$$\leq C R^{1-\frac{n}{p}} \|Du\|_p + \left| \fint_{B(x,R)} u \, dz - \fint_{B(y,R)} u \, dz \right| ,$$

since

$$\left| \fint_{B(x,R)} u \, dz - \fint_{B(y,R)} u \, dz \right| \leq \fint_{B(y,R)} |u - u_{B(x,R)}| \, dz$$

$$\leq \frac{|B(x,2R)|}{|B(y,R)|} \fint_{B(x,2R)} |u - u_{B(x,R)}| \, dz$$

$B(y,R) \subset B(x,2R)$

$$\leq 2^n \left[\fint_{B(x,2R)} |u - u_{B(x,2R)}| \, dz + \underbrace{\fint_{B(x,2R)} |u_{B(x,2R)} - u_{B(x,R)}| \, dz}_{\substack{= |u_{B(x,2R)} - u_{B(x,R)}| \\ \leq 2^n \fint_{B(x,2R)} |u - u_{B(x,2R)}| \, dz}} \right]$$

$$\leq C \fint_{B(x,2R)} |u - u_{B(x,2R)}| \, dz$$

$$\leq C R \fint_{B(x,2R)} |Du| \, dz \leq C R \left(\fint_{B(x,2R)} |Du|^p \, dz \right)^{\frac{1}{p}}$$

$$\leq C |x-y|^{1-\frac{n}{p}} \|Du\|_p .$$

$$\Rightarrow \quad |u^*(x) - u^*(y)| \le C \|Du\|_p \, |x-y|^{1-\frac{n}{p}}$$

Väite, että $u^* = u$ m.k. $\mathbb{R}^n$:ssä seuraa Lebesguen tiheyspistelauseesta.

$\square$

5.21. Esimerkki.

$u : B(0,1) \to [0,\infty]$, $u(x) = |x|^{-\alpha}$, $\alpha > 0$

$u \in C^\infty(B(0,1) \setminus \{0\})$

u:lla singulariteetti 0:ssa ($u(0) = \infty$)

$x \neq 0$

$$\Rightarrow \quad \frac{\partial u}{\partial x_i}(x) = -\alpha |x|^{-\alpha-1} \frac{x_i}{|x|} = -\alpha \frac{x_i}{|x|^{\alpha+2}}, \quad i = 1, \dots, n$$

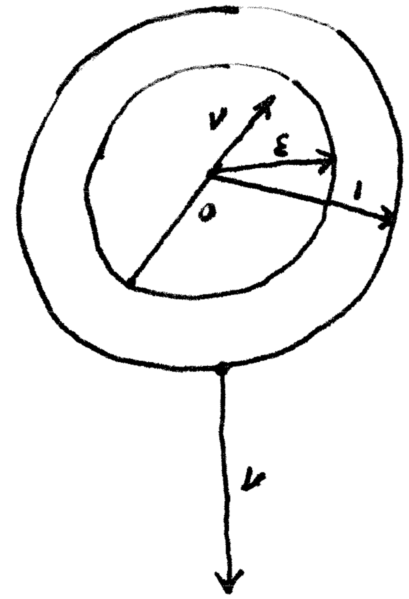
$$\Rightarrow \quad Du(x) = -\alpha \frac{x}{|x|^{\alpha+2}}$$

$\varphi \in C_0^\infty(B(0,1))$, $0 < \varepsilon < 1$

Gaussin lause $\Rightarrow$

$$\int_{B(0,1) \setminus \overline{B(0,\varepsilon)}} \frac{\partial}{\partial x_i}(u\varphi)\, dx = \int_{\partial(B(0,1) \setminus \overline{B(0,\varepsilon)})} u \varphi \nu^i \, dS,$$

missä $\nu = (\nu^1, \dots, \nu^n)$ on reunan ulkoyksikkönormaali



$$\Rightarrow \int_{B(0,1)\setminus\overline{B(0,\varepsilon)}} \frac{\partial u}{\partial x_i}\varphi\,dx + \int_{B(0,1)\setminus\overline{B(0,\varepsilon)}} u\frac{\partial \varphi}{\partial x_i}\,dx$$

