac{5}{2})h]}(f) \le \frac{5}{6} h^3 \max_{(i-\frac{3}{2})h \le x \le (i+\frac{5}{2})h} \mid f^{(3)}(x) \mid .$$

Example 8.9. For $m = 4$ (cubic splines), $p = 2$, $x \in [ih, (i+1)h]$ similar calculations yield $\gamma_1 = -\frac{1}{6}$,

$$(Q^{(1)}_{n,3} f)(x) = \sum_{k=i-3}^{i} \left(-\frac{1}{6} f((k+1)h)) + \frac{4}{3} f((k+2)h)) - \frac{1}{6} f((k+3)h)) \right) B_3(nx-k),$$

$$\mid f(x) - (Q^{(1)}_{n,3} f)(x) \mid \le \frac{8}{3} \mathcal{E}_{3,[(i-2)h,(i+3)h]}(f) \le \frac{625}{1152} h^4 \max_{(i-2)h \le x \le (i+3)h} \mid f^{(4)}(x) \mid .$$

Now we return to the Wiener interpolant $Q_{n,m-1} f$ determined by (8.1), (8.6), (8.9).

Theorem 8.10. If $f \in C^{(m)}(\mathbb{R})$ and $f^{(m)}(x)$ is bounded in $\mathbb{R}$ then

$$\sup_{x \in \mathbb{R}} \mid f(x) - (Q_{n,m-1} f)(x) \mid \le c''_m h^m \sup_{x \in \mathbb{R}} \mid f^{(m)}(x) \mid, \tag{8.26}$$

$$c''_m = c_m^{(\mu)} \frac{(m+2\mu-1)^m}{m! 2^{2m-1}} + c'_m 2^{-m}$$

with c'_m and $c_m^{(\mu)}$ defined in (8.21) and (8.25).

Proof. Theorem 8.5 with $r = m$ yields

$$\sup_{x \in \mathbb{R}} | (Q_{n,m-1} f) - (Q_{n,m-1}^{(\mu)} f)(x) | \leq c'_m 2^{-m} h^m \sup_{x \in \mathbb{R}} | f^{(m)}(x) | .$$

Together with (8.25) this implies (8.26). □

Remark 8.11. Modifying the choice of p and r we obtain the estimate

$$\sup_{x \in \mathbb{R}} | f(x) - (Q_{n,m-1} f)(x) | \leq c_{m,r} h^r \sup_{x \in \mathbb{R}} | f^{(r)}(x) |, \quad 0 \leq r \leq m,$$

where we assume that $f \in C^{(r)}(\mathbb{R})$ and $f^{(r)}(x)$ is bounded in $\mathbb{R}$. A detailed proof is left as an exercise.

Remark 8.12. If f is bounded and uniformly continuous on $\mathbb{R}$ then

$$\sup_{x \in \mathbb{R}} | f(x) - (Q_{n,m-1} f)(x) | \to 0 \text{ as } n \to \infty.$$

The proof can be constructed using the equalities $\sum_{k \in \mathbb{Z}} B_{m-1}(nx-k) \equiv 1$ and $\sum_{k \in \mathbb{Z}} a_k = 1$ for a_k in (8.6) implying

$$f(x) - (Q_{n,m-1} f)(x) = \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} a_{k-j} [f(x) - f((j + \frac{m}{2})h)] B_{m-1}(nx - k).$$

For $x \in [ih, (i+1)h]$, $i \in \mathbb{Z}$, this takes the form

$$f(x) - (Q_{n,m-1} f)(x) = \sum_{k=i-m+1}^{i} \sum_{j \in \mathbb{Z}} a_{k-j} [f(x) - f((j + \frac{m}{2})h)] B_{m-1}(nx - k).$$

For a given ε-accuracy, also the second sum can be reduced to a finite one with $| j - k | \leq N_\varepsilon$. The arguments x and $j + \frac{m}{2}$ become close to one another uniformly with respect to x, and we can make use of the uniform continuity of f. A detailed argument is left as an exercise.

8.7. **Interpolation of periodic functions.** Introduce the space $C_{\text{per}}(\mathbb{R})$ of continuous 1-periodic functions equipped with the usual supremum norm,

$$\| u \|_\infty = \max_{0 \leq x \leq 1} | u(x) | = \sup_{x \in \mathbb{R}} | u(x) | .$$

Introduce also the space $C^m_{\text{per}}(\mathbb{R}) = C^m(\mathbb{R}) \cap C_{\text{per}}(\mathbb{R})$ of C^m-smooth 1-periodic functions.

Previos results concerning the Wiener interpolant and qusi-interpolants, in particular Theorem 8.10, can be applied to 1-periodic functions. Formula (8.6) can be rewritten so that the summation extends only over one period. An equivalent form of Wiener interpolant can be presented using the periodization of B-splines as follows.

Introduce the B-splines $B_{n,m-1,j}(x) = B_{m-1}(nx - j)$, their periodization

$$\widetilde{B}_{n,m-1,j}(x) = \sum_{k\in\mathbb{Z}} B_{n,m-1,j}(x+k) = \sum_{k\in\mathbb{Z}} B_{m-1}(nx + nk - j), \;\; j \in \mathbb{Z},$$

and the subspace of 1-periodic splines

$$\widetilde{S}_{n,m-1} := \operatorname{span}\{\widetilde{B}_{n,m-1,j} : j \in \mathbb{Z}\} \subset C_{\text{per}}(\mathbb{R}).$$

Clearly, $\widetilde{B}_{n,m-1,j} = \widetilde{B}_{n,m-1,j+n}$, $j \in \mathbb{Z}$, hence $\dim \widetilde{S}_{n,m-1} = n$; a basis of $\widetilde{S}_{n,m-1}$ is constituted, for instance, by $\widetilde{B}_{n,m-1,j}$, $j = 1, ...n$. Define the interpolation projection operator $\widetilde{Q}_{n,m-1} : C_{\text{per}}(\mathbb{R}) \to \widetilde{S}_{n,m-1}$ by the conditions

$$\widetilde{Q}_{n,m-1}f \in \widetilde{S}_{n,m-1}, \;\; (\widetilde{Q}_{n,m-1}f)((i + \frac{m}{2})h) = f((i + \frac{m}{2})h), \;\; i \in \mathbb{Z}, \;\; f \in C_{\text{per}}(\mathbb{R}).$$

