Analyysi I (Ville Turunen, kevät 2006)

("Mitta ja integraali")

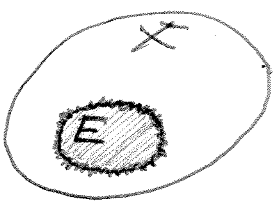
[GZ] Gariepy & Ziemer: Modern real analysis.

Ulkomitta ja mitta [GZ 4]

Joukko X

Potenssijoukko $\mathcal{P}(X) := \{E \mid E \subset X\}$.

Miten "punnita" $E \subset X$?



Määr.

Olkoon $\{\emptyset, X\} \subset \underbrace{\mathcal{A}}_{\text{"alkeisjoukot"}} \subset \mathcal{P}(X)$.

X:in "mitake" ("alkeismitta") on

$$m : \mathcal{A} \to [0,\infty],$$

jolle $m(\emptyset) = 0$.

Esim. 1. $w : \{\emptyset, X\} \to [0,\infty]$, $(X \neq \emptyset)$,

$$w(\emptyset) := 0, \qquad w(X) := 1.$$

Esim. 2. $(E \mapsto \underbrace{\#E}_{\text{joukon E alkioiden lukumäärä}}) : \mathcal{P}(X) \to [0,\infty]$.

[redacted]

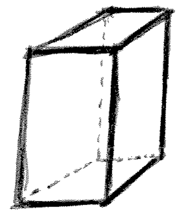
Esim. 3.

Ota $a, b \in \mathbb{R}^n$. Merk.

$$a \leq b \quad \text{joss} \quad \forall k: a_k \leq b_k .$$

Väli ("laatikko")

$$I_{ab} := \{c \in \mathbb{R}^n \mid a \leq c \leq b\}$$
$$= \prod_{k=1}^{n} [a_k, b_k] .$$



Olkoon

$$\mathcal{A} := \{\emptyset, \mathbb{R}^n\} \cup \{ I_{ab} \subset \mathbb{R}^n \mid a, b \in \mathbb{R}^n, a \leq b \} .$$

"Lebesgue-mitake"

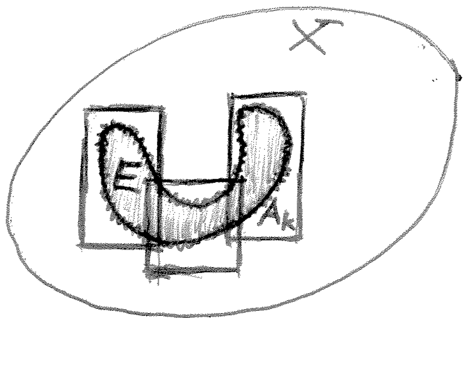
$$\lambda_{\mathbb{R}^n} : \mathcal{A} \to [0, \infty] ,$$

$$\lambda_{\mathbb{R}^n}(\emptyset) := 0 , \qquad \lambda_{\mathbb{R}^n}(\mathbb{R}^n) := \infty ,$$

$$\lambda_{\mathbb{R}^n}(I_{ab}) := \prod_{k=1}^{n} (b_k - a_k) .$$

Määr. X:n mitake $m : \mathcal{A} \to [0, \infty]$ synnyttää (indusoi/generoi) funktion $m^* : \mathcal{P}(X) \to [0, \infty]$,

$$m^*(E) := \inf \left\{ \sum_{k=1}^{\infty} m(A_k) : E \subset \bigcup_{k=1}^{\infty} A_k , A_k \in \mathcal{A} \right\} .$$



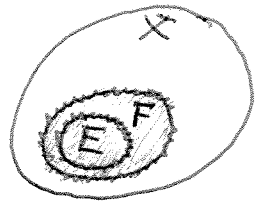
$\{A_k\}_{k=1}^{\infty} \subset \mathcal{A}$ numeroituva peite E:lle

"alkeisjoukot"

Lause

"Mitakkeen" $m : \mathcal{A} \to [0,\infty]$ synnyttämä m^* on <u>ulkomitta</u> eli

$$m^* : \mathcal{P}(X) \to [0,\infty]$$

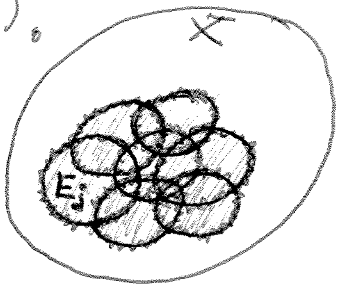


s.e.

