TKK Institute of Mathematics   Arponen/Pursiainen

Mat-1.3651 Numerical Linear Algebra, spring 2008
(Numeerinen matriisilaskenta, kevät 2008)

Exercise 1 (24.1.2008)

Please hand in the exercises marked with an asterisk (*) either to the assistant's folder in front of U313 or latest at the beginning of the exercise.

* 1. Let $A(a) := \begin{pmatrix} a & a \\ a & a \end{pmatrix}$ and $\mathcal{V}$ be the set

$$\mathcal{V} = \{A(a) \mid a \in \mathbb{R}\}.$$

(a) Show that $\mathcal{V}$ is an algebra, with respect to the usual operations (+, $*$, scalar multiplication) of the matrix algebra. Especially, what is its identity element?

(b) For which a is $A(a)$ invertible in $\mathcal{V}$? When it is invertible, give an expression for $A(a)^{-1}$.

(c) For which a is $A(a)$ invertible in $\mathbb{R}^{2\times 2}$?

* 2.

(a) Let B be a 4×4 matrix to which we apply the following operations:

(a) double column 1,
(b) halve row 3,
(c) add row 3 to row 1,
(d) interchange columns 1 and 4,
(e) subtract row 2 from each of the other rows,
(f) replace column 4 by column 3,
(g) delete column 1 (so that the column dimension is reduced by 1).

(b) Write the result as a product of eight matrices,

(c) Write it again as a product ABC (same B) of three matrices.

3. Show that if a matrix A is both triangular and unitary, then it is diagonal.

4. Let $A \in \mathbb{C}^{m\times m}$ be hermitian. An eigenvector of A is a nonzero vector $x \in \mathbb{C}^m$ such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue. Prove that

(a) all eigenvalues of A are real.

(b) if x and y are eigenvectors corresponding to different eigenvalues, then $x \perp y$.

5. Let $S \in \mathbb{C}^{m\times m}$ be skew-hermitian, that is, $S^* = -S$.

(a) Show that the eigenvalues of S are pure imaginary. (Hint: you can use the previous exercise.)

(b) Show that $I - S$ is non-singular.

(c) Show that the *Cayley transform* matrix $Q := (I - S)^{-1}(I + S)$ is unitary.

* 6. Suppose $u, v \in \mathbb{C}^m$. The matrix $A := I + uv^*$ is known as a rank-one perturbation of the identity. Show that if A is nonsingular, its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α. What is α in terms of u and v? When is A singular? When it is singular, what is the nullspace $N(A)$?