Lien ryhmät ja monistot — Harjoitus 3

(3.1) Olkoon $f : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}$

$$(z_1, \dots, z_n) \longmapsto (\mathrm{Re}\, z_1, \mathrm{Im}\, z_1, \dots, \mathrm{Re}\, z_n, \mathrm{Im}\, z_n)$$

<u>Väite:</u> $\langle X, Y\rangle_{\mathbb{C}} = \langle f(X), f(Y)\rangle_{\mathbb{R}} + i\,\langle f(X), f(iY)\rangle_{\mathbb{R}}$, $X, Y \in \mathbb{C}^n$

<u>Tod.</u>

$$\langle X, Y\rangle_{\mathbb{C}} = \langle X_r + iX_i, Y_r + iY_i\rangle_{\mathbb{C}}$$

$$= \langle X_r, Y_r\rangle_{\mathbb{R}} + \langle X_i, Y_i\rangle_{\mathbb{R}} + i\big(\langle X_i, Y_r\rangle_{\mathbb{R}} - \langle X_r, Y_i\rangle_{\mathbb{R}}\big)$$

$\langle\cdot,\cdot\rangle_{\mathbb{C}} = \langle\cdot,\cdot\rangle_{\mathbb{R}}$ $\mathbb{R}^n$:n vektoreille

$$= (X_r, X_i)\cdot\begin{pmatrix} Y_r \\ Y_i \end{pmatrix} + i\,(X_r, X_i)\cdot\begin{pmatrix} \mathrm{Re}(iY) \\ \mathrm{Im}(iY) \end{pmatrix}$$

indeksoi summat uudestaan

$$= \langle f(X), f(Y)\rangle_{\mathbb{R}} + i\,\langle f(X), f(iY)\rangle_{\mathbb{R}} \qquad \square$$

<u>Seuraus:</u> $|X|_{\mathbb{C}} = |f(X)|_{\mathbb{R}}$, eli f on isometria.

<u>Väite:</u> $\langle X, Y\rangle_{\mathbb{H}} = \langle h(X), h(Y)\rangle_{\mathbb{R}} + i\langle h(X), h(iY)\rangle_{\mathbb{R}} + j\langle h(X), h(jY)\rangle + k\langle h(X), h(kY)\rangle_{\mathbb{R}}$ $\quad X, Y \in \mathbb{H}^n$.

<u>Tod</u> Tarkastellaan ensin tapausta $n = 1$.

$X = a + bj$, $Y = c + dj$, $a, b, c, d \in \mathbb{C} \subset \mathbb{H}$

merkitään: $a = a_r + a_i i$, $b = b_r + b_i i$, $a + bj = a_r + a_i i + b_r j + b_i k$.

$$\langle X, Y\rangle_{\mathbb{H}} = \langle a + bj, c + dj\rangle$$

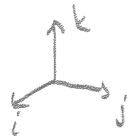
$$= \langle a, c\rangle_{\mathbb{C}} + \underbrace{\langle bj, dj\rangle_{\mathbb{H}}}_{= bj\bar{j}\bar{d} = \langle b, d\rangle_{\mathbb{C}}} + \langle bj, c\rangle_{\mathbb{H}} + \langle a, dj\rangle_{\mathbb{H}}$$

$$= \underbrace{\langle f(a), f(c)\rangle_{\mathbb{R}} + \langle f(b), f(d)\rangle_{\mathbb{R}}}_{= (a_r\; a_i\; c_r\; c_i)\begin{pmatrix} b_r \\ b_i \\ d_r \\ d_i \end{pmatrix} = \langle h(X), h(Y)\rangle_{\mathbb{R}}} + \overbrace{i\big(\langle f(a), f(ic)\rangle + \langle f(b), f(id)\rangle\big)}^{= \langle h(X), h(iY)\rangle_{\mathbb{R}}} + bj\bar{c} - aj\bar{d}$$