The coefficients d_j of $\widetilde{Q}_{n,m-1}f = \sum_{j=1}^{n} d_j \widetilde{B}_{n,m-1,j}$ can be determined solving the $n \times n$ system of linear algebraic equations

$$\sum_{j=1}^{n} d_j \widetilde{B}_{n,m-1,j}((i + \frac{m}{2})h) = f((i + \frac{m}{2})h), \quad i = 1, ..., n. \tag{8.27}$$

For $f \in C_{\text{per}}(\mathbb{R})$, it holds $\widetilde{Q}_{n,m-1}f = Q_{n,m-1}f$. Theorem 8.10 enables to establish the estimate

$$\| f - \widetilde{Q}_{n,m-1}f \|_\infty \le c''_m n^{-m} \| f^{(m)} \|_\infty, \quad f \in C^m_{\text{per}}(\mathbb{R}^m),$$

where the constant c''_m is independent of n and f. Using other ideas, the smallest possible value of the constant has been determined in the literature. We formulate this result without proof.

Theorem 8.13 ([4], p. 260). For $f \in C^m_{\text{per}}(\mathbb{R}^m)$ and even n, it holds

$$\| f - \widetilde{Q}_{n,m-1}f \|_\infty \le \gamma_m \pi^{-m} n^{-m} \| f^{(m)} \|_\infty \tag{8.28}$$

where γ_m *is* the *so-called Favard constant* defined by

$$\gamma_m = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k(m+1)}}{(2k+1)^{m+1}}$$

and satisfying the inequalities and the limit relation

$$\frac{\pi^2}{8} = \gamma_2 < \gamma_4 < ... < \frac{4}{\pi} < ... < \gamma_3 < \gamma_1 = \frac{\pi}{2}; \;\; \gamma_m \to \frac{4}{\pi} \text{ as } m \to \infty.$$

The constant $c_m = \gamma_m \pi^{-m}$ in (8.28) is the smallest possible for the class of functions $f \in C^m_{\text{per}}(\mathbb{R}^m)$. Moreover, (8.28) realizes the Kolmogorov n-width of the set $F^{(m)} := \{f \in C^m_{\text{per}}(\mathbb{R}) : \| f^{(m)} \|_\infty = 1\}$ in $C_{\text{per}}(\mathbb{R})$ which for even n occurs to be equal to $\gamma_m \pi^{-m} n^{-m}$ (see Section 1.1 for the definiion of the Kolmogorov n-width). Hence, for arbitrary n-dimensional subspace E_n of $C_{\text{per}}(\mathbb{R})$, no approximation procedure (linear or nonlinear!) exists that assignes to $f \in C^m_{\rm per}(\mathbb{R})$ an $f_n \in E_n$ so that $\| f - f_n \|_\infty \le \varepsilon_n \| f^{(m)} \|_\infty$ for all $f \in C^m_{\rm per}(\mathbb{R})$ with an $\varepsilon_n < \gamma_m \pi^{-m} n^{-m}$.

Probably, Theorem 8.13 can be used to reduce the constant c''_m also in Theorem 8.10. The argument idea could be as follows: for fixed $x_0 \in \mathbb{R}$ and given $f \in C^m(\mathbb{R})$, take a function $\widetilde{f} \in C^m_{\rm per}(\mathbb{R})$ such that $f(x) = \widetilde{f}(x)$ for $| x - x_0 | \le \delta =$ const, then the difference between $(Q_{n,m-1}f)(x_0)$ and $(\widetilde{Q}_{n,m-1}\widetilde{f})(x_0)$ is small due to the exponential decay of the sequence a_k defining the Wiener interpolant $Q_{n,m-1}f$; a problem arises about as small as possible norm $\| \widetilde{f}^{(m)} \|_\infty$.

9. Spline collocation and quasi-collocation for weakly singular integral equations

9.1. **Operator form of the quasi-collocation method.** Let us return to the weakly singular integral equation (5.1) , $u = Tu + f$, with $K \in \mathcal{S}^{m,\nu}$, $f \in C^{m,\nu}(0,1)$, $m \ge 2$, $0 < \nu < 1$. Using the smoothing change of variables we rewrite (5.1) in the form (5.15), $v = T_\varphi v + f_\varphi$, in the inteval $0 \le t \le 1$. Introduce the extension operator (already expoloited in Section 7)

$$E_\delta : C[0,1] \to C[-\delta, 1+\delta], \quad (E_\delta f)(t) = \left\{ \begin{array}{ll} f(0), & -\delta \le t \le 0 \\ f(t), & 0 \le t \le 1 \\ f(1), & 1 \le t \le 1+\delta \end{array} \right\}$$

and the spline quasi-interpolation operator

$$Q^{(\mu)}_{n,m-1} : C[-\delta, 1+\delta] \to C[0,1], \quad (Q^{(\mu)}_{n,m-1} v)(t) = \sum_{i=-m+1}^{n-1} \left(\sum_{q=0}^{\mu} \gamma_q D^q v_i \right) B_{m-1}(nt - i)$$

where $\mu = $ int $((m-1)/2)$, $D = D^+D^-$ is the second order central difference operator, γ_q is defined in (8.15) and $v_i = v((i + \frac{m}{2})h)$. We approximate (5.2) by the finite dimensional equation

$$v_n = Q^{(\mu)}_{n,m-1} E_\delta T_\varphi v_n + Q^{(\mu)}_{n,m-1} E_\delta f_\varphi. \tag{9.1}$$

In analogy to the collocation method, we call this method *spline quasicollocation method.* Note that $Q^{(\mu)}_{n,m-1} E_\delta : C[0,1] \to C[0,1]$ is not a projection operator but this is no obstacle to obtain an effective method.

