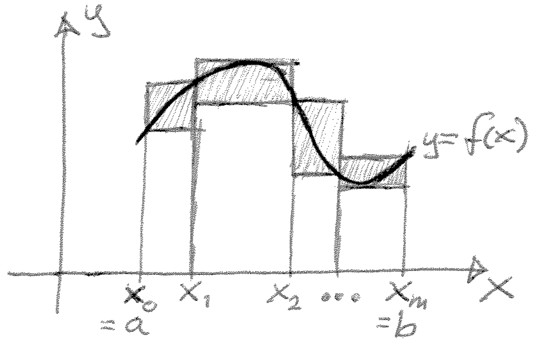
Riemann-integraali (kertaus)

Välin $[a,b] \subset \mathbb{R}$



ositus $\mathcal{P} = \{x_i\}_{i=0}^{m}$:

$$a = x_0 < x_1 < \dots < x_m = b.$$

Rajoitettu $f : [a,b] \to \mathbb{R}$,

"yläsumma" $U(f,\mathcal{P}) := \sum_{i=1}^{m} \left[\sup_{x_{i-1} \le x \le x_i} f(x) \right] (x_i - x_{i-1})$,

"alasumma" $L(f,\mathcal{P}) := \sum_{i=1}^{m} \left[\inf_{x_{i-1} \le x \le x_i} f(x) \right] (x_i - x_{i-1})$.

f Riemann-integroituva, jos

$$\sup_{\mathcal{P}} L(f,\mathcal{P}) = \inf_{\mathcal{P}} U(f,\mathcal{P})$$

$$\Big(=: \underset{\text{RIEMANN}}{\int_a^b f(x)\,dx} \text{ silloin} \Big).$$

[1854 [(Georg Friedrich) Bernhard Riemann (1826–1866)]].

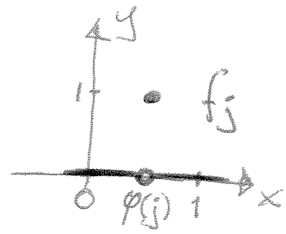
Ongelma:

voi olla $\underset{\text{RIEMANN}}{\int_a^b \sum_{j=1}^{\infty} f_j(x)\,dx} \ne \sum_{j=1}^{\infty} \underset{\text{RIEMANN}}{\int_a^b f_j(x)\,dx}$,

vaikka f_j "hyvin mukavia" ($j \in \mathbb{Z}^+$).

Esim. $\varphi : \mathbb{Z}^+ \to \mathbb{Q} \cap [0,1]$ bijektio,

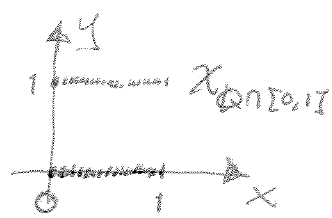
$f_j := \chi_{\{\varphi(j)\}}$.



$\sum_{j=1}^{\infty} \underset{\text{RIEMANN}}{\int_0^1 f_j(x)\,dx} = 0$, mutta

$\sum_{j=1}^{\infty} f_j = \chi_{\mathbb{Q} \cap [0,1]}$,

joka ei ole Riemann-integroituva!



6. Integraali (Lebesguen tapaan)

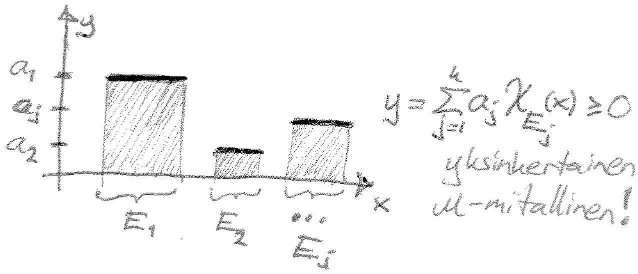
~ 1901/1902 – [Henri Lebesgue (1875–1941)]

6.1 Alkeita

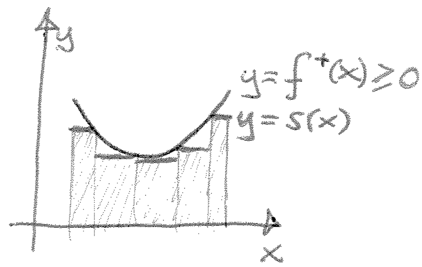
Seuraavassa $(X, \mathcal{M}, \mu)$ on mitta-avaruus.