(1) $m^*(\emptyset) = 0$,

(2) $E \subset F \Rightarrow m^*(E) \leq m^*(F)$ ja

(3) $m^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m^*(E_j)$.



Todistus

Tehtävä. ∎

Esim. 1'. $w^* : \mathcal{P}(X) \to [0,\infty]$,

$$w^*(E) = \begin{cases} 0, & \text{jos } E = \emptyset, \\ 1, & \text{jos } E \neq \emptyset. \end{cases}$$

Esim. 2'. $(E \mapsto \# E)^* = (E \mapsto \# E)$.

Esim! 3'.

<u>Lebesgue-ulkomitta</u> $\lambda^*_{\mathbb{R}^n} : \mathcal{P}(\mathbb{R}^n) \to [0,\infty]$.

Pätee mm. $\lambda^*_{\mathbb{R}^n}(I_{ab}) = \lambda_{\mathbb{R}^n}(I_{ab})$

[ModA], [GZ 4.3].

<u>Esim.</u> $\mathbb{R}^n$:n muita ulkomittoja:

- Hausdorff-ulkomitat $\mathcal{H}^s$ [GZ4.7] [ModA]
- kapasiteetit Cap_p [Gariepy-Evans]
- Lebesgue-Stieltjes-ulkomitat λ_f^* [GZ 4.6, n=1]

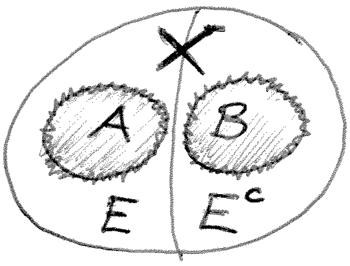
<u>Esim.</u> Lokaalisti kompaktin ryhmän Haar-ulkomitta (esim. $\lambda^*_{\mathbb{R}^n}$)

$\uparrow$ (ei käsitellä tällä kurssilla)

<u>Määr.</u> Olkoon $\psi : \mathcal{P}(X) \to [0,\infty]$ ulkomitta. $E \subset X$ on <u>ψ-mitallinen</u>, jos

$$\psi(A \cup B) = \psi(A) + \psi(B)$$

kaikilla $\begin{cases} A \subset E, \\ B \subset E^c. \end{cases}$



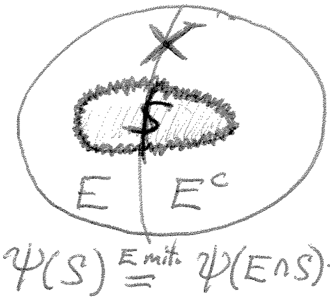
$\left(\begin{aligned} E^c &= X \setminus E \\ &= \{x \in X : x \notin E\}, \text{ E:n komplementti} \end{aligned} \right)$

Merk.

$$\mathcal{M}(\psi) := \{ E \subset X \mid E \text{ on } \psi\text{-mitallinen} \}.$$

Tulkinta:

mitallinen joukko jakaa avaruuden X "kauniisti" kahtia.



$$\psi(S) \overset{E \text{ mit.}}{=} \psi(E \cap S) + \psi(E^c \cap S).$$

<u>Esim. 1''</u> $\mathcal{M}(w^*) = \{\emptyset, X\}$.

<u>Esim. 2''</u> $\mathcal{M}(E \mapsto \#E) = \mathcal{P}(X)$.

<u>Esim. 3''</u> $\lambda^*_{\mathbb{R}^n}$-ei-mitallisten joukkojen olemassaolo voidaan todistaa <u>joss</u> valinta-aksiooma hyväksytään [GZ 4.5].