Theorem 9.1. Let $K \in \mathcal{S}^{m,\nu}$, $f \in C^{m,\nu}(0,1)$, $m \ge 3$, $\nu < 1$, and let $\varphi : [0,1] \to [0,1]$ satisfy the conditions of Theorem 5.3. Further, assume that $\mathcal{N}(I - T) = \{\mathbf{0}\}$ (or equivalently, $\mathcal{N}(I - T_\varphi) = \{\mathbf{0}\}$). Then there exists an n_0, such that for $n \ge n_0$ the quasicollocation equation (9.1) has a unique solution v_n. The error of v_n can be estimated by

$$\| v - v_n \|_\infty \le c n^{-m} \| v^{(m)} \|_\infty \tag{9.2}$$

where $v(t) = u_\varphi(t) = u(\varphi(t))$ is the solution of (5.15), $u(x)$ is the solution of (5.1); by Corollary 5.4, $v \in C^m[0,1]$. The constant c in (9.2) is independent of n and f (it depends on K, m and φ).

Proof. This formulation is almost identical to that in Theorem 7.5, the only difference is that now the claims concern the spline quasicollocation method (9.1). The proof of the theorem repeats the argument in the proof of Theorem 7.5. There is no need to reproduce all the details of the proof again. We comment only on details that are different from those in the proof of Theorem 7.5.

First of all, we have to justify the pointwise convergence of $Q^{(\mu)}_{n,m-1}E_\delta$ to I in $C[0,1]$. This follows by Banach–Steinhaus theorem (Theorem 2.2): (i) clearly

$$\| Q^{(\mu)}_{n,m-1}E_\delta \|_{C[0,1]\to C[0,1]} \le \| Q^{(\mu)}_{n,m-1} \|_{C[-\delta,1+\delta]\to C[0,1]} \le \text{const}, \quad n \in \mathbb{N};$$

(ii) the set

$$V^{(m)} := \{v \in C^m[0,1] : v^{(j)}(0) = v^{(j)}(1) = 0,\ j = 1, ..., m\}$$

is dense in $C[0,1]$, $E_\delta V^{(m)} \subset C^m[-\delta, 1+\delta]$ and by (8.25) $Q^{(\mu)}_{n,m-1}E_\delta v \to v$ for $v \in V^{(m)}$ as $n \to \infty$ with the estimate

$$\| v - Q^{(\mu)}_{n,m-1}E_\delta v \|_\infty \le ch^m \| v^{(m)} \|_\infty . \tag{9.3}$$

Now similarly as estimate (7.19) in the proof of Theorem 7.5, we obtain for the solution v of equation (5.2) and solution v_n of equation (9.1)

$$\| v - v_n \|_\infty \le \kappa_n \| v - Q^{(\mu)}_{n,m-1}E_\delta v \|_\infty \tag{9.4}$$

where

$$\kappa_n := \| (I - Q^{(\mu)}_{n,m-1}E_\delta T_\varphi)^{-1} \|_{C[0,1]\to C[0,1]} \le \frac{\kappa}{1-\kappa\varepsilon_n} \to \kappa \text{ as } n \to \infty,$$

$$\varepsilon_n := \| T_\varphi - Q^{(\mu)}_{n,m-1}E_\delta T_\varphi \|_{C[0,1]\to C[0,1]} \to 0 \quad \text{as } n \to \infty.$$

By Corollary 5.4, $v \in V^{(m)}$ for the solution of (5.15), thus estimate (9.4), (9.3) holds for the solution of (5.15) and (9.1) implying (9.2). □

9.2. **Matrix form of the spline quasicollocation method.** It is somewhat helpful for the implementation of quasi-interpolations to represent $Q^{(\mu)}_{n,m-1}v$ in the difference-free form

$$(Q^{(\mu)}_{n,m-1}v)(t) = \sum_{i=-m+1}^{n-1} \left(\sum_{q=-\mu}^{\mu} w_q v_{i+q} \right) B_{m-1}(nt - i), \quad 0 \le t \le 1,$$

with appropriate weights w_q. We already used this representation form in Examples 8.8 and 8.9.

The solution v_n of the quasi-collocation equation (9.1) has the form

$$v_n(t) = \sum_{i=-m+1}^{n-1} c_i B_{m-1}(nt - i) \tag{9.5}$$

in which we have to determine the $n+m-1$ unknown parameters c_i, $i=-m+1,...,n-1$. The two terms in the r.h.s. of (9.1) are

$$(Q^{(\mu)}_{n,m-1}E_\delta f_\varphi)(t) = \sum_{i=-m+1}^{n-1}\left(\sum_{q=-\mu}^{\mu} w_q (E_\delta f_\varphi)_{i+q}\right) B_{m-1}(nt-i)$$

and

$$(Q^{(\mu)}_{n,m-1}E_\delta T_\varphi v_n)(t) = \sum_{i=-m+1}^{n-1}\left(\sum_{q=-\mu}^{\mu} w_q (E_\delta T_\varphi v_n)_{i+q}\right) B_{m-1}(nt-i).$$

Here

$$(E_\delta T_\varphi v_n)(t) =$$

$$\left\{\begin{array}{ll} \int_0^1 K_\varphi(0,s)v_n(s)ds, & t<0 \\ \int_0^1 K_\varphi(t,s)v_n(s)ds, & 0\le t\le 1 \\ \int_0^1 K_\varphi(1,s)v_n(s)ds, & t>1 \end{array}\right\} = \sum_{j=-m+1}^{n-1}\left\{\begin{array}{ll} \int_0^1 K_\varphi(0,s)B_{m-1}(ns-j)ds, & t<0 \\ \int_0^1 K_\varphi(t,s)B_{m-1}(ns-j)ds, & 0\le t\le 1 \\ \int_0^1 K_\varphi(1,s)B_{m-1}(ns-j)ds, & t>1 \end{array}\right\} c_j,$$

thus

$$(E_\delta T_\varphi v_n)_i = \sum_{j=-m+1}^{n-1}\left\{\begin{array}{ll} \int_0^1 K_\varphi(0,s)B_{m-1}(ns-j)ds, & (i+\frac{m}{2})h<0 \\ \int_0^1 K_\varphi((i+\frac{m}{2})h,s)B_{m-1}(ns-j)ds, & 0\le (i+\frac{m}{2})h\le 1 \\ \int_0^1 K_\varphi(1,s)B_{m-1}(ns-j)ds, & (i+\frac{m}{2})h>1 \end{array}\right\} c_j.$$

