5. Sobolevin epäyhtälöitä

Tarkastellaan aluksi yksiulotteista tilannetta.

$u \in C_0^1(\mathbb{R})$

$\Rightarrow \exists\ [a,b] \subset \mathbb{R}$ s.e. $u(x) = 0 \quad \forall\, x \in \mathbb{R} \setminus [a,b]$

$$\Rightarrow u(x) = \underbrace{u(a)}_{=0} + \int_a^x u'(y)\,dy = \int_{-\infty}^x u'(y)\,dy$$

↑ Analyysin peruslause

$$\underbrace{u(b)}_{=0} = u(x) + \int_x^b u'(y)\,dy = u(x) + \int_x^\infty u'(y)\,dy$$

$$\Rightarrow u(x) = -\int_x^\infty u'(y)\,dy$$

$$\Rightarrow 2u(x) = \int_{-\infty}^x u'(y)\,dy - \int_x^\infty u'(y)\,dy$$

$$= \int_{-\infty}^x \frac{u'(y)(x-y)}{|x-y|}\,dy + \int_x^\infty \frac{u'(y)(x-y)}{|x-y|}\,dy$$

$$= \int_{\mathbb{R}} \frac{u'(y)(x-y)}{|x-y|}\,dy$$

$$\Rightarrow u(x) = \frac{1}{2}\int_{\mathbb{R}} \frac{u'(y)(x-y)}{|x-y|}\,dy \quad \forall\, x \in \mathbb{R}$$

Seuraavaksi yleistämme tämän esityskaavan $\mathbb{R}^n$:ään.

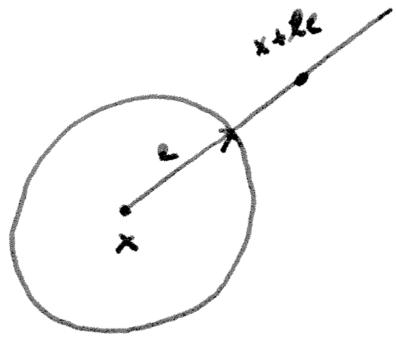
5.1. Lemma. Jos $u \in C_0^1(\mathbb{R}^n)$, niin

$$u(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{Du(y)\cdot(x-y)}{|x-y|^n}\, dy$$

$Du = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$
$= u$:n gradientti

kaikilla $x \in \mathbb{R}^n$, missä $\omega_{n-1} = n\Omega_n = \partial B(0,1)$:n $(n-1)$-ulotteinen mitta.

Tod: $x \in \mathbb{R}^n$, $e \in \mathbb{R}^n$, $|e| = 1$ $\quad (e \in \partial B(0,1))$



$$\Rightarrow u(x) \underset{\substack{\uparrow \\ \text{Anal. peruslause}}}{=} -\int_0^\infty \frac{\partial}{\partial t} u(x+te)\, dt$$

$$\underset{\substack{\uparrow \\ \text{Ketjusääntö}}}{=} -\int_0^\infty Du(x+te)\cdot e\, dt$$

$$\Rightarrow \omega_{n-1} u(x) = u(x) \int_{\partial B(0,1)} 1\, dS(e)$$

$$= -\int_{\partial B(0,1)} \int_0^\infty Du(x+te)\cdot e\, dt\, dS(e)$$

$$\underset{\substack{\uparrow \\ \text{Fubini}}}{=} -\int_0^\infty \int_{\partial B(0,1)} Du(x+te)\cdot e\, dS(e)\, dt$$

$$= -\int_0^\infty \int_{\partial B(0,t)} Du(x+y)\cdot \frac{y}{t}\,\frac{1}{t^{n-1}}\,dS(y)\,dt$$