Määr. 1.
$$\widetilde{\widetilde{\int}} \sum_{j=1}^{k} a_j \chi_{E_j} \, d\mu := \sum_{j=1}^{k} a_j \, \mu(E_j),$$
kun $k \in \mathbb{Z}^+$, $\{a_j\}_{j=1}^k \subset [0,\infty[$, $\{E_j\}_{j=1}^k \subset \mathcal{M}$ erilliset
(sopimuksena $0 \cdot \infty = 0$).

Huom. $\widetilde{\widetilde{\int}} \chi_E \, d\mu = \mu(E)$.



Määr. 2.
Jos $f^+ \geq 0$ $\mathcal{M}$-mitallinen,
$$\widetilde{\int} f^+ \, d\mu := \sup_{s \leq f^+} \widetilde{\widetilde{\int}} s \, d\mu.$$



Huom. $\widetilde{\widetilde{\int}} s \, d\mu = \widetilde{\int} s \, d\mu \in [0,\infty]$.

Määr. 3.
$\mathcal{M}$-mitallisen $f : X \to [-\infty,\infty]$ μ-integraali on
$$\int \underbrace{f}_{f = f^+ - f^-,} d\mu := \widetilde{\int} f^+ \, d\mu - \widetilde{\int} f^- \, d\mu,$$
$f^+ \geq 0$, $f^- \geq 0$

jos $\widetilde{\int} f^+ d\mu < \infty$ tai $\widetilde{\int} f^- d\mu < \infty$ (jolloin $\int f \, d\mu \in [-\infty,\infty]$).
Jos $\widetilde{\int} f^+ d\mu < \infty$ ja $\widetilde{\int} f^- d\mu < \infty$,
sanotaan, että f on μ-integroituva.

Huom. $\widetilde{\int} f^+ \, d\mu = \int f^+ d\mu \in [0,\infty]$.

Lause 6.6

Olkoot f, g μ-integroituvia. Silloin:

(1) $f(x) \in \mathbb{R}$ μ-m.k. $x \in X$.

(2) $\int \underbrace{a}_{\in \mathbb{R}} f \, d\mu = a \int f \, d\mu$, $\int (f+g)\, d\mu = \int f \, d\mu + \int g \, d\mu$.

(3) $f \le g$ $\Rightarrow$ $\int f \, d\mu \le \int g \, d\mu$.

(4) $E \in \mathcal{M} \Rightarrow f \chi_E$ μ-integroituva;

merkitään $\int_E f \, d\mu := \int f \chi_E \, d\mu$.

(5) $\mathcal{M}$-mitallinen h on μ-integroituva joss $|h|$ on μ-integroituva.

(6) $|\int f \, d\mu| \le \int |f| \, d\mu$.

Tod.

(1) $\infty \overset{f \text{ int.}}{>} \int f^+ \, d\mu \ge \int \underbrace{k}_{\ge 0} \cdot \chi_{\{f^+ = \infty\}} \, d\mu := k \cdot \mu(\{f^+ = \infty\})$

$\Rightarrow \mu(\{f^+ = \infty\}) = 0$. Samoin

$\mu(\{f^- = \infty\}) = 0$. $\blacksquare$ (1)

(2) Selvästi $\int 0 \cdot f \, d\mu = 0 = 0 \cdot \int f \, d\mu$. | af $\mathcal{M}$-mit.

Ota $a > 0$ $(a \in \mathbb{R})$.

Selvästi $\tilde{\tilde{\int}} a \cdot s \, d\mu = a \tilde{\int} s \, d\mu$ | $s \le f^+$ joss $as \le af^+$

$\Rightarrow \tilde{\int} a \cdot f^+ \, d\mu = a \tilde{\int} f^+ \, d\mu$

$\Rightarrow \int a f \, d\mu = \int (af^+ - af^-)\, d\mu = \int af^+ d\mu - \int af^- d\mu$

$= a\left(\int f^+ d\mu - \int f^- d\mu\right) = a \int f \, d\mu$,

$\Rightarrow \int -af \, d\mu = \int (af^- - af^+)\, d\mu = \ldots$ | $-af$ $\mathcal{M}$-mit.