From equality of coefficients by $B_{m-1}(nt-i)$, $i=-m+1,...,n-1$, in the l.h.s and r.h.s. of equation (9.1) we obtain the following system of linear equations for the determining of the parameters c_i, $i=-m+1,...,n-1$, of v_n:

$$c_i = \sum_{j=-m+1}^{n-1}\alpha_{i,j}c_j+\beta_i,\quad i=-m+1,...,n-1, \tag{9.6}$$

where

$$\beta_i=\sum_{q=-\mu}^{\mu} w_q\sigma_{i+q},\quad \alpha_{i,j}=\sum_{q=-\mu}^{\mu} w_q\tau_{i+q,j},\quad i,j=-m+1,...,n-1,$$

$$\sigma_i=\left\{\begin{array}{ll} f_\varphi(0), & (i+\frac{m}{2})h<0 \\ f_\varphi((i+\frac{m}{2})h), & 0\le (i+\frac{m}{2})h\le 1 \\ f_\varphi(1), & (i+\frac{m}{2})h>1 \end{array}\right\},$$

$$\tau_{i,j}=\left\{\begin{array}{ll} \int_0^1 K_\varphi(0,s)B(ns-j)ds, & (i+\frac{m}{2})h<0 \\ \int_0^1 K_\varphi((i+\frac{m}{2})h,s)B(ns-j)ds, & 0\le (i+\frac{m}{2})h\le 1 \\ \int_0^1 K_\varphi(1,s)B(ns-j)ds, & (i+\frac{m}{2})h>1 \end{array}\right\}.$$

Having found c_i, $i=-m+1,...,n-1$, by solving the system (9.6), the quasi-collocation solution v_n is given by (9.5).

9.3. **Periodization of weakly singular integral equations and collocation method.** Introduce the one dimensional projection operator

$$\Pi:\ C[0,1]\to C[0,1],\quad (\Pi v)(x)=[v(1)-v(0)]x$$

(clearly $\Pi^2 = \Pi$). Equation (5.15), $v = T_\varphi v + f_\varphi$, is equivalent to the system of two equations

$$\Pi v = \Pi T_\varphi \Pi v + \Pi T_\varphi (I-\Pi)v + \Pi f_\varphi,$$

$$(I-\Pi)v = (I-\Pi)T_\varphi(\Pi v) + (I-\Pi)T_\varphi(I-\Pi)v + (I-\Pi)f_\varphi$$

with unknowns Πv and $(I-\Pi)v =: \widetilde{v}$. With respect to the unknowns $\alpha := v(1) - v(0) \in \mathbb{R}$ and $\widetilde{v} \in C[0,1]$, this system can be written as

$$\alpha = \theta\alpha + \int_0^1 \sigma(s)\widetilde{v}(s)ds + \beta,\ \ \widetilde{v}(t) = \alpha\widetilde{\tau}(t) + \int_0^1 \widetilde{K}_\varphi(t,s)\widetilde{v}(s)ds + \widetilde{f}_\varphi(t) \tag{9.7}$$

where

$$\beta = f_\varphi(1) - f_\varphi(0),\ \ \widetilde{f}_\varphi(t) = f_\varphi(t) - \beta t,$$

$$\sigma(s) = K_\varphi(1,s) - K_\varphi(0,s),\ \ \theta = \int_0^1 \sigma(s)s ds,$$

$$\widetilde{\tau}(t) = \int_0^1 K_\varphi(t,s)s ds - \theta t,\ \ \widetilde{K}_\varphi(t,s) = K_\varphi(t,s) - t\sigma(s).$$

If $(\alpha, \widetilde{v})$ is a soluion to system (9.7) then $v(t) = \alpha t + \widetilde{v}(t)$ is a solution of equation (5.15). Observe that

$$\widetilde{\tau}(0) = \widetilde{\tau}(1),\ \ \widetilde{K}_\varphi(0,s) = \widetilde{K}_\varphi(1,s);$$

we extend $\widetilde{\tau}(t)$ and $\widetilde{K}_\varphi(t,s)$ into 1-periodic functions of t maintaining the same designations for the extensions. For $v \in C[0,1]$, $\widetilde{v} = (I-\Pi)v$ we also have $\widetilde{v}(0) = \widetilde{v}(1)$, and the may treat $\widetilde{v}$ and $\widetilde{f}_\varphi$ as 1-periodic functions. So we can consider system (9.7) as an equation in the space $X = \mathbb{R} \times C_{\mathrm{per}}(\mathbb{R})$,

$$\begin{pmatrix} \alpha \\ \widetilde{v} \end{pmatrix} = \begin{pmatrix} \theta & \Sigma \\ \widetilde{\tau} & \widetilde{T}_\varphi \end{pmatrix} \begin{pmatrix} \alpha \\ \widetilde{v} \end{pmatrix} + \begin{pmatrix} \beta \\ \widetilde{f}_\varphi \end{pmatrix} \tag{9.8}$$

where

$$\Sigma : C_{\mathrm{per}}(\mathbb{R}) \to \mathbb{R},\ \ \Sigma\widetilde{v} = \int_0^1 \sigma(s)\widetilde{v}(s)ds,$$

$$\widetilde{T}_\varphi : C_{\mathrm{per}}(\mathbb{R}) \to C_{\mathrm{per}}(\mathbb{R}),\ \ (\widetilde{T}_\varphi\widetilde{v})(t) = \int_0^1 \widetilde{K}_\varphi(t,s)\widetilde{v}(s)ds.$$