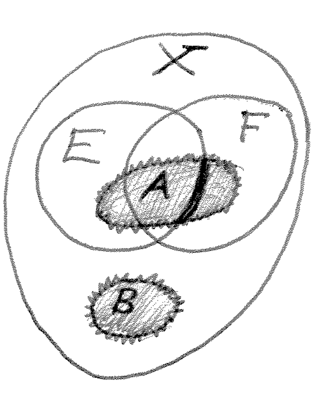
<u>Esim.</u> Jos $\psi(E)=0$, niin $E \in \mathcal{M}(\psi)$.

<u>Tod.</u> Ota $A \subset E$, $B \subset E^c$.

$$\psi(A\cup B) \overset{(3)}{\leq} \psi(A) + \psi(B) \overset{A\subset E,(2)}{\leq} \underbrace{\psi(E)}_{=0} + \psi(B) \overset{B\subset A\cup B}{\leq} \psi(A\cup B). \blacksquare$$

<u>Lemma</u> Jos $E, F \in \mathcal{M}(\psi)$, niin $E\cup F \in \mathcal{M}(\psi)$.

<u>Tod.</u> Ota $A \subset E\cup F$, $B \subset (E\cup F)^c = E^c \cap F^c$.



$$\psi(A) + \psi(B) \overset{E\text{ mit.}}{=} \psi(A\cap E) + \psi(\underbrace{A\cap E^c}_{\subset F}) + \psi(\underbrace{B}_{\subset F^c})$$

$$\overset{F\text{ mit.}}{=} \psi(\underbrace{A\cap E}_{\subset E}) + \psi(\underbrace{(A\cap E^c)\cup B}_{\subset E^c})$$

$$\overset{E\text{ mit.}}{=} \psi((A\cap E)\cup((A\cap E^c)\cup B))$$

$$= \psi(A\cup B). \blacksquare$$

Lause.

Olkoon $\psi : \mathcal{P}(X) \to [0,\infty]$ ulkomitta.

Silloin $\mathcal{M} = \mathcal{M}(\psi)$ on <u>σ-algebra</u> eli

(1) $\emptyset \in \mathcal{M}$,

(2) $X \setminus E = E^c \in \mathcal{M}$, jos $E \in \mathcal{M}$,

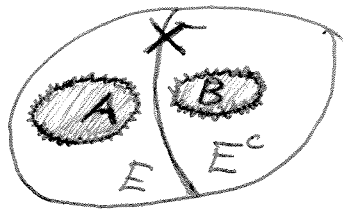
(3) $\bigcup_{j=1}^{\infty} E_j \in \mathcal{M}$, jos $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$.

Tod.

(1) Ota $\underbrace{A}_{=\emptyset} \subset \emptyset$, $B \subset \underbrace{\emptyset^c}_{=\cancel{X}}$.

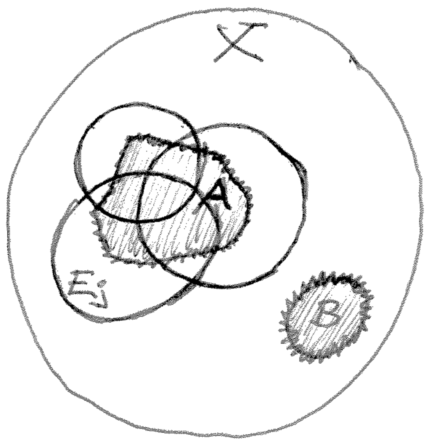
$$\Rightarrow \psi(\underbrace{A \cup B}_{=B}) = \underbrace{\psi(\overset{=\emptyset}{A})}_{=0} + \psi(B). \quad \boxtimes$$

(2) Mitallisen joukon määritelmä on komplementtisymmetrinen! $\boxtimes$



(3) Olkoon $E := \bigcup_{j=1}^{\infty} E_j$.

