Mat-1.3379 Dynaamisten systeemien erikoiskurssi **Eirola/Hannukainen**
Harjoitus 1, 26.1.2006 12-14 U345

Tehtävä 1 on kotitehtävä. Kotitehtävä palautetaan laskuharjoituksiin mennessä huoneen Y323b edessä sijaitsevaan lokeroon tai laskuharjoitusten alussa assistentille.

1. (Kotitehtävä, palautus 26.1.) (Problem 1.1 p. 53) Show that $f_x(x^0) \in \mathbb{R}^{n\times m}$ is onto if and only if it has a right inverse $f_x(x^0)^I \in \mathbb{R}^{m\times n}$

2. (Remark 1 p. 56) Show that, if $\boldsymbol{A}$ has a right inverse $\boldsymbol{A}^I$ and $\boldsymbol{I} + \boldsymbol{V}^T\boldsymbol{A}^I\boldsymbol{U}$ is invertible, then $\boldsymbol{A}^I - \boldsymbol{A}^I\boldsymbol{U}(\boldsymbol{I} + \boldsymbol{V}^T\boldsymbol{A}^I\boldsymbol{U})^{-1}\boldsymbol{V}^T\boldsymbol{A}^I$ is a right inverse of $\boldsymbol{A} + \boldsymbol{U}\boldsymbol{V}^T$.

3. (Problem 2.1 p.57) Suppose we have a QR–factorization of the matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, and wantto find a QR-factorization of $\boldsymbol{A}+\boldsymbol{u}\boldsymbol{v}^T$. Show how to achieve this with Givens rotations at a cost of $O(mn)$ flops. Conclude from this that one can implement Broyden's method by updating a sequence of QR–factorizations of matrices differing always by a rank one correction.

5. (Problem 5.4 p.70) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be invertible and $\boldsymbol{b}$, $\boldsymbol{c} \in \mathbb{R}^n$. Show the following handy identity
$$\det(\boldsymbol{A} + \boldsymbol{b}\boldsymbol{c}^T) = (1 + \boldsymbol{c}^T\boldsymbol{A}^{-1}\boldsymbol{b}) \det \boldsymbol{A} \,.$$
Then, deduce that if $\boldsymbol{b}$ is an eigenvector of $\boldsymbol{A}$ such that $\boldsymbol{b}^T\boldsymbol{c} = 0$, then $\boldsymbol{A}$ and $\boldsymbol{A} + \boldsymbol{b}\boldsymbol{c}^T$ have the same eigenvalues.

5. (Problem 5.7 p.71) The inverse of $\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{u}_0)_{|_{W^\perp}}$ in the proof of Theorem 4.1 is called the *Moore–Penrose pseudoinverse* of $\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{u}_0)$ (restricted to the subspace $W^\perp$). Generally, for a matrix $\boldsymbol{A}$ it is defined as follows. If $\boldsymbol{A} \in \mathbb{C}^{n\times m}$ then $N(\boldsymbol{A})^\perp$ and $R(\boldsymbol{A})$ have the same dimension (i.e., $\operatorname{rank}(\boldsymbol{A})$) and the pseudoinverse of $\boldsymbol{A}$, denoted by $\boldsymbol{A}^\dagger$, is defined to be the inverse of
$$\boldsymbol{A}_{|_{N(\boldsymbol{A})^\perp}} \; : \; N(\boldsymbol{A})^\perp \to R(\boldsymbol{A}) \,.$$
   (a) Let $\boldsymbol{A} \in \mathbb{C}^{m\times p}$, $m \geq p$. Show that $\boldsymbol{x} = \boldsymbol{A}^\dagger\boldsymbol{b}$ is the solution of the least-squares problem $\min_{\boldsymbol{x}} |\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}|$, that has the smallest norm. As usual, the norms are Euclidean. In particular, if $\boldsymbol{A}$ has full rank, then $\boldsymbol{A}^\dagger = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$. Also, show that $(\boldsymbol{A}^T)^\dagger = (\boldsymbol{A}^\dagger)^T$.
   (b) Let $\boldsymbol{A} \in \mathbb{C}^{m\times p}$, $m \geq p$, have rank r. Show that
$$\boldsymbol{A}^\dagger = \lim_{\varepsilon\to 0^+} (\boldsymbol{A}^*\boldsymbol{A} + \varepsilon I)^{-1}\boldsymbol{A}^* = \lim_{\varepsilon\to 0^+} \boldsymbol{A}^*(\boldsymbol{A}\boldsymbol{A}^* + \varepsilon I)^{-1} = \boldsymbol{V} \begin{bmatrix} \boldsymbol{S}^\dagger & 0 \end{bmatrix} \boldsymbol{U}^* \,,$$
   where $\boldsymbol{A} = \boldsymbol{U} \left[\begin{smallmatrix} \boldsymbol{S} \\ 0 \end{smallmatrix}\right] \boldsymbol{V}^*$ is the singular value decomposition of $\boldsymbol{A}$: $\boldsymbol{S} \in \mathbb{R}^{p\times p}$, $\boldsymbol{S} = \operatorname{diag}(\sigma_1, ..., \sigma_r, 0, ..., 0)$, $\sigma_i \neq 0$, $i = 1, \ldots, r$, and $\boldsymbol{S}^\dagger = \operatorname{diag}(1/\sigma_1, ..., 1/\sigma_r, 0, ..., 0)$.