We build the collocation solution of (9.8) with the help of periodic interpolation projection operator $\widetilde{Q}_{n,m-1}$ introduced in Section 8.7:

$$\begin{pmatrix} \alpha_n \\ \widetilde{v}_n \end{pmatrix} = \begin{pmatrix} \theta & \Sigma \\ \widetilde{Q}_{n,m-1}\widetilde{\tau} & \widetilde{Q}_{n,m-1}\widetilde{T}_\varphi \end{pmatrix} \begin{pmatrix} \alpha_n \\ \widetilde{v}_n \end{pmatrix} + \begin{pmatrix} \beta \\ \widetilde{Q}_{n,m-1}\widetilde{f}_\varphi \end{pmatrix}. \tag{9.9}$$

This is a system with repect to $\begin{pmatrix} \alpha_n \\ \widetilde{v}_n \end{pmatrix} \in \mathbb{R}\times\widetilde{S}_{n,m-1} \subset \mathbb{R}\times C_{\mathrm{per}}(\mathbb{R})$; the approximate solution v_n to equation (5.15) is given by

$$v_n(t) = \alpha_n t + \widetilde{v}_n(t),\ \ 0 \le t \le 1. \tag{9.10}$$

The matrix form of system (9.9) reads as follows: $\widetilde{v}_n = \sum_{j=1}^n c_j \widetilde{B}_{n,m-1,j}$,

$$\alpha_n = \theta\alpha_n + \sum_{j=1}^n \int_0^1 \sigma(s)\widetilde{B}_{n,m-1,j}(s)ds\, c_j + \beta, \tag{9.11}$$

$$\sum_{j=1}^n \widetilde{B}_{n,m-1,j}((i+\frac{m}{2})h)c_j = \alpha_n\widetilde{\tau}((i+\frac{m}{2})h) + \sum_{j=1}^n \int_0^1 [\widetilde{K}_\varphi((i+\frac{m}{2})h,s)\widetilde{B}_{n,m-1,j}(s)ds\, c_j$$
$$+\widetilde{f}_\varphi((i+\frac{m}{2})h), \quad i=1,...,n.$$

This is a system of $n+1$ linear algebraic equations with respect to $n+1$ unknowns α_n, $c_1,...,c_n$. Here $\widetilde{\tau}(t)$, $\widetilde{K}_\varphi(t,s)$ and $\widetilde{f}_\varphi(t)$ are understood as 1-periodic functions of t.

Theorem 9.2. Assume the conditions of Theorem 9.1. Then there exists an n_0, such that for $n \geq n_0$ the collocation system (9.11) has a unique solution α_n, $c_1,...,c_n$. The accuracy of v_n defined by (9.10) can be estimated by (9.2).

Proof. $X := \mathbb{R}\times C_{\text{per}}(\mathbb{R})$ is a Banach space with the norm $\| (\alpha,\widetilde{v}) \| = | \alpha | + \| \widetilde{v} \|_\infty$. Denote by $I_X = \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix}$ where I is the identity operator in $C_{\text{per}}(\mathbb{R})$ and

$$\mathcal{T} = \begin{pmatrix} \theta & \Sigma \\ \widetilde{\tau} & \widetilde{T}_\varphi \end{pmatrix} : X\to X, \ \mathcal{T}_n = \begin{pmatrix} \theta & \Sigma \\ \widetilde{Q}_{n,m-1}\widetilde{\tau} & \widetilde{Q}_{n,m-1}\widetilde{T}_\varphi \end{pmatrix} : X \to X.$$

The operators $\mathcal{T} : X \to X$ and $\mathcal{T}_n : X \to X$ are compact that easily follows from the compactness of the operator $\widetilde{T}_\varphi : C_{\text{per}}(\mathbb{R}) \to C_{\text{per}}(\mathbb{R})$. If

$$\begin{pmatrix} \alpha \\ \widetilde{v} \end{pmatrix} = \begin{pmatrix} \theta & \Sigma \\ \widetilde{\tau} & \widetilde{T}_\varphi \end{pmatrix}\begin{pmatrix} \alpha \\ \widetilde{v} \end{pmatrix}$$

for some $(\alpha,\widetilde{v}) \in X$ then $v := \alpha t + \widetilde{v}$ is the solution of the homogenous equation $v = T_\varphi v$, and by the assumption of the Theorem, $v = 0$ that implies $\alpha = 0$, $\widetilde{v} = 0$. Hence $I_X - \mathcal{T} : X \to X$ has a bounded inverse $(I_X - \mathcal{T})^{-1} : X \to X$. Further, since $\widetilde{T}_\varphi : C_{\text{per}}(\mathbb{R}) \to C_{\text{per}}(\mathbb{R})$ is compact and $\| \widetilde{v} - \widetilde{Q}_{n,m-1}\widetilde{v} \|_\infty \to 0$ for every $\widetilde{v} \in C_{\text{per}}(\mathbb{R})$, we have by Theorem 2.6 $\| (I - \widetilde{Q}_{n,m-1})\widetilde{T}_\varphi \|_{C_{\text{per}}(\mathbb{R})\to C_{\text{per}}(\mathbb{R})} \to 0$ that implies $\| \mathcal{T} - \mathcal{T}_n \|_{X\to X} \to 0$ as $n\to\infty$. For n such that $\| (I_X - \mathcal{T})^{-1} \|_{X\to X} \| \mathcal{T} - \mathcal{T}_n \|_{X\to X} < 1$, also $I_X - \mathcal{T}_n$ is invertible and

$$\zeta_n := \| (I_X - \mathcal{T}_n)^{-1} \|_{X\to X}$$
$$\leq \frac{\| (I_X - \mathcal{T})^{-1} \|_{X\to X}}{1 - \| (I_X - \mathcal{T})^{-1} \|_{X\to X} \| \mathcal{T} - \mathcal{T}_n \|_{X\to X}} \to \| (I_X - \mathcal{T})^{-1} \|_{X\to X} =: \zeta$$