$= -a \int f \, d\mu$.

$$\int (f^+ + g^+)\,d\mu = \sup_{t \le f^+ + g^+} \tilde{\tilde{\int}} \underbrace{t}_{\substack{=r+s,\\ r\le f^+,\, s\le g^+}} d\mu \qquad \Big|\ f^+ + g^+ \ \mu\text{-mit.}$$

$$= \sup_{\substack{r \le f^+\\ s\le g^+}} \left(\tilde{\tilde{\int}} r\,d\mu + \tilde{\tilde{\int}} s\,d\mu \right)$$

$$\le \sup_{r\le f^+} \tilde{\tilde{\int}} r\,d\mu + \sup_{s\le g^+} \tilde{\tilde{\int}} s\,d\mu$$

$$= \int f^+ d\mu + \int g^+ d\mu \qquad \Big\|\ \text{Ota } \varepsilon > 0. \text{ Ota } r \le f^+ \text{ ja } s \le g^+ \text{ s.e. } \ldots$$

$$< \left(\tilde{\tilde{\int}} r\,d\mu + \varepsilon \right) + \left(\tilde{\tilde{\int}} s\,d\mu + \varepsilon \right)$$

$$= \tilde{\tilde{\int}} (r+s)\,d\mu + 2\varepsilon \qquad \Big\|\ r+s \le f^+ + g^+$$

$$\le \int (f^+ + g^+)\,d\mu + 2\varepsilon$$

$$\Rightarrow \int (f^+ + g^+)\,d\mu = \int f^+ d\mu + \int g^+ d\mu. \quad (*)$$

$$f+g = \begin{cases} (f+g)^+ - (f+g)^-, \\ (f^+ - f^-) + (g^+ - g^-) \end{cases} \qquad \Big|\ f+g \ \mu\text{-mit.}$$

$$\Rightarrow (f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

$$\overset{(*)}{\Rightarrow} \int (f+g)^+ d\mu + \int f^- d\mu + \int g^- d\mu = \int (f+g)^- d\mu + \int f^+ d\mu + \int g^+ d\mu \quad (**)$$

$$\Rightarrow \int (f+g)\,d\mu := \int (f+g)^+ d\mu - \int (f+g)^- d\mu$$

$$\overset{(**)}{=} \int f^+ d\mu - \int f^- d\mu + \int g^+ d\mu - \int g^- d\mu$$

$$= \int f\,d\mu + \int g\,d\mu. \qquad \boxtimes\ (2)$$

...

(3) $\int f^+ d\mu = \sup_{s \le f^+} \tilde{\tilde{\int}} s \, d\mu \overset{f^+ \le g^+}{\le} \sup_{t \le g^+} \tilde{\tilde{\int}} t \, d\mu = \int g^+ d\mu,$

$$\int f \, d\mu = \underbrace{\int \underbrace{f^+}_{\le g^+} d\mu}_{\le \int g^+ d\mu} - \underbrace{\int \underbrace{f^-}_{\ge g^-} d\mu}_{\ge \int g^- d\mu}$$

$$\le \int g^+ d\mu - \int g^- d\mu = \int g \, d\mu. \quad \boxtimes (3)$$

(4) f $\mathcal{M}$-mitallinen, $E \in \mathcal{M}$
$\Rightarrow f \cdot \chi_E$ $\mathcal{M}$-mitallinen.

$$\int \underbrace{(f\chi_E)^+}_{= f^+ \chi_E} d\mu = \sup_{s \le f^+ \chi_E} \tilde{\tilde{\int}} s \, d\mu$$

$$\le \sup_{t \le f^+} \tilde{\tilde{\int}} t \, d\mu = \int f^+ d\mu \overset{f \text{ int.}}{<} \infty,$$

samoin $\int (f\chi_E)^- d\mu \le \int f^- d\mu < \infty$. $\boxtimes$ (4)

(5) Jos h μ-integroituva, niin

$$|h| = \underbrace{h^+}_{h \cdot \chi_{\{h>0\}}} + \underbrace{h^-}_{-h \cdot \chi_{\{h<0\}}} \quad \mu\text{-integroituva.}$$

Jos $|h|$ μ-integroituva, niin

$$h = |h| \cdot \chi_{\{h>0\}} - |h| \cdot \chi_{\{h<0\}} \quad \mu\text{-integroituva.}$$

$\boxtimes$ (5)

(6) $$\left| \int f \, d\mu \right| = \left| \int f^+ d\mu - \int f^- d\mu \right|$$

$$\le \left| \int f^+ d\mu \right| + \left| \int f^- d\mu \right|$$

$$= \int f^+ d\mu + \int f^- d\mu$$

$$= \int (f^+ + f^-) \, d\mu = \int |f| \, d\mu. \quad \blacksquare$$

Huom.