$$bj\bar{c} - aj\bar{d} = (b_r + ib_i)j(c_r - ic_i) - (a_r + ia_i)j(d_r - id_i)$$

$$= (b_rc_r + b_ic_i - a_rd_r + a_id_i)j + (-b_rc_i + b_ic_r + a_rd_i - a_id_r)k$$

$$= (a_r\ a_i\ b_r\ b_i)\underbrace{\begin{pmatrix}-d_r\\ d_i\\ c_r\\ -c_i\end{pmatrix}}_{=h(jY)} j + (a_r\ a_i\ b_r\ b_i)\underbrace{\begin{pmatrix}-d_i\\ -d_r\\ c_i\\ c_r\end{pmatrix}}_{=h(kY)} k$$



$$jY = j(c_r + ic_i + d_rj + d_ik) = -d_r + d_ii + c_rj - c_ik \qquad kY = k(c_r + ic_i + d_rj + d_ik) = -d_i - d_ri + c_ij + c_rk$$

$$= \langle h(X), h(jY)\rangle_{\mathbb{R}}\, j + \langle h(X), h(kY)\rangle_{\mathbb{R}}\, k$$

Jos $n > 1$

$$\langle X, Y\rangle_{\mathbb{H}} = \underbrace{\sum_{s=1}^{n} X_s \bar{Y}_s} = (h(X_1), \dots, h(X_n))\begin{pmatrix}h(Y_1)\\ \vdots\\ h(Y_n)\end{pmatrix}$$

$$= \sum_{s=1}^{n} \langle h(X_s), h(Y_s)\rangle_{\mathbb{R}} + i\langle h(X_s), h(iY_s)\rangle_{\mathbb{R}} + j \cdots$$

$$= \langle h(X), h(Y)\rangle_{\mathbb{R}} + i\langle h(X), h(iY)\rangle_{\mathbb{R}} + \cdots \qquad \square$$

Seuraus: $|X|_{\mathbb{H}} = |h(X)|_{\mathbb{R}}$.

Tämä seuraa sillä tiedetään, että $\langle X, X\rangle \overset{s3.1}{\geq} 0$ joten $\langle X, X\rangle$:llä ei ole i, j, k komponenttia ja $\langle X, X\rangle_{\mathbb{H}} = \langle h(X), h(X)\rangle_{\mathbb{R}}$.

3.2 Olkoon $A \in O(2) \setminus SO(2)$, eli luentojen s. 3.9 perusteella:

$$A = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix} \quad \text{jollain } \theta \in [0, 2\pi)$$

Väite: A on peilaus suoran $\{\frac{\theta}{2} = \text{vakio}\}$ suhteen

Esim: $\theta = 0$: $(x,y) \mapsto (x,-y)$

$\theta = \pi$: $(x,y) \mapsto (-x,y)$

Olkoon R_α tason kierto α radiaania myötäpäivään. Tällöin $R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$

Peilaus suoran $\frac{\theta}{2} = \text{vakio}$ on

$$R_{-\theta/2} \circ \{(x,y) \mapsto (x,-y)\} \circ R_{\theta/2}$$

$$= \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \circ (x,y) \mapsto (x,-y) \circ \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ \sin\theta/2 & -\cos\theta/2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \qquad \begin{cases} \sin 2\alpha = 2\sin\alpha \cdot \cos\alpha \\ \cos 2\alpha = \cos^2\alpha - \sin^2\alpha \end{cases}$$

3.2b Olkoon $B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2)$ $\quad \theta \in (0,\pi) \cup (\pi, 2\pi)$

(Huom: $B = \pm I$ kun $\theta = 0, \pi$).

<u>Väite:</u> $BA \neq AB \quad \forall A \in O(2) \setminus SO(2)$.

Olkoon $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ jollain $\alpha \in [0, 2\pi)$.

