Lien algebrat ja monistot Harjoitus 10

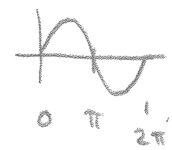
10.1 a) $g \in SO(3)$ ja T $SO(3)$ standardi maksimaalinen torus. Jos $gt = tg$ jollain säännöllisellä $t \in T$ niin $g \in T$.

Huom: väite ei päde kun $n > 3$ (tai $n=2$) (kts. vastaesimerkkiä liitteenä.)

Lemma: Olkoon $A \in GL_2$ ja $AR_\theta = R_\theta A$ jollain $\theta \in [0, 2\pi) \setminus \{0, \pi\}$. Tällöin $A = \lambda R_\alpha$, $\lambda > 0$, $\alpha \in [0, 2\pi)$ tai $A = 0$

Tod: Kirjoitetaan $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ jolloin

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$



$\Rightarrow$ $-b \sin\theta = c \sin\theta$ $a \sin\theta = d \sin\theta$

$\sin\theta \neq 0 \Rightarrow b = -c$ $a = d$

$\Rightarrow A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $\det A = a^2 + b^2 > 0$ (tai $A = 0$).

$\underset{A \neq 0}{\Rightarrow}$ $A = (a^2+b^2)\begin{pmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ \frac{-b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{pmatrix} = \lambda \cdot R_\lambda$. □

$\underline{n=3}$ $g = \begin{pmatrix} \alpha & u \\ v^T & A \end{pmatrix} \in SO(3)$ ja $gt = tg$, $t = \begin{pmatrix} 1 & \\ & R_\theta \end{pmatrix}$ $\theta \in (0,\pi) \cup (\pi, 2\pi)$

$$\Rightarrow \begin{cases} \alpha = \alpha \\ u = u \cdot R_\theta \\ v^T = R_\theta \cdot v^T \\ AR_\theta = R_\theta \cdot A \end{cases} \quad \Rightarrow u = v = 0 \quad \Rightarrow \alpha \neq 0,\ A \in GL_2$$

Lemma $\Rightarrow$ $A = \lambda \cdot R_\alpha$ $\Rightarrow$ $g = \begin{pmatrix} \alpha & \\ & \lambda R_\theta \end{pmatrix} \in SO(3)$

$g \in SO(3) \Rightarrow \alpha = \pm 1$, $\lambda = \pm 1$

$\left.\begin{array}{l} \alpha = 1 \text{ sillä muuten } \det g = -1 \\ -R_\theta = R_{\theta+\pi} \end{array}\right\} \Rightarrow g \in T.$

Vastaesimerkki SO(4):ssa

In[1]:=
```
M = {
    {0, 0, Cos[a], Sin[a]},
    {0, 0, Sin[a], -Cos[a]},
    {Cos[a], Sin[a], 0, 0},
    {Sin[a], -Cos[a], 0, 0}
  };
```

In[2]:=
```
FullSimplify[Det[M]]
Simplify[M.Transpose[M]] // MatrixForm
```

Out[2]= 1

Out[3]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In[6]:=
```
(* M on SO (4) alkio *)
a = Pi;
M // MatrixForm
```

Out[7]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

```
(* Määritellään säännöllinen alkio joka kommutoi M:n kanssa *)
```

In[8]:=
```
t = {
    {Cos[r], Sin[r], 0, 0},
    {-Sin[r], Cos[r], 0, 0},
    {0, 0, Cos[s], Sin[s]},
    {0, 0, -Sin[s], Cos[s]}
  };
```

In[9]:=
```
t // MatrixForm
```

Out[9]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[r] & \mathrm{Sin}[r] & 0 & 0 \\ -\mathrm{Sin}[r] & \mathrm{Cos}[r] & 0 & 0 \\ 0 & 0 & \mathrm{Cos}[s] & \mathrm{Sin}[s] \\ 0 & 0 & -\mathrm{Sin}[s] & \mathrm{Cos}[s] \end{pmatrix}$$

In[12]:=
```
r = Pi - Pi / 4;
s = Pi / 4 - Pi + 2 Pi;
t // MatrixForm
```

$r = \frac{3\pi}{4}$

$s = \frac{5\pi}{4}$

Out[14]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
In[15]:= FullSimplify[t.M - M.t] /. {r → a - d, s → d - a} // MatrixForm
```

ei tee mitään

Out[15]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Vastaesimerkki SO(5):ssa

```
In[16]:= M = {
           {0, 0, Cos[a], Sin[a], 0},
           {0, 0, Sin[a], -Cos[a], 0},
           {Cos[a], Sin[a], 0, 0, 0},
           {Sin[a], -Cos[a], 0, 0, 0},
           {0, 0, 0, 0, 1}
          };

