Lien ryhmät ja monistot Harj. 2.

(2.1) Määritellään

$$f_n : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}$$
$$(z_1, \dots, z_n) \longmapsto (\operatorname{Re} z_1, \operatorname{Im} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_n)$$
$$\varrho_n : M_n(\mathbb{C}) \longrightarrow M_{2n}(\mathbb{R})$$
$$(z_{ij}) \longmapsto \begin{pmatrix} \operatorname{Re} z_{ij} & \operatorname{Im} z_{ij} \\ -\operatorname{Im} z_{ij} & \operatorname{Re} z_{ij} \end{pmatrix}$$

$2n \times 2n$ lohkomatriisi

ts. f tulkitsee $\mathbb{C}$-vektorin reaalisena vektorina
ϱ —"— matriisin —"— matriisina.

Näytetään, että tämä identifikaatio on hyvin käyttäytyvä vektori-matriisitulon suhteen.

Väite: Jos $A \in M_n(\mathbb{C})$ niin $R_{\varrho_n(A)} \circ f_n = f_n \circ R_A$.

Ts. $f_n(X) \cdot \varrho_n(A) = f_n(X \cdot A)$, $X \in \mathbb{C}^n$

tai $X \cdot A = f_n^{-1}\left(f_n(X) \cdot \varrho_n(A) \right)$

Ensin tapaus $n=1$: Jos $z = z_r + i z_i$, $a = a_r + i a_i \in \mathbb{C}$,

$$f_1(z)\, \varrho_1(a) = (z_r, z_i) \begin{pmatrix} a_r & a_i \\ -a_i & a_r \end{pmatrix}$$
$$= (z_r a_r - z_i a_i \,,\, z_r a_i + z_i a_r) = (\operatorname{Re} za, \operatorname{Im} za)$$
$$= f_1(za)$$

$n > 1$ Nyt $f_n(X) \cdot \varrho_n(A)$ on $2n$-vektori joka koostuu n:stä 1×2-lohkosta. Jos $X = (z_1, \dots, z_n) \in \mathbb{C}^n$, $A = (a_{ij}) \in M_n(\mathbb{C})$,

$$f_n(X) \cdot \varrho_n(A) \Big|_{k:s \text{ lohko}} = \left(f_1(z_1) \middle| \cdots \middle| f_1(z_n) \right) \left(\begin{array}{c|c|c} \varrho_1(a_{11}) & \cdots & \varrho_1(a_{1n}) \\ \hline \vdots & \ddots & \vdots \\ \hline \varrho_1(a_{n1}) & \cdots & \varrho_1(a_{nn}) \end{array} \right) \Bigg|_{k:s \text{ lohko}}$$

$$= \sum_{r=1}^{n} f_1(z_r)\cdot \varrho_1(a_{rk}) \quad \Big(\in M_2(\mathbb{R})\Big)$$

$$= \sum_{r=1}^{n} f_1(\underbrace{z_r \cdot a_{rk}}_{\in \mathbb{C}}) = f_1\Big((X\cdot A)_k\Big) = k\text{:s lohko } f_n(XA)\text{:ssa.} \quad \square$$

(2.2) Olk. $\tilde{f}_n : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}, \quad X_r + iX_i \longmapsto (X_r, X_i)$

$$\tilde{\varrho}_n : M_n(\mathbb{C}) \longrightarrow M_{2n}(\mathbb{R}) \quad M_r + iM_i \longmapsto \begin{pmatrix} M_r & M_i \\ -M_i & M_r \end{pmatrix}$$

Tällöin $\quad R_{\tilde{\varrho}_n(A)} \circ \tilde{f}_n = \tilde{f}_n \circ R_A \qquad A \in M_n(\mathbb{C})$

eli $\quad X\cdot A = \tilde{f}_n^{-1}\Big(\tilde{f}_n(X)\cdot \tilde{\varrho}_n(A)\Big), \quad$ kun $X \in \mathbb{C}^n, A \in M_n(\mathbb{C}^n)$

$$\underset{X_r + iX_i}{\tilde{f}_n(X)} \cdot \underset{A_r + iA_i}{\tilde{\varrho}_n(A)} = (X_r, X_i)\begin{pmatrix} A_r & A_i \\ -A_i & A_r \end{pmatrix} =$$

$$= (X_rA_r - X_iA_i,\ X_rA_i + X_iA_r)$$

$$= \Big(\mathrm{Re}(X\cdot A),\ \mathrm{Im}(X\cdot A)\Big) = \tilde{\varrho}(X\cdot A)$$