$$= \int_{\partial B(0,\varepsilon)} u\varphi\nu^i\,dS$$

$\varphi = 0$ $\partial B(0,1)$:lla

$$\Rightarrow \int_{B(0,1)\setminus\overline{B(0,\varepsilon)}} u\frac{\partial \varphi}{\partial x_i}\,dx = -\int_{B(0,1)\setminus\overline{B(0,\varepsilon)}} \frac{\partial u}{\partial x_i}\varphi\,dx + \int_{\partial B(0,\varepsilon)} u\varphi\nu^i\,dS$$

$$x \in \partial B(0,\varepsilon) \Rightarrow \nu(x) = -\frac{x}{|x|} \Rightarrow \nu^i(x) = -\frac{x_i}{|x|},\ i=1,\dots,n$$

$$\Rightarrow \left|\int_{\partial B(0,\varepsilon)} u\varphi\nu^i\,dS\right| \le \|\varphi\|_{L^\infty(B(0,1))}\int_{\partial B(0,\varepsilon)} \varepsilon^{-\alpha}\,dS$$

$$= \underbrace{\|\varphi\|_{L^\infty(B(0,1))}\,\omega_{n-1}\,\varepsilon^{n-1-\alpha}}_{\to 0,\ \text{kun}\ \varepsilon\to 0,\ n-1-\alpha>0}$$

$$\Rightarrow \int_{B(0,1)} u \frac{\partial \varphi}{\partial x_i} dx$$

$$= \lim_{\varepsilon \to 0} \int_{B(0,1) \setminus \overline{B(0,\varepsilon)}} u \frac{\partial \varphi}{\partial x_i} dx$$

LDK, $\left| u \frac{\partial \varphi}{\partial x_i} \chi_{B(0,1) \setminus \overline{B(0,\varepsilon)}} \right| \leq \underbrace{\|D\varphi\|_\infty |u| \chi_{B(0,1)}}_{\in L^1(\mathbb{R}^n),\ n-\alpha>0}$

$$= -\lim_{\varepsilon \to 0} \int_{B(0,1) \setminus \overline{B(0,\varepsilon)}} \frac{\partial u}{\partial x_i} \varphi \, dx + \underbrace{\lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} u \varphi \nu^i \, dS}_{=0}$$

$$= - \int_{B(0,1)} \frac{\partial u}{\partial x_i} \varphi \, dx \quad \forall \varphi \in C_0^\infty(B(0,1))$$

LDK, $\left| \frac{\partial u}{\partial x_i} \varphi \chi_{B(0,1) \setminus \overline{B(0,\varepsilon)}} \right|$

$$\leq \underbrace{|Du| \, \|\varphi\|_{L^\infty(B(0,1))} \chi_{B(0,1)}}_{\in L^1(\mathbb{R}^n),\ n-\alpha-1>0}$$

$\Rightarrow$ Du on u:n heikko gradientti

$$|Du| \in L^p(B(0,1)) \iff -p(\alpha+1)+n > 0$$

$$\iff \alpha < \frac{n-p}{p}$$

$$u \in L^p(B(0,1)) \iff -p\alpha + n > 0$$

$$\iff \alpha < \frac{n}{p}$$

$\Rightarrow$ $u \in W^{1,p}(B(0,1))$, jos $0 < \alpha < \dfrac{n-p}{p}$

Olkoon (q_i) numeroituva tiheä joukko $B(0,1)$:ssä ja $v : B(0,1) \to [0,\infty]$,

$$v(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} |x - q_i|^{-\alpha}$$

$\Rightarrow$ $v \in W^{1,p}(B(0,1))$, jos $0 < \alpha < \dfrac{n-p}{p}$

ja v:llä on singulariteetteja tiheässä osassa.

<u>Huomautus</u>. Kun $p > n$, niin Sobolevin funktiot ovat Hölder-jatkuvia ja näin ollen lokaalisti rajoitettuja. Kun $1 < p < n$, niin edellä olleen esimerkin nojalla Sobolevin funktioilla voi olla singulariteetteja tiheässä osassa. Kun $p = n$, niin Sobolevin funktion ei tarvitse olla rajoitettu (ja näin ollen sillä voi olla singulariteetteja tiheässä osassa).

<u>Esim</u>: $u : B(0,1) \to [0,\infty]$,

$$u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$$

$$\Rightarrow u \in W^{1,n}(B(0,1)),$$

$$u \notin L^\infty(B(0,1)).$$

$$\boxed{\text{LOPPU}}$$