Käyttämällä

$$\cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y$$
$$\sin(x \pm y) = \sin x \cdot \cos y \pm \sin y \cdot \cos x$$

Saadaan

$$AB = \begin{pmatrix} \cos(\alpha+\theta) & \sin(\alpha+\theta) \\ \sin(\alpha+\theta) & -\cos(\alpha+\theta) \end{pmatrix} = \text{peilaus suoran } \frac{\alpha+\theta}{2} \text{ yli}$$

$$BA = \begin{pmatrix} \cos(\alpha-\theta) & \sin(\alpha-\theta) \\ \sin(\alpha-\theta) & -\cos(\alpha-\theta) \end{pmatrix} = \text{peilaus suoran } \frac{\alpha-\theta}{2} \text{ yli}$$

eli $AB = BA \implies \begin{cases} \cos(\alpha+\theta) = \cos(\alpha-\theta) \\ \sin(\alpha+\theta) = \sin(\alpha-\theta) \end{cases} \implies e^{i(\alpha+\theta)} = e^{i(\alpha-\theta)}$

$\implies e^{i2\theta} = 1 \qquad k \in \mathbb{Z}$

$\implies \cos 2\theta = 1 \implies \theta = k\pi$ ↯

$\square$

3.2.c O(2):n alkiot ovat muotoa

$$R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \text{ tai } P_\alpha = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$$

det = 1   det = -1

$\alpha \in [0, 2\pi)$ $\beta \in [0, 2\pi)$

Olkoon $A, B \in S(2)$. Tietokoneella saadaan:

$R_\alpha \cdot R_\beta = R_{\alpha+\beta}$ kiertoja voi yhdistellä.

$R_\alpha \cdot P_\beta = P_{\beta-\alpha}$

$P_\alpha \cdot R_\beta = P_{\alpha+\beta}$ } b- kohta

$P_\alpha \cdot P_\beta = R_{\beta-\alpha}$ esim $P_\alpha \cdot P_\alpha = I$ $\square$

3.3 Olkoon n pariton, ja

$$f: O(n) \longrightarrow SO(n) \times \{\pm 1\}$$
$$A \longmapsto (\det A \cdot A, \det A)$$

Jos $A \in O(n)$, $\det A = \pm 1$ (Lause 3.3.1), $\det A \cdot A \in O(n)$ ja $\det(\det A \cdot A) = (\det A)^n \cdot \det A = (\det A)^{n+1} = 1$,

joten $f(A) \in SO(n) \times \{\pm 1\}$.

Huom: $SO(n) \times \{\pm 1\}$ on ryhmä $(a,b) \cdot (c,d) = (ac, bd)$.

Väite: f on isomorfism:

Tod Olkoon $(B, \sigma) \in SO(n) \times \{\pm 1\}$. Tällöin $\det B = 1$, $\sigma^n = \sigma$

ja
$$\begin{aligned} f(\sigma B) &= (\det \sigma B \cdot \sigma B, \det \sigma B) \\ &= (\sigma^n \cdot 1 \cdot \sigma \cdot B, \sigma^n \det B) \\ &= (B, \sigma) \rightsquigarrow f \text{ on surjektio} \end{aligned}$$

$$f(A) = f(B) \Rightarrow \begin{cases} \det A \cdot A = \det B \cdot B \Rightarrow A = B \\ \det A = \det B \ (\neq 0) \end{cases} \rightsquigarrow f \text{ bijektio}$$

Jos $A, B \in O(n)$ niin

$$\begin{aligned} f(AB) &= (\det(AB)\,AB,\ \det(AB)) \\ &= (\det A \cdot A,\ \det A) \cdot (\det B \cdot B,\ \det B) \\ &= f(A) \cdot f(B) \quad \rightsquigarrow f \text{ ryhmähomomorfismi} \ \square \end{aligned}$$

Huom Jos $(B, \sigma) \in SO(n) \times \{\pm 1\}$ niin

$$f^{-1}(B, \sigma) = \sigma \cdot B \in O(n).$$

b) Olkoon $X \subset \mathbb{R}^n$ symmetrinen ($p \in X \Rightarrow -p \in X$).