$y = t e \Leftrightarrow e = \frac{y}{t}$, $dS(e) = \frac{1}{t^{n-1}}\,dS(y)$

muuttujanvaihto

$$= -\int_0^\infty \int_{\partial B(0,t)} Du(x+y)\cdot \frac{y}{|y|^n}\,dS(y)\,dt$$

$$= -\int_{\mathbb{R}^n} \frac{Du(x+y)\cdot y}{|y|^n}\,dy$$

Napakoordinaatisto (co-area-kaava)

$$= -\int_{\mathbb{R}^n} \frac{Du(z)\cdot(z-x)}{|z-x|^n}\,dz$$

$z = x+y \Leftrightarrow y = z-x$, $dy = dz$

$$= \int_{\mathbb{R}^n} \frac{Du(y)\cdot(x-y)}{|x-y|^n}\,dy. \qquad \square$$

$u \in C_0^1(\mathbb{R}^n)$

$$\underset{\text{Lemma 5.1}}{\Rightarrow} |u(x)| \underset{\text{Cauchy-Schwarz}}{\leq} \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{|Du(y)||x-y|}{|x-y|^n}\, dy$$

$$= \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy$$

$$= \frac{1}{\omega_{n-1}} I_1(|Du|)(x)$$

Rieszin ydin I_α, $0<\alpha<n$, määritellään

$$I_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$$

ja Rieszin potentiaali $I_\alpha f$ määritellään konvoluutiona

$$I_\alpha f(x) = (I_\alpha * f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

5.2. Lemma. Jos $0<\alpha<n$, niin on olemassa $c = c(n,\alpha) > 0$ s.e.

$$\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \leq c\, r^\alpha M f(x)$$

Hardy-Littlewoodin maksimaalifunktio

kaikilla $x \in \mathbb{R}^n$, $r > 0$.

Tod:

$$\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy = \sum_{i=0}^{\infty} \int_{B(x,\frac{r}{2^i})\setminus B(x,\frac{r}{2^{i+1}})} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy$$

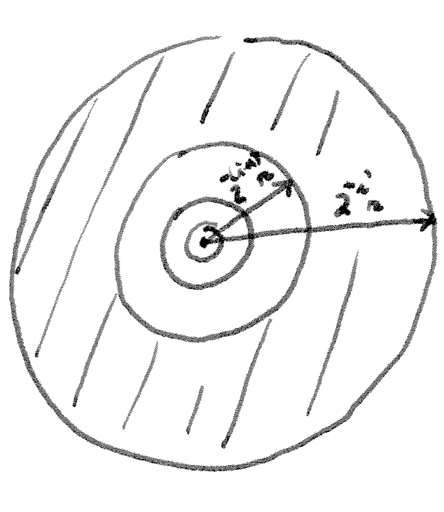
$$\leq \sum_{i=0}^{\infty} \left(\frac{r}{2^{i+1}}\right)^{\alpha-n} \int_{B(x,\frac{r}{2^i})} |f(y)|\,dy$$

$$= \Omega_n \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{\alpha-n} \left(\frac{r}{2^i}\right)^{\alpha} \frac{1}{\Omega_n} \left(\frac{r}{2^i}\right)^{-n} \int_{B(x,\frac{r}{2^i})} |f(y)|\,dy$$

$$= \Omega_n \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{\alpha-n} \left(\frac{r}{2^i}\right)^{\alpha} \underbrace{\frac{1}{|B(x,\frac{r}{2^i})|} \int_{B(x,\frac{r}{2^i})} |f(y)|\,dy}_{\leq Mf(x)}$$

$$= c\, Mf(x)\, r^{\alpha} \underbrace{\sum_{i=0}^{\infty} \left(\frac{1}{2^i}\right)^{\alpha}}_{\text{geometrinen sarja}}$$

$$= c\, r^{\alpha} Mf(x). \quad \square$$



5.3. <u>Lause</u>. (Sobolevin epäyhtälö Rieszin potentiaaleille)

Oletetaan, että $\alpha > 0$, $1 < p < n$, $\alpha p < n$. Silloin on olemassa $c = c(n, p, \alpha) > 0$ s.e. kaikille $f \in L^p(\mathbb{R}^n)$ pätee

$$\| I_\alpha f \|_{p^*} \leq c \| f \|_p ,$$

missä $p^* = \dfrac{pn}{n - \alpha p}$.