Määritellään $\tilde{J} \in M_{2n}(\mathbb{R})$ s.e. seuraava kaavio kommutoi:

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\tilde{f}_n} & \mathbb{R}^{2n} \\ {\scriptstyle X \mapsto iX}\downarrow & & \downarrow {\scriptstyle R_{\tilde{J}}} \\ \mathbb{C}^n & \xrightarrow{\tilde{f}_n} & \mathbb{R}^{2n} \end{array} \qquad (*)$$

Jos $X, Y \in \mathbb{R}^n$, niin $\tilde{f}_n \circ (Z \mapsto iZ) \circ \tilde{f}_n^{-1}(X, Y) = \cdots = (X, Y)\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

joten asetetaan $\quad \tilde{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \rightsquigarrow \quad (*)$ kommutoi.

Edellä on motivoitu $\tilde{J}$:n valinta.

<u>Väite</u> Olkoon $B \in M_{2n}(\mathbb{R})$. Tällöin

$$\underbrace{B \text{ on } \mathbb{C}\text{-lineaarinen}}_{} \iff B\tilde{J} = \tilde{J}B.$$

$$\overset{}{\underset{\text{def.}}{\iff}} B = \tilde{\rho}_n(A) \text{ jollain } A \in M_n(\mathbb{C}).$$

<u>Tod.</u> $\boxed{\Rightarrow}$ $B = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$, joillakin $X, Y \in M_n(\mathbb{R})$

suora lasku $\rightsquigarrow$ väite.

$\boxed{\Leftarrow}$ Jos $B \in M_{2n}(\mathbb{R})$ se voidaan jakaa neljään lohkoon:

$$B = \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right) \quad \text{missä } X, Y, Z, W \in M_n(\mathbb{R}).$$

$$B\tilde{J} = \tilde{J}B \Rightarrow \begin{pmatrix} Z & W \\ -X & -Y \end{pmatrix} = \begin{pmatrix} -Y & X \\ -W & Z \end{pmatrix} \Rightarrow X = W, \quad Y = -Z$$

$$\Rightarrow B = \left(\begin{array}{c|c} X & Y \\ \hline -Y & X \end{array}\right) = \tilde{\rho}(X + iY) \qquad \square$$

2.3 <u>Väite:</u> Jos $B \in M_{2n}(\mathbb{R})$ ja $B \cdot J_{2n} = J_{2n} \cdot B$ missä

$$J_{2n} = \left(\begin{array}{c|c|c} J_2 & & \\ \hline & \ddots & \\ \hline & & J_2 \end{array}\right) \qquad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{R}),$$

Luennot s2.5

niin B on $\mathbb{C}$-lineaarinen. (Lause 2.5)

<u>Tod.</u> Kirjoitetaan B lohkomatriisina

$$B = \left(\begin{array}{c|c|c} \beta_{11} & \cdots & \beta_{1n} \\ \hline \vdots & \ddots & \vdots \\ \hline \beta_{n1} & \cdots & \beta_{nn} \end{array}\right) \quad \text{missä jokainen } \beta_{ij} \in M_2(\mathbb{R})$$

$$BJ\Big|_{ij\text{:s lohko}} = \beta_{ij} \cdot J_2 \qquad JB\Big|_{ij\text{:s lohko}} = J_2 \cdot \beta_{ij}$$

Jos $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d \in \mathbb{R}$,

niin $a = d$, $b = -c$. Seuraa, että kaikki β_{ij}-lohkot ovat muotoa $\beta_{ij} = \varsigma_1(z_{ij})$ $z_{ij} \in \mathbb{C}$.