Ota $\begin{cases} A \subset E, \\ B \subset E^c. \end{cases}$



Olkoon $A_k := A \cap F_k$, missä

$$F_1 := E_1,$$

$$F_{k+1} := E_{k+1} \setminus \bigcup_{j=1}^{k} E_j$$

$$= E_{k+1} \cap \bigcap_{j=1}^{k} E_j^c .$$

Nyt $\{F_k\}_{k=1}^{\infty} \subset \mathcal{P}(X)$ pistevieras (eli $k \neq l \Rightarrow F_k \cap F_l = \emptyset$),

$\bigcup_{k=1}^{\infty} F_k = E$.

(2) & edellinen lemma

$$\Rightarrow \{F_k\}_{k=1}^{\infty} \subset \mathcal{M}(\Psi) \text{ pistevieras perhe.} \quad (*)$$

$$\Psi(A \cup B) \overset{A \supset A_k}{\geq} \Psi\Big(\bigcup_{k=1}^{N} A_k \cup B\Big) \qquad \Bigg\| \begin{array}{l} A_k \subset F_k \\ B \subset F_k^c \end{array}$$

$$\overset{(*)}{=} \sum_{k=1}^{N} \Psi(A_k) + \Psi(B)$$

$$\xrightarrow[N \to \infty]{} \sum_{k=1}^{\infty} \Psi(A_k) + \Psi(B) \qquad \Bigg\| \; A = \bigcup_{k=1}^{\infty} A_k$$

$$\geq \Psi(A) + \Psi(B)$$

$$\overset{\text{triv.}}{\geq} \Psi(A \cup B)$$

$$\Rightarrow \Psi(A \cup B) = \Psi(A) + \Psi(B)$$

$$\Rightarrow E = \bigcup_{j=1}^{\infty} E_j \in \mathcal{M}(\Psi). \quad \blacksquare$$

<u>Huom.</u> Yllä $\bigcup_{j=1}^{\infty} \underbrace{E_j}_{\in \mathcal{M}(\Psi)} = \bigcup_{k=1}^{\infty} \underbrace{F_k}_{\in \mathcal{M}(\Psi)}$,

$$F_k \subset E_k, \qquad \underbrace{k \neq l \Rightarrow F_k \cap F_l = \emptyset}_{\{F_k\}_{k=1}^{\infty} \text{ pistevieras perhe.}}.$$

<u>Huom.</u>

Jos $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}(\Psi)$ pistevieras,

niin $\Psi\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \Psi(E_j)$

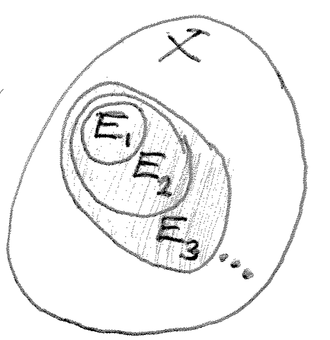
(nähdään ed. todistuksesta.).

Seuraus

Jos $E_k \subset E_{k+1} \in \mathcal{M}(\Psi)$ $(\forall k \in \mathbb{Z}^+)$,

niin

$$\circledast \quad \Psi\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \lim_{k\to\infty} \Psi(E_k).$$

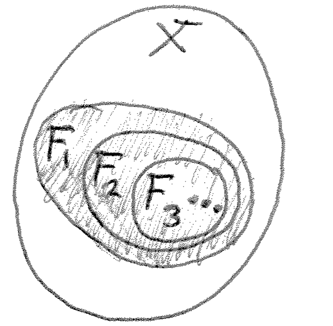


Jos $F_k \supset F_{k+1} \in \mathcal{M}(\Psi)$ $(\forall k \in \mathbb{Z}^+)$

ja $\Psi(F_1) < \infty$,

niin

$$\Psi\Big(\bigcap_{k=1}^{\infty} F_k\Big) = \lim_{k\to\infty} \Psi(F_k).$$



Tod. Tapaus $\forall k: \Psi(E_k) < \infty$ riittää (miksi?).