<u>Tod</u>:

$$\int_{\mathbb{R}^n \setminus B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy$$

$$\leq \left(\int_{\mathbb{R}^n \setminus B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n \setminus B(x,r)} |x-y|^{(\alpha-n)p'} \, dy \right)^{\frac{1}{p'}},$$

Hölder, $\frac{1}{p} + \frac{1}{p'} = 1$

$$\int_{\mathbb{R}^n \setminus B(x,r)} |x-y|^{(\alpha-n)p'} \, dy = \int_r^\infty \int_{\partial B(x,\varrho)} |x-y|^{(\alpha-n)p'} \, dS(y)\, d\varrho$$

Napakoordinaatit

$$= \int_r^\infty \varrho^{(\alpha-n)p'} \underbrace{\int_{\partial B(x,\varrho)} 1 \, dS(y)}_{= \omega_{n-1}\varrho^{n-1}} \, d\varrho$$

$$= \omega_{n-1} \int_r^\infty \varrho^{(\alpha-n)p'+n-1} \, d\varrho = - \frac{\omega_{n-1}}{(\alpha-n)p'+n} r^{(\alpha-n)p'+n}$$

$$= \frac{\omega_{n-1}}{(n-\alpha)p'-n} r^{n-(n-\alpha)p'}$$

$$n-(n-\alpha)p' = n - (n-\alpha)\frac{p}{p-1} = n\left(1-\frac{p}{p-1}\right) + \alpha\frac{p}{p-1}$$

$$= -\frac{n}{p-1} + \frac{\alpha p}{p-1} = \frac{\alpha p - n}{p-1}$$

$$\Rightarrow \int_{\mathbb{R}^n \setminus B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}} \, dy \le C r^{\frac{\alpha p - n}{p-1} \cdot \frac{p-1}{p}} \|f\|_p$$

$$= C r^{\alpha - \frac{n}{p}} \|f\|_p .$$

Lemma 5.2 $\Rightarrow$

$$|I_\alpha f(x)| \le \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} \, dy$$

$$= \int_{B(x,r)} \dots \, dy + \int_{\mathbb{R}^n \setminus B(x,r)} \dots \, dy$$

$$\leq c \left(r^{\alpha} Mf(x) + r^{\alpha - \frac{n}{p}} \|f\|_p \right)$$

Valitaan

$$r = \left(\frac{Mf(x)}{\|f\|_p} \right)^{-\frac{p}{n}}$$

$$\Rightarrow |I_\alpha f(x)| \leq c \left[\left(\frac{Mf(x)}{\|f\|_p} \right)^{-\frac{\alpha p}{n}} Mf(x) + \left(\frac{Mf(x)}{\|f\|_p} \right)^{-\frac{p}{n}(\alpha - \frac{n}{p})} \|f\|_p \right]$$

$$= c \left[Mf(x)^{1 - \frac{\alpha p}{n}} \|f\|_p^{\frac{\alpha p}{n}} + Mf(x)^{1 - \frac{\alpha p}{n}} \|f\|_p^{1 + \frac{p}{n}(\alpha - \frac{n}{p})} \right]$$

$$= c \, Mf(x)^{1 - \frac{\alpha p}{n}} \|f\|_p^{\frac{\alpha p}{n}}$$

$$\Rightarrow |I_\alpha f(x)|^{p^*} \leq c \, Mf(x)^{\frac{n - \alpha p}{n} \cdot \frac{np}{n - \alpha p}} \|f\|_p^{\frac{\alpha p}{n} \cdot p^*}$$

$$\Rightarrow \int_{\mathbb{R}^n} |I_\alpha f|^{p^*}\,dx \le c\,\|f\|_p^{\frac{\alpha p}{n}p^*} \int_{\mathbb{R}^n} (Mf)^p\,dx$$

$$\le c\,\|f\|_p^{\frac{\alpha p}{n}p^*}\,\|f\|_p^{p}$$

Hardy-Littlewood II (Lause 3.9), $p>1$

$$\Rightarrow \|I_\alpha f\|_{p^*} \le c\,\|f\|_p^{\frac{\alpha p}{n}+\frac{p}{p^*}}$$

$$= c\,\|f\|_p .$$

$$\frac{\alpha p}{n}+\frac{p}{p^*} = \frac{\alpha p}{n}+\cancel{p}\cdot\frac{n-\alpha p}{n\cancel{p}} = 1$$

$\square$

5.4. <u>Lause</u>. Jos $1<p<n$, niin on olemassa vakio $c=c(n,p)>0$ s.e. kaikille $u\in C_0^1(\mathbb{R}^n)$ pätee

$$\|u\|_{p^*} \le c\,\|Du\|_p ,$$

missä $p^* = \dfrac{pn}{n-p}$.