$\operatorname{Symm}(X) \subset O(n)$.

Väite: $\operatorname{Symm} X \cong \operatorname{Symm}^+ X \times \{\pm 1\}$

Tod. $\operatorname{Symm} X = \{ u \mapsto u \cdot A + v \mid A \in O(n),\ \overset{X \cdot A = X}{v \in \mathbb{R}^n} \}$

$$\overset{\operatorname{Symm} X \subset O(n)}{=} \{ A \mid \underbrace{A \in O(n)}, X \cdot A = X \}$$

$\Leftrightarrow A \in f^{-1}(SO(n) \times \{\pm 1\})$ (a-kohta)

$\Leftrightarrow A = \sigma B,\ B \in SO(n),\ \sigma \in \{\pm 1\}$

$$= \{ \sigma B \mid B \in SO(n), \sigma \in \{\pm 1\}, X \cdot B = \underbrace{\sigma X}_{= \pm X = X} \}$$

$\cong f\{ \sim \}$ koska $\{\sim\}$ on $O(n)$:n aliryhmä

$$= \{ (\underbrace{\det \sigma B}_{= \sigma^n \cdot \det B = \sigma} \cdot \sigma B,\ \underbrace{\det \sigma B}_{\sigma}) \mid B \in SO(n), \sigma \in \{\pm\}\ XB = X \}$$

$$= \{ (B, \sigma) \mid B \in SO(n)\ XB = X,\ \sigma \in \{\pm 1\} \}$$

$$= \operatorname{Symm}^+(X) \times \{\pm 1\}.$$

c) Ei ole olemassa homomorfismia

$$O(2) \longrightarrow SO(2) \times \{\pm 1\}$$

Tod Todetaan ensin: Jos $B \in SO(2)$ ja $B^2 = I$ niin

$B = \pm I$. Olkoon $f : O(2) \longrightarrow SO(2) \times \{\pm 1\}$ ryhmähomomorfismi.

Tehl. 2.c

Jos $A \in O(2) \setminus SO(2)$ niin A on peilaus, $A^2 = I$ ja

$$f(A) = (B, \pm 1) \text{ jollain } B \in SO(2) \text{ ja } B^2 = I$$
$$= (\pm I, \pm 1) \quad \text{(merkit riippumattomia)}$$

eli löytyy bijektio joukkojen $O(2) \setminus SO(2)$ ja $\{\pm I, \pm 1\}$ välillä. ⚡

3.4 $\mathrm{Aff}_n(\mathbb{K}) = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline v & 1 \end{array} \right) \middle| A \in GL_n(\mathbb{K}),\ v \in \mathbb{K}^n \right\} \subset GL_{n+1}(\mathbb{K})$

a) $\mathrm{Aff}_n(\mathbb{K})$ on aliryhmä.

- $A = I_n,\ v = 0 \Rightarrow \mathrm{Id}_{n+1} \in \mathrm{Aff}_n(\mathbb{K})$
- Kahden $\mathrm{Aff}_n(\mathbb{K})$:n alkion tulolle saadaan

$$\left(\begin{array}{c|c} A & 0 \\ \hline v & 1 \end{array} \right) \left(\begin{array}{c|c} B & 0 \\ \hline w & 1 \end{array} \right) = \left(\begin{array}{c|c} A \cdot B & 0 \\ \hline v \cdot B + w & 1 \end{array} \right) \rightsquigarrow \mathrm{Aff}_n(\mathbb{K}) \text{ suljettu matriisitulon suhteen}$$