Jos $\mu(E)=0$, niin

$$0 = \tilde{\tilde{\int}} s\cdot\chi_E \, d\mu = \int_E s \, d\mu,$$

$$\left.\begin{aligned} &\Rightarrow \int_E f^+ d\mu = 0, \\ &\Rightarrow \int_E f \, d\mu = 0 \end{aligned}\right\} \text{kun } f \ \mathcal{M}\text{-mitallinen.}$$

$$\int f \, d\mu = \int (f\cdot\chi_{\{f=0\}} + f\cdot\chi_{\{f\neq 0\}}) \, d\mu$$
$$= \int_{\{f\neq 0\}} f \, d\mu;$$

$$\boxed{\begin{array}{l} \text{jos } f=0 \ \mu\text{-m.k., niin } \int f \, d\mu = 0. \\ \text{Jos taas } \int |f| \, d\mu = 0, \text{ niin } f=0 \ \mu\text{-m.k.:} \end{array}}$$

$$0 \le \mu(\{f\neq 0\}) = \mu\Big(\bigcup_{k=1}^{\infty} \{|f| > 1/k\}\Big)$$
$$\le \sum_{k=1}^{\infty} \mu(\{|f| > \tfrac{1}{k}\})$$
$$\le \sum_{k=1}^{\infty} \int_{\{|f|>\frac{1}{k}\}} k\cdot|f| \, d\mu \le \sum_{k=1}^{\infty} k \underbrace{\int |f| \, d\mu}_{=0} = 0.$$

Huom.

Jos f on $\mathcal{M}$-mitallinen,

$|f| \le g$ ja g μ-integroituva,

niin

f on μ-integroituva.

6.2 Rajankäyntiä

Seuraavassa $(X, \mathcal{M}, \mu)$ on mitta-avaruus.

Lause 6.11 (Monotonisen konvergenssin lause)
Olkoot $f_k \geq 0$ $\mathcal{M}$-mitallisia,
$f_k \leq f_{k+1}$ $(k \in \mathbb{Z}^+)$.
Silloin
$$\lim_{k\to\infty} \int f_k \, d\mu = \int \lim_{k\to\infty} f_k \, d\mu.$$

Tod.
Nyt $f := \lim_{k\to\infty} f_k$ on $\mathcal{M}$-mitallinen.

"$\leq$" $\underbrace{\int f_k \, d\mu}_{\xrightarrow[k\to\infty]{} \lim_{k\to\infty} \int f_k d\mu.} \overset{f_k \leq f_{k+1} \to f}{\leq} \int f \, d\mu$ $\boxtimes$

"$\geq$" Olkoon $s \leq f$, $s = \sum_{j=1}^{n} \underbrace{a_j}_{\geq 0} \chi_{A_j} \geq 0$ $\{A_j\}_{j=1}^{k} \subset \mathcal{M}$ erilliset.
(yksinkertainen $\mathcal{M}$-mitallinen).
Olkoon $0 < \varepsilon < 1$,
$E_k := \{ f_k > (1-\varepsilon) \cdot s \} \in \mathcal{M}$, $E_k \subset E_{k+1}$, $\bigcup_{k=1}^{\infty} E_k = X$. $\circledast$

$$\Rightarrow \int f_k \, d\mu \geq \int_{E_k} f_k \, d\mu \geq (1-\varepsilon) \underbrace{\int_{E_k} s \, d\mu}_{= \int s \chi_{E_k} d\mu = \int \sum_{j=1}^{n} a_j \chi_{A_j \cap E_k} d\mu}$$
$$= (1-\varepsilon) \sum_{j=1}^{n} a_j \, \mu(A_j \cap E_k)$$
$$\xrightarrow[k\to\infty]{\circledast} (1-\varepsilon) \sum_{j=1}^{n} a_j \mu(A_j) = (1-\varepsilon) \int s \, d\mu$$
$$\overset{\varepsilon \to 0}{\Longrightarrow} \lim_{k\to\infty} \int f_k \, d\mu \geq \int s \, d\mu$$
$$\Rightarrow \lim_{k\to\infty} \int f_k \, d\mu \geq \int f \, d\mu. \quad \blacksquare$$