<u>Tod:</u> Lemma 5.1 $\Rightarrow$

$$|u(x)| \leq \frac{1}{\omega_{n-1}} I_1(|Du|)(x) \quad \forall\, x \in \mathbb{R}^n$$

$$\Rightarrow \|u\|_{p^*} \leq c\, \|I_1(|Du|)\|_{p^*} \leq c\, \|Du\|_p$$

Lause 5.3, $\alpha = 1$. □

<u>Huomautus</u>. $u \in C_0^1(\Omega)$, $\Omega \subset \mathbb{R}^n$ avoin $\Rightarrow$

$$\left(\int_\Omega |u|^{p^*} dx \right)^{\frac{1}{p^*}} \leq c \left(\int_\Omega |Du|^p dx \right)^{\frac{1}{p}}$$

<u>Syy</u>: Määritellään $u(x) = 0$ $\forall\, x \in \mathbb{R}^n \setminus \Omega$ (nollajatko)

Seuraavaksi tarkastelemme lokaaleja estimaatteja.

<u>Merkintä</u>: $u \in L^1_{loc}(\mathbb{R}^n)$

$$u_{B(x,r)} = \fint_{B(x,r)} u(y)\, dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy$$

= integraalikeskiarvo

Tarkastellaan aluksi yksiulotteista tilannetta.

$u \in C^1(\mathbb{R})$

$z \in B(x,r) = (x-r, x+r)$, $y \in B(x,r)$

$$u(z) - u(y) = \int_z^y u'(y)\,dy$$

$$\Rightarrow |u(z) - u(y)| \leq \left| \int_z^y u'(y)\,dy \right|$$

$$\leq \int_{x-r}^{x+r} |u'(y)|\,dy$$

$$\Rightarrow \left| u(z) - \fint_{B(x,r)} u(y)\,dy \right| \leq \fint_{B(x,r)} |u(z) - u(y)|\,dy$$

$$\leq \int_{B(x,r)} |u'(y)|\,dy$$

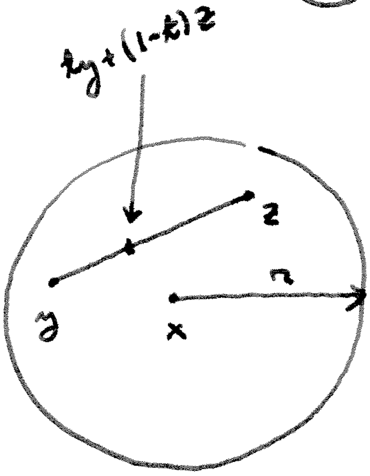
Seuraavaksi yleistämme tämän $\mathbb{R}^n$:ään.

5.5. <u>Lemma</u>. On olemassa $c = c(n) > 0$ s.e.

$$\left| u(z) - \fint_{B(x,r)} u(y)\,dy \right| \leq c \int_{B(x,r)} \frac{|Du(y)|}{|z-y|^{n-1}}\,dy$$

kaikilla $B(x,r) \subset \mathbb{R}^n$, $z \in B(x,r)$ ja $u \in C^1(\mathbb{R}^n)$.