as $n\to\infty$. In particular, collocation system (9.11) is uniquely solvable for all sufficiently large n. For the solution $(\alpha,\widetilde{v})$ of (9.8) and the solution $(\alpha_n,\widetilde{v}_n)$ of (9.9) we have

$$(I_X - \mathcal{T}_n)\{(\alpha,\widetilde{v}) - (\alpha_n,\widetilde{v}_n)\} = (I_X - \mathcal{T}_n)(\alpha,\widetilde{v}) - (\beta, \widetilde{Q}_{n,m-1}\widetilde{f}_\varphi)$$
$$= (I_X - \mathcal{T})(\alpha,\widetilde{v}) + (\mathcal{T} - \mathcal{T}_n)(\alpha,\widetilde{v}) - (\beta,\widetilde{Q}_{n,m-1}\widetilde{f}_\varphi)$$

$$= (\beta, \widetilde{f}_\varphi) + (0,\ \alpha(I - \widetilde{Q}_{n,m-1})\widetilde{\tau} + (I - \widetilde{Q}_{n,m-1})\widetilde{T}_\varphi \widetilde{v}) - (\beta, \widetilde{Q}_{n,m-1}\widetilde{f}_\varphi)$$

$$= (0, \alpha(I - \widetilde{Q}_{n,m-1})\widetilde{\tau} + (I - \widetilde{Q}_{n,m-1})\widetilde{T}_\varphi \widetilde{v} + (I - \widetilde{Q}_{n,m-1})\widetilde{f}_\varphi) = (0, (I - \widetilde{Q}_{n,m-1})\widetilde{v})$$

that implies

$$\mid \alpha - \alpha_n \mid + \parallel \widetilde{v} - \widetilde{v}_n \parallel_\infty = \parallel (\alpha, \widetilde{v}) - (\alpha_n, \widetilde{v}_n) \parallel_X \leq \zeta_n \parallel (I - \widetilde{Q}_{n,m-1})\widetilde{v} \parallel_\infty .$$

By Theorem 8.13, $\parallel (I - \widetilde{Q}_{n,m-1})\widetilde{v} \parallel_\infty \leq \gamma_m \pi^{-m} h^m \parallel \widetilde{v}^{(m)} \parallel_\infty$. For the solution $v(t) = \alpha t + \widetilde{v}(t)$ of equation (5.15) and the collocation solution $v_n(t) = \alpha_n t + \widetilde{v}_n(t)$, these estimate yield $\parallel v - v_n \parallel_\infty \leq c h^m \parallel v^{(m)} \parallel_\infty$ completing the proof of the Theorem. $\square$

Exercises and Problems

1. Prove the compactness in $C[0,1]$ of the Fredholm and Volterra integral operators with a continuous kernel, see Section 2.4.

2. Prove the Faa di Bruno differentiation formula (2.1). *Hint*: induction.

3. Establish the Leibnitz rule for $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^l$:

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^l [a(x,y)b(x,y)] = \sum_{j=0}^{l} \begin{pmatrix} l \\ j \end{pmatrix} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j a(x,y) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{l-j} b(x,y).$$

4. Show that the kernel (3.1) with $0 < \nu < 1$ belongs to $\mathcal{S}^{m,\nu}$ if $a \in C^m([0,1] \times [0,1])$ or, more generally, if $a \in \mathcal{S}^{m,-\delta}$, $\delta > 0$.

5. Show that the kernel $K(x,y) = a(x,y) \log \mid x - y \mid$ with $a \in C^m([0,1] \times [0,1])$ belongs to $\mathcal{S}^{m,0}$ or, more generally, if $a \in \mathcal{S}^{m,-\delta}$, $\delta > 0$.

6. Prove the claims of Lemma 3.1.

7. Prove Lemma 4.1.

8. Present a detailed proof of Theorem 4.3.

9. Prove the compactness of the imbedding $C^{m,\nu}(0,1) \subset C[0,1]$, $m \geq 1$, $\nu < 1$.

10. Prove (5.4) and the compactness of the imbedding operator.

11. Prove that the spaces $C^{m,\nu}(0,1)$ and $C^{m,\nu}(0,1]$ are complete.

12. Prove that $uv \in C^{m,\nu}(0,1)$ for $u, v \in C^{m,\nu}(0,1)$ and

$$\parallel uv \parallel_{C^{m,\nu}(0,1)} \leq c \parallel u \parallel_{C^{m,\nu}(0,1)} \parallel v \parallel_{C^{m,\nu}(0,1)}$$

with a constant c that is independent of u and v.

13. Prove that $\parallel u' \parallel_{C^{m-1,\nu+1}(0,1)} \leq \parallel u \parallel_{C^{m,\nu}(0,1)}$ for $u \in C^{m,\nu}(0,1)$, $m \geq 1$, $\nu < 0$.

14. Prove that equation (6.1) with $K \in \mathcal{S}^{m,\nu}(\triangle)$, $m \geq 0$, $\nu < 1$, $f \in C[0,1]$ has a unique solution $u \in C[0,1]$.

15. Present a detailed proof of Lemma 7.1.

16. Assume the conditions of Theorem 7.5 but purely $f_\varphi \in C[0,1]$. Prove that $\parallel v - v_n \parallel_\infty \to 0$ as $n \to \infty$. Relax the condition also for K assuming that $K \in \mathcal{S}^{0,\nu}$, $\nu < 1$.

17*. Present and prove a counterpart of Theorem 7.5 in case $m = 1$ using piecewise constant interpolants with the central dislocation of the interpolation points, cf. Section 7.3. Examine the superconvergence of the collocation solution at the collocation points, i.e., the convergence with a speed exceeding the global convergence speed $\| v - v_n \|_\infty \leq ch \| v' \|_\infty$, cf. [12]. Examine full discretizations of the method and two grid iteration schemes of complexity $O(n^2)$ flops to solve the discretized collocation system . Solve numerical examples and comment on them.