$$\Rightarrow B = \varrho_n\left((z_{ij})_{ij}\right) \qquad \square$$

2.4 $\psi_1 : \mathbb{H} \longrightarrow M_2(\mathbb{C})$

$$(a,b,c,d) \longmapsto \begin{pmatrix} r & s \\ -\bar{s} & \bar{r} \end{pmatrix} \qquad r = a+ib, \quad s = c+id$$

$$\psi_n : M_n(\mathbb{H}) \longrightarrow M_{2n}(\mathbb{C})$$

$$u_{ij} \longmapsto \psi_1(u_{ij})$$

Väite: ψ_n $\mathbb{R}$-lineaarinen (koska $\bar{\ }$ on). $\mathbb{C}$-lin: $L(\lambda A + \mu B) = \lambda L(A) + \mu L(B)$ kun $\lambda, \mu \in \mathbb{C}$.

Väite: ψ_n ei $\mathbb{C}$-lineaarinen.

$$\psi_1(\underset{\in\mathbb{C}}{i}\ \underset{\in\mathbb{H}}{j}) = \psi_1(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$i\,\psi_1(j) = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \square$$

Väite: $\psi_n(A \cdot B) = \psi_n(A) \cdot \psi_n(B)$ $\quad A, B \in M_n(\mathbb{H})$.

Tod. $n = 1$

$$\psi_1\left((a,b,c,d)\cdot(x,y,z,w)\right) = \psi_1(a,b,c,d)\cdot\psi_1(x,y,z,w)$$

kts. Harj 1.

n mielivaltainen: $\psi_n(AB)$ ja $\psi_n(A)\cdot\psi_n(B)$ ovat $2n$ lohkomatriiseja jotka koostuvat 2×2 lohkoista.

$\psi_n(AB)$:n ij-s lohko $= \psi_1\left((A\cdot B)_{ij}\right)$

$$= \psi_1\left(\sum_{k=1}^{n} A_{ik}B_{kj}\right) \underset{\substack{\text{lin.} \\ \text{ja } n=1.}}{=} \sum_{k=1}^{n} \psi_1(A_{ik})\cdot\psi_1(B_{kj})$$

$$= \left(\begin{pmatrix} \Psi_1(A_{11}) & \cdots & \Psi_1(A_{1n}) \\ \vdots & \ddots & \vdots \\ \Psi_1(A_{n1}) & \cdots & \Psi_1(A_{nn}) \end{pmatrix} \begin{pmatrix} \Psi_1(B_{11}) & \cdots & \Psi_1(B_{1n}) \\ \vdots & \ddots & \vdots \\ \Psi_1(B_{n1}) & \cdots & \Psi_1(B_{nn}) \end{pmatrix} \right)_{ij\text{:s lohko}}$$

$$= \left(\Psi_n(A) \cdot \Psi_n(B) \right)_{ij\text{:s lohko.}} \qquad \square$$

(2.5) Lause 2.7 Olkoon $B \in M_{4n}(\mathbb{R})$. Tällöin ehdot

(1) B on $\mathbb{H}$-lineaarinen (eli $\exists A \in M_n(\mathbb{H})$ s.e. $B = \rho_{2n} \circ \Psi_n(A)$)

(2) B kommutoi I_{4n} ja J_{4n}:n kanssa.

(3) $(f_{2n} \circ g_n)^{-1} \circ R_B \circ (f_{2n} \circ g_n)$ on $\mathbb{H}$-lineaarinen.

ovat yhtäpitäviä.

Matriisit $I_{4n}, J_{4n} \in M_{4n}(\mathbb{R})$ ovat määritelty s.e. kaaviot

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{f_{2n} \circ g} & \mathbb{R}^{4n} \\ {\scriptstyle u \mapsto iu} \downarrow & & \downarrow {\scriptstyle R_{I_{4n}}} \\ \mathbb{H}^n & \xrightarrow[f_{2n} \circ g_n]{} & \mathbb{R}^{4n} \end{array} \qquad \begin{array}{ccc} \mathbb{H}^n & \xrightarrow{f_{2n} \circ g} & \mathbb{R}^{4n} \\ {\scriptstyle u \mapsto ju} \downarrow & & \downarrow {\scriptstyle R_{J_{4n}}} \\ \mathbb{H}^n & \xrightarrow[f_{2n} \circ g_n]{} & \mathbb{R}^{4n} \end{array}$$