$$\Psi\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \Psi\Big(\underbrace{\underbrace{E_1}_{\in \mathcal{M}(\Psi)} \cup \bigcup_{k=1}^{\infty} (\underbrace{E_{k+1} \setminus E_k}_{\in \mathcal{M}(\Psi)})}_{\text{pistevieras yhdiste}}\Big)$$

$$= \Psi(E_1) + \sum_{k=1}^{\infty} \underbrace{\Psi(E_{k+1} \setminus E_k)}_{= \Psi(E_{k+1}) - \Psi(E_k), \text{ koska...}}$$

$$= \lim_{k\to\infty} \Psi(E_{k+1}) \quad \boxed{\diagup}$$

$$\underbrace{\Psi(F_1)}_{<\infty} = \Psi\Big(\underbrace{\Big(\bigcap_{k=1}^{\infty} F_k\Big)}_{\in \mathcal{M}(\Psi)} \cup \underbrace{\bigcup_{l=1}^{\infty} (F_1 \setminus F_l)}_{\in \mathcal{M}(\Psi)}\Big) \quad \text{erilliset}$$

$$= \Psi\Big(\bigcap_{k=1}^{\infty} F_k\Big) + \Psi\Big(\bigcup_{l=1}^{\infty} \underbrace{(F_1 \setminus F_l)}_{=: E_l}\Big)$$

$$\overset{\circledast}{=} \Psi\Big(\bigcap_{k=1}^{\infty} F_k\Big) + \lim_{l\to\infty} \underbrace{\Psi(F_1 \setminus F_l)}_{= \underbrace{\Psi(F_1)}_{<\infty} - \Psi(F_l)}$$

$\blacksquare$

Lause

Olkoon $\Psi : \mathcal{P}(X) \to [0,\infty]$ ulkomitta ja $\mathcal{M} := \mathcal{M}(\Psi)$. Silloin

$$\mu = \Psi|_{\mathcal{M}} : \mathcal{M} \to [0,\infty]$$

(eli $\mu(E) = \Psi(E)$, jos $E \in \mathcal{M}$)

on <u>mitta</u>:

toisin sanoen $\mathcal{M}$ on σ-algebra ja

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{k=1}^{\infty} \mu(E_k),$$

jos $\{E_k\}_{k=1}^{\infty} \subset \mathcal{M}$ pistevieras.

Lisäksi mitta μ on <u>täydellinen</u>:

$$F \in \mathcal{M},\ \mu(F) = 0,\ E \subset F$$
$$\Rightarrow E \in \mathcal{M}.$$

Tod.

Seuraa aiemmasta suoraan. $\blacksquare$

Määr.

<u>Mitta-avaruus</u> on $(X, \mathcal{M}, \mu)$, missä

$$\mu : \underbrace{\mathcal{M}}_{\subset \mathcal{P}(X)} \to [0,\infty]$$

on mitta.

$\{\emptyset, X\} \subset \mathcal{A} \subset \mathcal{P}(X)$

"Mitake" $m : \mathcal{A} \to [0,\infty]$ synnyttää ulkomitan $m^* : \mathcal{P}(X) \to [0,\infty]$, josta saadaan mitta $\mu = m^*|_{\mathcal{M}(m^*)} : \mathcal{M}(m^*) \to [0,\infty]$,

joka synnyttää ulkomitan μ^*,
josta saadaan mitta $\mu^*|_{\mathcal{M}(\mu^*)}$,
joka synnyttää ...

[Eikö tälle ole loppua ???]

Lause (Carathéodory–Hahn -laajennus)
Olkoon $\mu : \mathcal{M} \to [0,\infty]$ mitta.
Silloin $\mu^*|_{\mathcal{M}} = \mu$ ja $\mathcal{M} \subset \mathcal{M}(\mu^*)$.

Tod.
Tehtävä. $\blacksquare$

Lause
Olkoon $\mu : \mathcal{M} \to [0,\infty]$ mitta.
Silloin $\mu^* = (\mu^*|_{\mathcal{M}(\mu^*)})^*$.

Tod.
Tehtävä. $\blacksquare$

Ajatus: mitan $\mu : \mathcal{M} \to [0,\infty]$ "luonnollisin" laajennus on

$$\mu^*|_{\mathcal{M}(\mu^*)} : \mathcal{M}(\mu^*) \to [0,\infty].$$