18. The nonvanishing coefficients $b_k = b_k^m = B_{m-1}(k + \frac{m}{2})$, $| k | \leq \mu = \mathrm{int}((m-1/2)$, can be computed recursively using the the recursive definition of B_{m-1} (see Section 8.1). Establish for $m \geq 3$ the recursion formula

$$b_k^m = \frac{1}{4(m-1)(m-2)} \left((m-2k)^2 b_{k-1}^{m-2} + 2(m^2 - 2m - 4k^2) b_k^{m-2} + (m+2k)^2 b_{k+1}^{m-2} \right).$$

19. Compute b_k^5, $| k | \leq 2$. Find characteristic roots.

20. Prove that $B_{m-1}^{(m-1)}(x) = (-1)^i \begin{pmatrix} m-1 \\ i \end{pmatrix}$ for $ih < x < (i+1)h$, $i = 0, ..., m-1$.

21. Prove the estimate $\| Q_{n,m-1} \|_{BC(\mathbb{R}) \to BC(\mathbb{R})} \leq \sum_{k \in \mathbb{Z}} | a_k |$, see Theorem 8.1.

22. Show that for $m \geq 3$, the null space $\mathcal{N}(\mathfrak{B})$ of the matrix $\mathfrak{B} = (b_{k-j})_{k,j \in \mathbb{Z}}$ in the space X of all bi-infinite vectors $(d_j)_{j \in \mathbb{Z}}$ is of the dimension 2μ and is spanned by $(z_\nu^j)_{j \in \mathbb{Z}}$, $\nu = 1, ..., 2\mu$, where z_ν, $\nu = 1, ..., 2\mu$, are the characteristic roots.

23. Present a detailed proof of (8.11).

24. Prove that $J\underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$ for $\underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$.

25. Present the formula for the quasi-interpolant $Q_{n,5}^2 f$ and an error estimate for it in the spirit of Examples 8.8 and 8.9. The characteristic roots for $m = 6$ $(m - 1 = 5)$ are given in Section 8.3.

26. Prove Remark 8.11. Characrerize the constant $c_{m,r}$.

27. Present a detailed proof of Remark 8.12.

28. The function $\beta_{m-1}(x) := \sum_{j \in \mathbb{Z}} a_j B_{m-1}(x - j)$ with a_j given in (8.9) is the so called *fundamental spline:* $\beta_{m-1}(i + \frac{m}{2}) = \delta_{i,0}$, $i \in \mathbb{Z}$ (prove this!), and hence the Wiener interpolant $Q_{n,m-1} f$ of $f \in BC(\mathbb{R})$ can be represented in the form (show this!)

$$(Q_{n,m-1} f)(x) = \sum_{k \in \mathbb{Z}} f_k \beta_{m-1}(nx - k), \quad f_k = f((k + \frac{m}{2})h).$$

Observe that $\mathrm{supp} \beta_{m-1} = \mathbb{R}$ but β_{m-1} decays exponentially as $| x | \to \infty$. Finally, prove that

$$\| Q_{n,m-1} \|_{BC(\mathbb{R}) \to BC(\mathbb{R})} = \max_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} | \beta_{m-1}(x + k) |$$

(the function $\gamma_{m-1}(x) := \sum_{k \in \mathbb{Z}} | \beta_{m-1}(x + k) |$ is 1-periodic). Thus $\| Q_{n,m-1} \|_{BC(\mathbb{R}) \to BC(\mathbb{R})}$ can be determined numerically. One can conjecture that $\beta_{m-1}(x)$ vanishes only at points $x = i + \frac{m}{2}$, $0 \neq i \in \mathbb{Z}$, and $\beta'_{m-1}(x) \neq 0$ at these points.

29*. Establish the counterpart of Theorem 9.1 for the collocation method

$$v_n = Q_{n,m-1} E_\delta T_\varphi v_n + Q_{n,m-1} E_\delta f_\varphi.$$

Present the matrix form of the method. Examine suitable full discretizations of the collocation and quasi-collocation methods and present two grid iterations to solve the systems trying to restrict all the computations to $O(n^2)$ flops; of course, the accuracy $O(h^m)$ should be maintained by the approximate solution. Solve numerical examples an comment on them.

30*. Establish a counterpart of Theorem 9.2 for the the quasi-collocation method and present the matrix form of the method. Examine suitable full discretizations of the collocation and quasi-collocation methods for the periodized problem and present two grid iterations to solve the systems trying to restrict all the computations to $O(n^2)$ flops; the accuracy $O(h^m)$ should be maintained by the approximate solution. Solve numerical examples and comment on them.

Exercises labelled by $\star$ propose topics for master theses, Exercises 29* and 30* even for doctoral theses. Also Exercise 28 can be extended up to a master thesis topic proving or disproving the formulated conjecture, presenting computer graphs of $\beta_{m-1}(x)$ and $\gamma_{m-1}(x)$, determining $\| Q_{n,m-1} \|_{BC(\mathbb{R})\to BC(\mathbb{R})}$ numerically for $m = 3, ..., 10$ and examining the asymptotics of $\| Q_{n,m-1} \|_{BC(\mathbb{R})\to BC(\mathbb{R})}$ as $m \to \infty$.

See the Comments and Bibliographical Remarks for further radical open problems.

Comments and Bibliographical remarks

With proofs, the results of Sections 2.1–2.4, except Theorem 2.8, can be found in any text book on functional analysis. The proof of Theorem 2.8 , in its full extent, is based on the Fredholm theory for compact operators, see [18] for details. Strange enought, the Faa di Bruno's differentiation formula (Theorem 2.9) is not included into standard text books on calculus although when the formula is already formulated, its proof by induction is instructive and simple .