kommutoivat

Eli

$$(u \mapsto iu) = (f_{2n} \circ g_n)^{-1} \circ R_{I_{4n}} \circ (f_{2n} \circ g_n)$$

$$(u \mapsto ju) = (f_{2n} \circ g_n)^{-1} \circ R_{J_{4n}} \circ (f_{2n} \circ g_n)$$

$$(u \mapsto ku) = (u \mapsto iju) = (u \mapsto iu) \circ (u \mapsto ju)$$

$$= (f_{2n} \circ g_n)^{-1} \circ R_{J_{4n} I_{4n}} \circ (f_{2n} \circ g_n)$$

Jos $A \in M_n(\mathbb{H})$ seuraava kaavio kommutoi:

$$\begin{tikzcd}
\mathbb{H}^n \arrow[r, "g_n"] \arrow[d, "R_A"'] & \mathbb{C}^{2n} \arrow[r, "f_{2n}"] \arrow[d, "R_{\psi_n(A)}"] & \mathbb{R}^{4n} \arrow[d, "R_{(\varrho_{2n} \circ \psi_n)(A)}"] \\
\mathbb{H}^n \arrow[r, "g_n"] & \mathbb{C}^{2n} \arrow[r] & \mathbb{R}^{4n}
\end{tikzcd}$$

1 — 2 — 3

$\boxed{1 \Rightarrow 3}$ Ol. $B = \varrho_{2n} \circ \psi_n(A)$, $A \in M_n(\mathbb{H})$

$$(f_{2n} \circ g_n)^{-1} \circ R_B \circ (f_{2n} \circ g_n) =$$

y.o. kaavio

$$g_n^{-1} \circ f_{2n}^{-1} \circ R_{(\varrho_{2n} \circ \psi_n)(A)} \circ f_{2n} \circ g_n = R_A = \mathbb{H}\text{-lin.}$$

$\boxed{3 \Rightarrow 2}$ Oletuksen mukaan

$$(f_{2n} \circ g_n)^{-1} \circ R_B \circ \underbrace{(f_{2n} \circ g_n) \circ (u \mapsto iu)}_{= R_{I_{4n}} \circ (f_{2n} \circ g)} = \underbrace{(u \mapsto iu) \circ (f_{2n} \circ g_n)^{-1}}_{= (f_{2n} \circ g_n)^{-1} \circ R_{I_{4n}}} \circ R_B \circ (f_{2n} \circ g_n)$$

$$\Rightarrow R_B \circ R_{I_{4n}} = R_{I_{4n}} \circ R_B \quad \Rightarrow \quad R_{I_{4n} \cdot B} = R_{B I_{4n}}$$

$$\Rightarrow I_{4n} \cdot B = B \cdot I_{4n}. \quad \text{vastaavasti} \quad (u \longmapsto ju)$$

$\boxed{3 \Rightarrow 1}$ Lause 1.4.1 $\Rightarrow \exists A \in M_n(\mathbb{H})$ s.e.

$$R_A = (f_{2n} \circ g_n)^{-1} \circ R_B \circ (f_{2n} \circ g_n)$$

$$\Rightarrow R_B = f_{2n} \circ g_n \circ R_A \circ g_n^{-1} \circ f_{2n}^{-1}$$

$$= R_{(\varrho_{2n} \circ \psi_n)(A)} \quad \Rightarrow \quad B = (\varrho_{2n} \circ \psi_n)(A)$$

y.o. kaavio

$\boxed{2 \Rightarrow 3}$ Ol. $L : \mathbb{H}^n \longrightarrow \mathbb{H}^n$

$$X \longmapsto (f_{2n} \circ g)^{-1} \circ R_B \circ (f_{2n} \circ g_n)(X)$$

Selvä $L(X+Y) = L(X) + L(Y)$, $X, Y \in \mathbb{H}^n$.