In[17]:= FullSimplify[Det[M]]
         Simplify[M.Transpose[M]] // MatrixForm
```

Out[17]= 1

Out[18]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
In[19]:= (* M on SO (4) alkio *)
         a = Pi;
         M // MatrixForm
```

Out[20]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
(* Määritellään säännöllinen alkio joka kommutoi M:n kanssa *)

In[29]:= t = {
          {Cos[r], Sin[r], 0, 0, 0},
          {-Sin[r], Cos[r], 0, 0, 0},
          {0, 0, Cos[s], Sin[s], 0},
          {0, 0, -Sin[s], Cos[s], 0},
          {0, 0, 0, 0, 1}
         };
        t // MatrixForm
```

Out[30]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$t.M - M.t = 0$

b, c, d ..

Olkoon $g \in M_n(\mathbb{K})$, (myöhemmin $U(n)$, $SU(n)$, $Sp(n)$)

ja $t = \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \in M_n(\mathbb{K})$ $\theta_i \in \mathbb{R}$.

$\theta_i \neq \theta_j$ kun $i \neq j$

$$(gt)_{lm} = \sum_{s=1}^{n} g_{ls} \underbrace{t_{sm}}_{= e^{i\theta_s}\delta_{sm}}$$

$$= g_{lm} e^{i\theta_m}$$

$$(tg)_{lm} = e^{i\theta_l} g_{lm}$$

$\underline{\mathbb{K} = \mathbb{C}}$

$\Rightarrow$ $g_{lm}(e^{i\theta_m} - e^{i\theta_l}) = 0$

$\Rightarrow$ $g_{lm} = 0$ kun $l \neq m$

b) $g \in U(n)$

$\Rightarrow$ $gg^{\#} = I$ $\Rightarrow$ $|g_{mm}| = 1$ $m = 1, 2, \dots, n$

$\Rightarrow$ $g = \mathrm{diag}(e^{i\alpha_1}, \dots, e^{i\alpha_n}) \in T$, $\alpha_i \in \mathbb{R}$

c) $g \in SU(n) \subset U(n)$

$\Rightarrow$ $g = \mathrm{diag}(e^{i\alpha_1}, \dots, e^{i\alpha_n})$

$1 = \det g = e^{i(\alpha_1 + \dots + \alpha_{n-1})} \cdot e^{i\alpha_n}$

$\Rightarrow$ $e^{i\alpha_n} = e^{-i(\alpha_1 + \dots + \alpha_{n-1})}$

Voidaan olettaa, että $\alpha_n = -(\alpha_1 + \dots + \alpha_{n-1})$

d) $g \in Sp(n)$, $\mathbb{K} = \mathbb{H}$

Huom: Oletetaan, että $\theta_i \neq 0, \pi$.

Esim. $k \cdot \underbrace{e^{i \cdot 0}}_{=1} = \underbrace{e^{i0}}_{=1} k$ mutta $k \neq e^{i\theta}$, $\theta \in \mathbb{R}$

n=1 $e^{i\theta} g = g e^{i\theta}$ $\quad g \in \mathbb{H}$ $\quad g = x + yj$, $x, y \in \mathbb{C}$

$$e^{i\theta}(x + yj)$$
$$= x e^{i\theta} + y \underbrace{e^{i\theta} j}_{= j e^{-i\theta}} = x e^{i\theta} + y j e^{-i\theta}$$
$$= x e^{i\theta} + y j e^{i\theta}$$

$\Rightarrow$ $y e^{-i\theta} = y e^{i\theta}$

$\Rightarrow$ $y = 0$ (tai $2\theta = 2\pi k$ $\quad k \in \mathbb{Z}$ $\lightning$)

$\Rightarrow$ $g \in \mathbb{C}$ eli $g = e^{i\theta}$ $\quad \theta \in \mathbb{R}$.

$n \longmapsto n+1$ $\quad g = \begin{pmatrix} \alpha & v \\ u^T & A \end{pmatrix} \in Sp(n+1)$

$$t = \left(\begin{array}{c|c} e^{i\theta} & \\ \hline & T \end{array} \right) \in T(Sp(n+1)) \quad \text{säännöllinen}$$

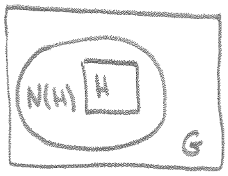
$$gt = tg \Rightarrow \begin{cases} v = v \cdot (e^{-i\theta} \cdot T) \\ (e^{-i\theta} T) u^T = u^T \\ TA = AT \end{cases}$$

$\Rightarrow$ $u = v = 0$ (sillä muuten $k \in \mathbb{R}$, $e^{-i\theta} T \cdot u^T \in \mathbb{C}^n \setminus \mathbb{R}^n$)

$g \cdot g^{\#} = I \Rightarrow \alpha \in Sp(1) \quad A \in Sp(n)$

T säännöllinen $\underset{\text{induktio-oletus}}{\Rightarrow}$ $A \in T(Sp(n)) \Rightarrow g \in T(Sp(n+1))$ $\square$

(10.2) G ryhmä, $H \subset G$ aliryhmä



Määritellään: H:n <u>normalisaattori</u> on

$$N(H) = \{ g \in G \mid gHg^{-1} = H \}$$

<u>Väite</u> $N(H)$ on G:n aliryhmä.

<u>Tod.</u> $eHe^{-1} = H \Rightarrow e \in N(H)$

$a, b \in N(H) \Rightarrow aHa^{-1} = H,\ bHb^{-1} = H$

$\Rightarrow abHb^{-1}a^{-1} = aHa^{-1} = H \Rightarrow ab \in N(H).$

$a \in N(H) \Rightarrow aHa^{-1} = H \quad | \cdot a^{-1}(\cdot\cdot)a$

$\Rightarrow H = a^{-1}H(a^{-1})^{-1} \Rightarrow a^{-1} \in N(H).$

$\square$

<u>Väite:</u> H on $N(H)$:n normaali aliryhmä.

<u>Tod.</u> $H \subset N(H)$, eli jos $h \in H$, niin $hHh^{-1} = H$.

$\boxed{\subset}$ Jos $a \in H$ niin $hah^{-1} \in H$ sillä H aliryhmä.

$\boxed{\supset}$ Olkoon $a \in H$. $a = h\underbrace{(h^{-1}ah)}_{\in H}h^{-1} \in hHh^{-1}$

$\Rightarrow H$ on $N(H)$:n aliryhmä.

H on $N(H)$:n normaali aliryhmä