- $$\left(\begin{array}{c|c} A & 0 \\ \hline v & 1 \end{array} \right) \left(\begin{array}{c|c} A^{-1} & 0 \\ \hline -v \cdot A^{-1} & 1 \end{array} \right) = I_{n+1} \rightsquigarrow \mathrm{Aff}_n(\mathbb{K}) \text{ aliryhmä.}$$

b) Samaistetaan $\left(\begin{array}{c|c} A & 0 \\ \hline v & 1 \end{array} \right) \in \mathrm{Aff}_n(\mathbb{K})$ ja kuvaus

$$f: \mathbb{K}^n \longrightarrow \mathbb{K}^n$$
$$x \longmapsto x \cdot A + v.$$

$L \subset \mathbb{K}^n$ on <u>suora</u> $\Leftrightarrow L = \{v_0 + \lambda e \mid \lambda \in \mathbb{R}\}$ joillakin $v_0, e \in \mathbb{K}^n$.

<u>f kuvaa suorat suoriksi</u>

$$f\{v_0 + \lambda e \mid \lambda \in \mathbb{R}\} = \{(v_0 \cdot A + v) + \lambda e \cdot A \mid \lambda \in \mathbb{R}\}$$
$$= \text{suora.}$$

c) $\mathrm{Aff}_1(\mathbb{R}) = \left\{ \begin{pmatrix} a & 0 \\ v & 1 \end{pmatrix} \middle| a, v \in \mathbb{R} \right\}$ vaihdannainen?

$$(x \mapsto ax + v) \circ (x \mapsto bx + w) = (x \mapsto abx + v + aw)$$
$$(x \mapsto bx + w) \circ (x \mapsto ax + v) = (x \mapsto abx + w + bv)$$

$\Rightarrow$ vaihdannainen joss $v(b-1) = w(a-1)$. esim $a = b = 1$.

Ongelmallisia ovat tässä termit aw, bv missä ensimmäisten kuvauksien siirrot skaalataan.

(3.5) a) $\mathrm{Aff}_n(\mathbb{K}) \subset GL_{n+1}(\mathbb{K})$ on matriisiryhmä.

Määritellään

$$Z_1 : M_{n+1}(\mathbb{K}) \longrightarrow \mathbb{K}^n$$
$$\left(\begin{array}{c|c} A & w \\ \hline v & \alpha \end{array}\right) \longmapsto w$$

$$Z_2 : M_{n+1}(\mathbb{K}) \longrightarrow \mathbb{K}$$
$$\left(\begin{array}{c|c} A & w \\ \hline v & \alpha \end{array}\right) \longmapsto \alpha$$

$$Z_3 : M_{n+1}(\mathbb{K}) \longrightarrow \mathbb{K}$$
$$M \longmapsto \det M.$$

Sitten $Z_1^{-1}(0) \cap Z_2^{-1}(1) \cap Z_3^{-1}(\mathbb{K}\setminus\{0\})$

$$= \left\{ \left(\begin{array}{c|c} A & w \\ \hline v & \alpha \end{array}\right) \in M_{n+1}(\mathbb{K}) \;\middle|\; w = 0,\ \alpha = 1,\ \overset{v\in\mathbb{K}^n,\ A\in M_n(\mathbb{K})}{\underbrace{\det\left(\begin{array}{c|c} A & w \\ \hline v & \alpha \end{array}\right) \neq 0}_{\overset{\text{Lause 2.9}}{\iff} \left(\begin{array}{c|c} A & w \\ \hline v & \alpha \end{array}\right) \in GL_{n+1}(\mathbb{K})}} \right\}$$

$$= \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline v & 1 \end{array}\right) \;\middle|\; v \in \mathbb{K}^n,\ \underbrace{\left(\begin{array}{c|c} A & 0 \\ \hline v & 1 \end{array}\right) \in GL_{n+1}(\mathbb{K})}_{\iff \det A \cdot \det 1 \neq 0 \iff \det A \neq 0 \iff A \in GL_n(\mathbb{K})} \right\} = \mathrm{Aff}_n(\mathbb{K})$$