Tod: $y, z \in B(x,r)$



$$u(y) - u(z) = \int_0^1 \frac{\partial}{\partial t} u(ty + (1-t)z)\, dt$$

Analyysin peruslause

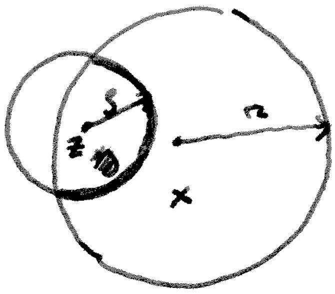
$$= \int_0^1 Du(ty + (1-t)z) \cdot (y - z)\, dt$$

ketjusääntö

$$\Rightarrow |u(y) - u(z)| \leq |y - z| \int_0^1 |Du(ty + (1-t)z)|\, dt$$

Cauchy-Schwarz

$$\Rightarrow \int_{B(x,r) \cap \partial B(z,\rho)} |u(y) - u(z)|\, dS(y)$$



$$\leq \rho \int_0^1 \int_{B(x,r) \cap \partial B(z,\rho)} |Du(ty + (1-t)z)|\, dS(y)\, dt$$

$w = ty + (1-t)z \iff y = \frac{1}{t}(w - (1-t)z)$,

$dS(y) = \frac{1}{t^{n-1}} dS(w)$, $|w - z| = t|y - z|$

$$= \rho \int_0^1 \frac{1}{t^{n-1}} \int_{B(x,r) \cap \partial B(z, t\rho)} |Du(w)|\, dS(w)\, dt$$

$$= \varrho^n \int_0^1 \int_{B(x,r)\cap \partial B(z,t\varrho)} \frac{|Du(w)|}{|z-w|^{n-1}}\, dS(w)\, dt$$

$$|z-w| = t\varrho \iff t^{n-1} = \left(\frac{|z-w|}{\varrho}\right)^{n-1}$$

$$= \varrho^{n-1} \int_0^\varrho \int_{B(x,r)\cap \partial B(z,s)} \frac{|Du(w)|}{|z-w|^{n-1}}\, dS(w)\, ds$$

$$s = t\varrho \Rightarrow dt = \frac{1}{\varrho}\, ds$$

$$= \varrho^{n-1} \int_{B(x,r)\cap B(z,\varrho)} \frac{|Du(w)|}{|z-w|^{n-1}}\, dw$$

Napakoordinaatit

$$\Rightarrow \left| u(z) - \fint_{B(x,r)} u(y)\, dy \right| \le \fint_{B(x,r)} |u(z)-u(y)|\, dy$$

$$= \frac{1}{|B(x,r)|} \int_0^{2r} \int_{B(x,r)\cap \partial B(z,\varrho)} |u(y)-u(z)|\, dS(y)\, d\varrho$$

Napakoordinaatit, $B(x,r) \subset B(z,2r)$

$$\leq \frac{1}{|B(x,r)|} \int_0^{2r} \varrho^{n-1} \int_{B(x,r) \cap B(z,\varrho)} \frac{|Du(y)|}{|z-y|^{n-1}} \, dy \, d\varrho$$

$$\leq \frac{1}{|B(x,r)|} \int_{B(x,r)} \frac{|Du(y)|}{|z-y|^{n-1}} \, dy \underbrace{\int_0^{2r} \varrho^{n-1} \, d\varrho}_{= \frac{1}{n} \Big/_0^{2r} \varrho^n = \frac{2^n}{n} r^n}$$

$$= c(n) \int_{B(x,r)} \frac{|Du(y)|}{|z-y|^{n-1}} \, dy . \qquad \square$$

5.6. __Lause.__ (Sobolev–Poincarén epäyhtälö)

Jokaisella $1<p<n$ on olemassa vakio $c = c(n,p) > 0$ s.e.

$$\left(\fint_{B(x,r)} |u - u_{B(x,r)}|^{p^*} \, dy \right)^{\frac{1}{p^*}} \leq c r \left(\fint_{B(x,r)} |Du|^p \, dy \right)^{\frac{1}{p}}$$

kaikilla $B(x,r) \subset \mathbb{R}^n$ ja $u \in C^1(\mathbb{R}^n)$.

__Tod:__ Lemma 5.5 $\Rightarrow$

$$|u(z) - u_{B(x,r)}| \leq c \int_{B(x,r)} \frac{|Du(y)|}{|z-y|^{n-1}} \, dy$$

$$= c \, I_1 \left(|Du| \chi_{B(x,r)} \right)(z), \quad z \in B(x,r)$$