Seuraus 6.12, 6.13

Jos $\{E_k\}_{k=1}^{\infty} \subset \mathcal{M}$ pistevieras, $X = \bigcup_{k=1}^{\infty} E_k$ ja

f μ-integroituva,

niin
$$\int f\,d\mu = \sum_{k=1}^{\infty} \int_{E_k} f\,d\mu.$$

Tod.

$$\int f\,d\mu = \int f^+ d\mu - \int f^- d\mu,$$

$$\int f^+ d\mu = \int \lim_{l\to\infty} \sum_{k=1}^{l} f^+ \chi_{E_k}\, d\mu$$

$$\overset{\text{mon. konv.}}{=} \lim_{l\to\infty} \int \sum_{k=1}^{l} f^+ \chi_{E_k}\, d\mu = \lim_{l\to\infty} \sum_{k=1}^{l} \int f^+ \chi_{E_k}\, d\mu$$

$$= \sum_{k=1}^{\infty} \int_{E_k} f^+ d\mu, \quad \text{samoin}$$

$$\int f^- d\mu = \sum_{k=1}^{\infty} \int_{E_k} f^- d\mu. \quad \blacksquare$$

Lause 6.10 (Fatoun lemma)

Olkoot $f_k \geq 0$ $\mathcal{M}$-mitallisia.

Silloin
$$\int \liminf_{k\to\infty} f_k\, d\mu \leq \liminf_{k\to\infty} \int f_k\, d\mu.$$

(epäyhtälö usein aito: harjoitustehtävä,...)

Tod.
$$\int \liminf_{k\to\infty} f_k\, d\mu = \int \underbrace{\sup_{k\geq 1} \underbrace{\inf_{l\geq k} f_l}_{=:g_k \geq 0\ \mathcal{M}\text{-mit.},\ g_k \leq g_{k+1}}}_{=\lim_{k\to\infty} g_k} d\mu$$

$$\overset{\text{mon. konv.}}{=} \lim_{k\to\infty} \int g_k\, d\mu$$

$$= \liminf_{k\to\infty} \int g_k\, d\mu$$

$$\overset{g_k \leq f_k}{\leq} \liminf_{k\to\infty} \int f_k\, d\mu. \quad \blacksquare$$

Lause 6.14 (Lebesguen dominoidun konvergenssin lause)

Olkoot $f_k : X \to [-\infty, \infty]$ $\mathcal{M}$-mitallisia,

$f_k \to f$ μ-m.k., f $\mathcal{M}$-mitallinen,

$|f_k| \leq g$ μ-m.k., g μ-integroituva.

Silloin

$$\int |f_k - f| \, d\mu \to 0,$$

$$\int f_k \, d\mu \to \int f \, d\mu .$$

Tod.

Nyt $|f_k| \leq g$, $|f| \leq g$ μ-m.k., f_k, f $\mathcal{M}$-mitallisia

$\Rightarrow$ f_k, f μ-integroituvia.

$$\int 2g \, d\mu = \int \liminf_{k\to\infty} (2g - |f_k - f|) \, d\mu$$

$$\overset{\text{Fatou}}{\leq} \liminf_{k\to\infty} \int (2g - |f_k - f|) \, d\mu$$

$$= \int 2g \, d\mu + \underbrace{\liminf_{k\to\infty} - \int |f_k - f| \, d\mu}_{= - \limsup_{k\to\infty} \int |f_k - f| d\mu}$$

$$\Rightarrow \limsup_{k\to\infty} \int |f_k - f| \, d\mu = 0 . \quad \boxtimes$$

$$\left| \int f_k \, d\mu - \int f \, d\mu \right| = \left| \int (f_k - f) \, d\mu \right|$$

$$\leq \int |f_k - f| \, d\mu$$

$$\xrightarrow[k\to\infty]{} 0 . \quad \blacksquare$$

Seuraus (Lause 6.15, 6.16)

Rajoitettu $f: [a,b] \to \mathbb{R}$
on Riemann-integroituva
jos ja vain jos
f on jatkuva $\lambda_{\mathbb{R}}$-m.k. $x \in [a,b]$.
Silloin f on myös Lebesgue-integroituva ja

$$\underset{\text{RIEMANN}}{\int_a^b f(x)\,dx} = \int_{[a,b]} f \, d\lambda_{\mathbb{R}}.$$

Tod.
Ks. [GZ 6.3, s. 142–144]. ■

Esim. $\int_{[0,1]} \chi_{\mathbb{Q}} \, d\lambda_{\mathbb{R}} = 0$

kaikkialla epäjatkuva,
ei Riemann-integroituva.