Koska Z_1, Z_2, Z_3 jatkuvia niin $Z_1^{-1}(0) \cap Z_2^{-1}$ on suljettu $M_{n+1}(\mathbb{K})$:ssa

Pätee myös $Z_3^{-1}(\mathbb{K}\setminus\{0\}) = GL_{n+1}(\mathbb{K})$, eli

$$\mathrm{Aff}_n(\mathbb{K}) = \{\text{suljettu joukko } M_{n+1}\text{:ssä}\} \cap GL_{n+1}(\mathbb{K})$$
$$= \text{suljettu } GL_{n+1}(\mathbb{K})\text{:ssa}. \qquad \square$$

- $\left(\begin{array}{c|c} \varepsilon I & 0 \\ \hline 0 & 1 \end{array}\right) \in \mathrm{Aff}_n(\mathbb{K})$ $\forall \varepsilon > 0$ mutta $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right) \notin \mathrm{Aff}_n(\mathbb{K})$ joten $\mathrm{Aff}_n(\mathbb{K})$ ei suljettu $M_{n+1}(\mathbb{K})$:ssa.

... mutta: Jos $M \in GL_{n+1}(\mathbb{K})$, $M_i \in \mathrm{Aff}_n(\mathbb{K})$ ja $M_i \longrightarrow M$ niin $M \in \mathrm{Aff}_n(\mathbb{K})$. (S 4.1)

b) $\mathrm{Isom}(\mathbb{R}^n) = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline v & 1 \end{array} \right) \;\middle|\; A \in O(n),\ v \in \mathbb{R}^n \right\}$ suljettu $GL_{n+1}(\mathbb{R})$

Kuten edellä: Jos $M = \left(\begin{array}{c|c} A & w \\ \hline v & \alpha \end{array} \right) \in M_n(\mathbb{R})$ määritellään

$$Z_1 : M_n(\mathbb{R}) \longrightarrow \mathbb{R} \qquad\qquad Z_2 : M_n(\mathbb{R}) \longrightarrow \mathbb{R}^n$$
$$M \longmapsto \alpha \qquad\qquad M \longmapsto w$$

$$Z_3 : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R}) \qquad\qquad Z_4 : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$M \longmapsto A \cdot A^T \qquad\qquad M \longmapsto \det M$$

Sitten $\underbrace{Z_1^{-1}(1) \cap Z_2^{-1}(0) \cap Z_3^{-1}(I)}_{\text{suljettu } M_n(\mathbb{R})\text{:ssä}} \cap \underbrace{Z_4^{-1}(\mathbb{R}\setminus\{0\})}_{= Gl_{n+1}(\mathbb{R})}$

$$= \left\{ \left(\begin{array}{c|c} A & w \\ \hline v & \alpha \end{array} \right) \;\middle|\; \alpha = 1,\ w = 0,\ \underbrace{\det \left(\begin{array}{c|c} A & 0 \\ \hline v & 1 \end{array} \right) \neq 0}_{\Leftrightarrow\ \det A \neq 0},\ A \in O(n),\ v \in \mathbb{R}^n \right\}$$

$$= \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline v & 1 \end{array} \right) \;\middle|\; A \in O(n),\ v \in \mathbb{R}^n \right\} = \mathrm{Isom}(\mathbb{R}^n) \quad \square$$

c) $\mathrm{Isom}\,\mathbb{R}^n$ ei kompakti.

Jos $K \subset \mathbb{R}^d$ kompakti ja $S \subset K$ suljettu, niin S kompakti

Eli jos $\mathrm{Isom}\,\mathbb{R}^n$ on kompakti: niin $\left\{ \left(\begin{array}{c|c} I & 0 \\ \hline (r,0,\dots,0) & 1 \end{array} \right) \;\middle|\; r \in \mathbb{Z} \right\} = R$

on kompakti. Nyt voidaan peittää R pienillä kuulilla

s.e. peitteellä ei ole äärellistä osa-peitettä $\lightning$ $\square$