Walter Rudin: Real and Complex Analysis, 3rd ed., McGraw-Hill, 1987.

pp. 199-200 & 235-6.

10.7 Theorem *Suppose μ is a complex (finite) measure on a measurable space X, φ is a complex measurable function on X, Ω is an open set in the plane which does not intersect $\varphi(X)$, and*

$$f(z) = \int_X \frac{d\mu(\zeta)}{\varphi(\zeta) - z} \qquad (z \in \Omega). \tag{1}$$

Then f is representable by power series in Ω.

PROOF Suppose $D(a; r) \subset \Omega$. Since

$$\left| \frac{z-a}{\varphi(\zeta) - a} \right| \le \frac{|z-a|}{r} < 1 \tag{2}$$

for every $z \in D(a; r)$ and every $\zeta \in X$, the geometric series

$$\sum_{n=0}^{\infty} \frac{(z-a)^n}{(\varphi(\zeta) - a)^{n+1}} = \frac{1}{\varphi(\zeta) - z} \tag{3}$$

converges uniformly on X, for every fixed $z \in D(a; r)$. Hence the series (3) may be substituted into (1), and $f(z)$ may be computed by interchanging summation and integration. It follows that

$$f(z) = \sum_0^{\infty} c_n (z-a)^n \qquad (z \in D(a; r)) \tag{4}$$

where

$$c_n = \int_X \frac{d\mu(\zeta)}{(\varphi(\zeta) - a)^{n+1}} \qquad (n = 0, 1, 2, \ldots). \tag{5}$$

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Note: The convergence of the series (4) in $D(a; r)$ is a consequence of the proof. We can also derive it from (5), since (5) shows that

$$|c_n| \le \frac{|\mu|(X)}{r^{n+1}} \qquad (n = 0, 1, 2, \ldots). \tag{6}$$

11.9 Theorem *Suppose u is a continuous real function on the closed unit disc $\bar{U}$, and suppose u is harmonic in U. Then (in U) u is the Poisson integral of its restriction to T, and u is the real part of the holomorphic function*

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u(e^{it})\, dt \qquad (z \in U). \tag{1}$$

PROOF Theorem 10.7 shows that $f \in H(U)$. If $u_1 = \operatorname{Re} f$, then (1) shows that u_1 is the Poisson integral of the boundary values of u, and the theorem will be proved as soon as we show that $u = u_1$.

Put $h = u - u_1$. Then h is continuous on $\bar{U}$ (apply Theorem 11.8 to u_1), h is harmonic in U, and $h = 0$ at all points of T. Assume (this will lead to a contradiction) that $h(z_0) > 0$ for some $z_0 \in U$. Fix ϵ so that $0 < \epsilon < h(z_0)$, and define

$$g(z) = h(z) + \epsilon |z|^2 \qquad (z \in \bar{U}). \tag{2}$$

Then $g(z_0) \ge h(z_0) > \epsilon$. Since $g \in C(\bar{U})$ and since $g = \epsilon$ at all points of T, there exists a point $z_1 \in U$ at which g has a local maximum. This implies that $g_{xx} \le 0$ and $g_{yy} \le 0$ at z_1. But (2) shows that the Laplacian of g is $4\epsilon > 0$, and we have a contradiction.

Thus $u - u_1 \le 0$. The same argument shows that $u_1 - u \le 0$. Hence $u = u_1$, and the proof is complete. ////

11.10 So far we have considered only the unit disc $U = D(0; 1)$. It is clear that the preceding work can be carried over to arbitrary circular discs, by a simple change of variables. Hence we shall merely summarize some of the results:

If u is a continuous real function on the boundary of the disc $D(a; R)$ and if u is defined in $D(a; R)$ by the Poisson integral

$$u(a + re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u(a + Re^{it})\, dt \tag{1}$$

then u is continuous on $\bar{D}(a; R)$ and harmonic in $D(a; R)$.

If u is harmonic (and real) in an open set Ω and if $\bar{D}(a; R) \subset \Omega$, then u satisfies (1) in $D(a; R)$ and there is a holomorphic function f defined in $D(a; R)$ whose real part is u. This f is uniquely defined, up to a pure imaginary additive constant. For if two functions, holomorphic in the same region, have the same real part, their difference must be constant (a corollary of the open mapping theorem, or the Cauchy-Riemann equations).

We may summarize this by saying that *every real harmonic function is locally the real part of a holomorphic function.*

Consequently, every harmonic function has continuous partial derivatives of all orders.