The smoothness/singularity problem for the solutions of weakly singular integral equations has a long history, see [1], [5–8], [12], [17,18] and the literature quoted there; the results of Section 5 can be extended to multidimensional weakly singular integral equations, see [8], [12]. In very last time, the smoothness/singularity results have been extended to integral equations of the type

$$u(x) = \int_0^1 K(x,y) y^{-\lambda} (1-y)^{-\mu} u(y) dy + f(x)$$

where $K \in \mathcal{S}^{m,\nu}$, $m \geq 1$, $\nu < 1$, $\nu + \lambda < 1$, $\nu + \mu < 1$. It occurs that the boundary singularities $y^{-\lambda}(1-y)^{-\mu}$ by the kernel shift the solutions from $C^{m,\nu}(0,1)$ into the space $C^{m,\nu+\lambda,\nu+\mu}(0,1)$ of functions that have the singularities of the type $C^{m,\nu+\lambda}$ in a vicinity of 0 and of the type $C^{m,\nu+\mu}$ in a vicinity of 1. See [7] for precise (and more general) formulations and for proofs.

Piecewise polynomial collocation method can be applied to integral equation (5.1) directly, without a smoothing transformation. The optimal convergence order $O(n^{-m})$ can be achieved by using a suitable graded grid of the type

$$x_i = \frac{1}{2}\left(\frac{i}{n}\right)^r, \ \ i = 0, ..., n, \ \ x_{n+i} = 1 - x_{n-i}\,, \ \ i = 1, ..., n,$$

where $r \geq 1$ is the grading parameter. For $r = 1$ the grid is uniform; for greater r the grid points x_i are more densely located near the end points of the interval $[0, 1]$. On every subinterval $[x_i, x_{i+1}]$, $i = 0, ..., 2n-1$, take m interpolation points $\xi_{i,l} = x_i + b_l(x_{i+1} - x_i)$, $l = 1, ..., m$, where $0 \leq b_1 < ... < b_m \leq 1$ are parameters that are independent of i and n. Using these interpolation points we can build a polynomial interpolant of degree $m-1$ of a given function $f \in C[0, 1]$ on every interval $[x_i, x_{i+1}]$, $i = 0, ..., 2n-1$, independently and compose from those partial interpolants a piecewise polynomial function on $[0, 1]$ that we denote by $Q_n f$. It occurs that for $f \in C^{m,\nu}(0, 1)$ and sufficiently large $r = r(m, \nu)$ described in [12],[18], $\| f - Q_n f \|_\infty \leq cn^{-m} \| w_{m+\nu-1} f^{(m)} \|_\infty$. Assuming that $f \in C^{m,\nu}(0, 1)$, $K \in \mathcal{S}^{m,\nu}$ and $\mathcal{N}(I - T) = \{0\}$, the collocation method

$$u_n = Q_n T u_n + Q_n f$$

applied to equation (5.1) converges with the optimal accuracy order

$$\| u - u_n \|_\infty \leq cn^{-m} \| w_{m+\nu-1} u^{(m)} \|_\infty \,.$$

In [5,6], this method is combined with the smoothing change of variables to reduce the restriction on the grading parameter r. In particular, the uniform grid ($r = 1$) can be used setting suitable conditions on φ; the collocation is performed still at points $\xi_{i,l}$, $l = 1, ..., m$, $i = 0, ..., 2n-1$; the boundary behaviour (5.20) of the solution is not exploited. The collocation method introdoced and examined in Section 7 of the present lecture notes is different. This method similarly as the two methods of Section 9 seem to be new. The periodization of the problem allows to use not only periodic splines (as in Section 9.3) but also trigonometric or wavelet trial functions, cf. [9], [15].

The spline interpolation problem has been found much attention in the literature, see, in particular, the monographs [3], [4], [10], [11], [19]. Usually the interpolation problem is formulated for an interval, say, for $[0, 1]$, but [11] starts from the interpolation on $\mathbb{R}$. For us the interpolation on $\mathbb{R}$ is suitable since, due to boundary conditions (5.20) satisfied by the solution of the transformed integral equation (5.15) on $[0, 1]$, we have a simple way to extend the solution onto $\mathbb{R}$ maintaining the C^m-smoothness. Our idea to use the Wiener theorem for the construction of the (Wiener) interpolant is equivalent to the idea of constructing the interpolant with bounded derivative of order $m-1$ exploited in [11]. Technically, our approach is simpler than this in [11] but equivalent to it, so we finally arrive at the same formula (8.10) as in [11].

In the literature, the spline quasi-interpolants have been usually introduced through the condition that they reproduce the polynomials of degree $\leq m-1$, without any connection to the real interpolant, see [3], [10], [19]. In [10], the quasi-interpolants are systematically exploited to estimate, in a varity of norms, the distance of a given function from the subspace of splines. This approach leads to optimal convergence orders but the constants in estimates remain undetermined or are rather coarse, for instance, $\| Q_{n,m-1} \|_{C\to C} \leq (2m)^m$ for the quasi-interpolation operators constructed in [10]. Our treatment of quasi-interpolants based on the difference representation of the Wiener solution is different from that in the literature. An advantage of our approach is that we obtain simple closed formulae for the quasi-interpolants of any approximation degree and at least for $m \leq 10$ the norms of the interpolation and quasi-interpolation operators are quite acceptable to be sure that the numerical schemes are stable with respect to rounding errors.

The problem of a full discretization of the collocation schemes and of a fast solution of the collocation systems remained untouched in these lectures. In the section Exercises and Problems we formulated some problems to construnt fully discrete schemes of the optimal accuracy order and complexity $O(n^2)$ flops for their implementation. Similarly as in [13–16] in case of smooth kernels without singularities, a **challenging problem** is to reduce the arithmetical work to $O(n)$ flops maintaining the optimal accuracy $\| v - v_n \|_\infty \leq ch^m \| v^{(m)} \|_\infty$ of the approximate solution under assumptions that $f \in C^{m,\nu}(0,1)$, $K \in \mathcal{S}^{2m,\nu}$, $m \geq 2$, $\nu < 1$ and $\mathcal{N}(I-T) = \{0\}$. Actually a radical **open problem** is whether (or under which further conditions on the kernel) this is possible at all; note that, in analogy to the case of smooth kernels, we already strengthened the condition $K \in \mathcal{S}^{m,\nu}$ up to $K \in \mathcal{S}^{2m,\nu}$ but it is not clear whether this is sufficient.

Fast solution is supremely important in the case of **multidimensinal** integral equations.

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