Ol. $X \in \mathbb{H}^n$. Tällöin

$$L(\lambda X) = \lambda L(X) \quad \text{kun} \quad \lambda \in \mathbb{R}$$

$$
\begin{aligned}
L(\underset{\substack{\uparrow \\ \mathbb{H}}}{i} X) &= L \circ (u \mapsto iu)(X) \\
&= (f_{2n} \circ g_n)^{-1} \circ R_B \circ \underbrace{(f_{2n} \circ g) \circ (u \mapsto iu)}_{= R_{I_{4n}} \circ f_{2n} \circ g_n}(X) \\
&= (f_{2n} \circ g_n)^{-1} \circ \underbrace{R_{I_{4n} \cdot B}}_{\overset{\uparrow}{\underset{\text{oletus}}{=}} R_{B I_{4n}} = R_{I_{4n}} \circ R_B} \circ (f_{2n} \circ g_n)(X) \\
&= \underbrace{(f_{2n} \circ g_n)^{-1} \circ R_{I_{4n}}}_{= (u \mapsto iu) \circ (f_{2n} \circ g_n)^{-1}} \circ R_B \circ (f_{2n} \circ g_n)(X) \\
&= (u \mapsto iu) \circ L(X) = i\, L(X)
\end{aligned}
$$

Vastaavalla tavalla $L(jX) = j\,L(X)$ ja

$L(kX) = L(ijX) = \dots = k\,L(X)$ $\square$

Luennolla:

$$
I_4 = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right) \qquad J_4 = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)
$$

$$
\rho_2 \circ \varphi_n \underbrace{(a, b, c, d)}_{\in \mathbb{H}} = \begin{pmatrix} r & s \\ -\bar{s} & \bar{r} \end{pmatrix} \qquad \begin{aligned} r &= a + ib \\ s &= c + id \end{aligned}
$$

$$
= \left(\begin{array}{cc|cc} a & b & c & d \\ -b & a & -d & c \\ \hline -c & d & a & -b \\ -d & -c & b & a \end{array}\right) = M
$$

$M = I_4 \Rightarrow c = d = 0, \quad b = 1 = -1$ ↯

$M = J_4 \Rightarrow a = b = 0 \quad c = 1 = -1$ ↯

eli I_4 ja J_4 eivät ole $\mathbb{H}$ lineaarisia.

Elegantisti: $B = I_4$ ja J_4 eivät kommutoi joten I_4 ei $\mathbb{H}$-lin.

(2.6) $Gl_n(\mathbb{H}) = \{ A \in M_n(\mathbb{H}) \mid \exists B \in M_n(\mathbb{H}) \text{ s.e. } AB = BA = I \}$

a) Jos $A \in Gl_1(\mathbb{H})$ niin $\det A > 0$

$A \in Gl_1(\mathbb{H}) \Rightarrow A \in \mathbb{H}$ ja $\exists A^{-1}$ joten $A \neq 0$.

Jos $A = a + bi + cj + dk$ niin

$$\det A = \det \Psi_1(A) = \det \begin{pmatrix} r & s \\ -\bar{s} & \bar{r} \end{pmatrix} \qquad \begin{array}{l} r = a + ib \\ s = c + id \end{array}$$

$$= |r|^2 + |s|^2 = a^2 + b^2 + c^2 + d^2 > 0$$

b) $SL_n(\mathbb{H}) = \{ A \in GL_n(\mathbb{H}) \mid \det A = 1 \}$ on aliryhmä.

Jos $A, B \in SL_n(\mathbb{H})$ niin

$$\begin{aligned} \det A \cdot B &= \det \Psi_n(A \cdot B) \\ &= \det \Psi_n(A) \cdot \Psi_n(B) \\ &= \det \Psi_n(A) \cdot \det \Psi_n(B) = \det A \cdot \det B = 1 \end{aligned}$$

c)
$$\begin{aligned} SL_1(\mathbb{H}) &= \{ u \in GL_1(\mathbb{H}) \mid |u|^2 = 1 \} \\ &= \{ a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1 \} \\ &= \Psi(S^3) \end{aligned}$$

missä $\Psi : S^3 \longrightarrow \mathbb{H}$ on bijektio $\square$

$(a,b,c,d) \longmapsto a + bi + cj + dk$