$\Leftrightarrow nHn^{-1} = H \quad \forall n \in N(H)$

$\square$

10.3 $\quad T = \{ e^{i\theta} \mid \theta \in \mathbb{R} \} \subset Sp(1)$

Väite:

$$N(T) = \{ g \in Sp(1) \mid gT = Tg \}$$
$$= T \cup \{T \cdot j\}$$

$\boxed{\supset}$ 1° $t \in T \Rightarrow t \in N(T)$

2° $tj \in T \cdot j \Rightarrow tjT = Ttj$?

$\Leftrightarrow t \underbrace{jTj}_{} = Tt$

$= \{ j e^{i\theta} j^{-1} \mid \theta \in \mathbb{R} \}$

$= \{ e^{-i\theta} \mid \theta \in \mathbb{R} = T$ /ok

$q \in \mathbb{H} \setminus \mathbb{R}$

$q(\cos + i \sin)$

$= \cos q + qi \sin$

$= (\cos - i \sin) q$

$\boxed{\subset}$ $g \in N(T)$. Olkoon $e^{i\theta} \in T$ säännöllinen

$gT \subset Tg \Rightarrow g e^{i\theta} = e^{iz} g$ jollain $z \in \mathbb{R}$

$g = x + yj \quad x, y \in \mathbb{C}$:

$x e^{i\theta} + yj e^{i\theta} = x e^{i\theta} + y e^{-i\theta} j$

$e^{iz} x + e^{iz} yj = x e^{iz} + y e^{iz} j$

$$\Rightarrow \begin{cases} x(e^{i\theta} - e^{iz}) = 0 \\ y(e^{-i\theta} - e^{iz}) = 0 \end{cases}$$

$x = 0$ tai $\theta = z + 2\pi k \quad k \in \mathbb{Z}$

$y = 0$ tai $-\theta = z + 2\pi k \quad k \in \mathbb{Z}$

1° $e^{i\theta} = e^{iz} \Rightarrow g$ kommutoi säännöllisen alkion kanssa

2° $x = 0 \Rightarrow y \neq 0 \Rightarrow e^{-i\theta} = e^{iz}$ $\Rightarrow g \in T$.

$\Rightarrow yj e^{i\theta} = e^{-i\theta} yj$

$\Rightarrow y \underbrace{e^{jij^{-1}\theta}}_{e^{-i\theta}} = e^{-i\theta} y$ (L6.3.7) $\Rightarrow y$ kommutoi säännöllisen alkion kanssa

$\Rightarrow y = gj^{-1} \in T \Rightarrow g \in T \cdot j$ $\square$

10.4 $SO(3)$:n standardi maksimaalinen torus

$$T = \left\{ \left(\begin{array}{c|c} R_\alpha & \\ \hline & 1 \end{array} \right) \;\middle|\; \alpha \in \mathbb{R} \right\}$$

Merk. $R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ $\quad P_\alpha = \begin{pmatrix} -\cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

Väite:

$$N(T) = \{ g \in SO(3) \mid gT = Tg \} = T \cup \left\{ \left(\begin{array}{c|c} P_\alpha & \\ \hline & -1 \end{array} \right) \;\middle|\; \alpha \in \mathbb{R} \right\}$$

$\boxed{\supset}$ 1° $t \in T$

$\Rightarrow t \in N(T)$.

2° $t = \left(\begin{array}{c|c} P_\alpha & \\ \hline & -1 \end{array} \right) \quad \alpha \in \mathbb{R}$

$$t T t^{-1} = \left\{ \left(\begin{array}{c|c} P_\alpha & \\ \hline & -1 \end{array} \right) \left(\begin{array}{c|c} R_\theta & \\ \hline & 1 \end{array} \right) \underbrace{\left(\begin{array}{c|c} P_\alpha & \\ \hline & -1 \end{array} \right)}_{= t^{-1}} \;\middle|\; \theta \in \mathbb{R} \right\}$$

$$= \left\{ \left(\begin{array}{c|c} P_\alpha R_\theta P_\alpha & \\ \hline & 1 \end{array} \right) \;\middle|\; \theta \in \mathbb{R} \right\} = T$$

sillä $\underbrace{P_\alpha R_\theta}_{P_{\alpha+\theta}} P_\alpha = P_{\alpha+\theta} P_\alpha = R_{\alpha - (\alpha+\theta)} = R_{-\theta}$

(kts. T 3.2. c) $\square$

$\boxed{\subset}$ Olkoon $g \in SO(3)$ s.e. $gT = Tg$

Kirjoitetaan $g = \left(\begin{array}{c|c} A & v^T \\ \hline u & r \end{array}\right)$

$$gT = Tg \Rightarrow \left\{ \left(\begin{array}{c|c} AR_\alpha & v^T \\ \hline u \cdot R_\alpha & r \end{array}\right) \middle| \alpha \in \mathbb{R} \right\}$$

$$= \left\{ \left(\begin{array}{c|c} R_\beta \cdot A & R_\beta \cdot v^T \\ \hline u & r \end{array}\right) \middle| \beta \in \mathbb{R} \right\}$$

Saadaan

$$\{ u \cdot R_\alpha \mid \alpha \in \mathbb{R} \} = u$$
$$\{ R_\beta \cdot v^T \mid \beta \in \mathbb{R} \} = v^T$$

eli $u = v = 0$, ja g on muotoa

$$g = \left(\begin{array}{c|c} A & \\ \hline & r \end{array}\right) \qquad A \in O(2), \quad r \in \{\pm 1\}.$$

$$O(2) = SO(2) \cup \{ P_\alpha \mid \alpha \in \mathbb{R} \}$$

$\underline{1^\circ}$ $A \in SO(2) \Rightarrow A = R_\alpha$ jollain $\alpha \in \mathbb{R}$

$\Rightarrow r = +1$

$\Rightarrow g \in T.$

$\underline{2^\circ}$ $A = P_\alpha$ jollain $\alpha \in \mathbb{R}$

$\Rightarrow r = -1$

$\Rightarrow g = \left(\begin{array}{c|c} P_\alpha & \\ \hline & -1 \end{array}\right)$

$\square$

10.5


```
In[12]:= E1 // MatrixForm
         E2 // MatrixForm
         E3 // MatrixForm
         E4 // MatrixForm
         E5 // MatrixForm
         E6 // MatrixForm
```

Out[12]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[13]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[14]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[15]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[16]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[17]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[18]:= Comm[A_, B_] := A.B - B.A
```

$\mathfrak{heis}_4 = \mathfrak{nil}_4$

$= \mathrm{span}\{E_1, \dots, E_6\}$

$= \overset{\text{span}}{\langle} E_1, E_2, E_3 \rangle$

$\uparrow$

$[\cdot,\cdot]$ toimii generaattorina.

```
In[143]:=
  Norm[Comm[E1, E2] - E4]
  Norm[Comm[E1, E3]]
  Norm[Comm[E1, E4]]
  Norm[Comm[E1, E5] - E6]
  Norm[Comm[E1, E6]]
  Norm[Comm[E2, E3] - E5]
  Norm[Comm[E2, E4]]
  Norm[Comm[E2, E5]]
  Norm[Comm[E2, E6]]
  Norm[Comm[E3, E4] + E6]
  Norm[Comm[E3, E5]]
  Norm[Comm[E3, E6]]
  Norm[Comm[E4, E5]]
  Norm[Comm[E5, E6]]
```

```
Out[143]= 0
Out[144]= 0
Out[145]= 0
Out[146]= 0
Out[147]= 0
Out[148]= 0
Out[149]= 0
Out[150]= 0
Out[151]= 0
Out[152]= 0
Out[153]= 0
Out[154]= 0
Out[155]= 0
Out[156]= 0
```

$[E_1, E_2] = E_4$

$[E_1, E_5] = E_6$

$[E_2, E_3] = E_5$

$[E_3, E_4] = -E_6$

$\square$