Non-Uniform Time Discretization and Lower Bounds for Stochastic Heat Equations

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Complexity of Numerical Problems

1. **Computational problem**: approximation of SPDEs, SDEs, ...
2. **Computational means**: class $\mathcal{A}$ of algorithms.
3. **Quality criterion**: error and cost of an algorithm.
4. **Minimal error and complexity**:

$$e(n) = \inf\{\mathrm{error}(A) : A \in \mathcal{A} \text{ such that } \mathrm{cost}(A) \leq n\},$$
$$\mathrm{comp}(\varepsilon) = \inf\{\mathrm{cost}(A) : A \text{ algorithm with } \mathrm{error}(A) \leq \varepsilon\}.$$

Leads to

- benchmarks for existing algorithms,
- definition of optimal algorithms,
- construction of new algorithms (sometimes).

Typical result

$$e(n) \asymp n^{-\alpha},$$

consists of

- **upper bound**: existence (construction) of algorithms $A_n \in \mathcal{A}$ such that
$$\operatorname{cost}(A_n) \le n \quad \wedge \quad \operatorname{error}(A_n) = O\left(n^{-\alpha}\right),$$
- **lower bound**: $\exists\, c > 0$ $\forall$ algorithm $A \in \mathcal{A}$ $\forall\, n \in \mathbb{N}$:
$$\operatorname{cost}(A) \le n \quad \Rightarrow \quad \operatorname{error}(A) \ge c \cdot n^{-\alpha}.$$

See, e.g., *Traub, Wasilkowski, Woźniakowski (1988), Novak (1988), Ritter (2000), . . .*

Strong and weak approximation of SDEs

- **upper bounds**:
  extensively studied
- **lower bounds**:
  Hofmann, Müller-Gronbach, R (2000, . . .),
  Müller-Gronbach (2002, . . .), Neuenkirch (2006),
  Petras, R (2006), Creutzig, Dereich, Müller-Gronbach, R (2008)

Strong approximation of SPDEs

- **upper bounds**:
  Grecksch, Kloeden (1996), Gyöngy, Nualart (1997),
  Gyöngy (1998, . . .), Shardlow (1999, . . .) **. . .** ,
- **lower bounds**:
  Davie, Gaines (2001), Müller-Gronbach, R (2007),
  Müller-Gronbach, R, Wagner (2007)

Outline

I. **Strong Approximation of Stochastic Heat Equations**

computational problem, class of algorithms, error and cost

II. **Results and Remarks for Strong Approximation**

asymptotics of minimal errors, asymptotically optimal algorithms

III. **Quadrature for Stochastic Heat Equations**

multilevel Monte Carlo algorithms, some results and numerical experiments

I. Strong Approximation of Stoch. Heat Equations

$$dX(t) = \Delta X(t)\,dt + B(t, X(t))\,dW(t),$$
$$X(0) = x_0.$$

Assumptions

(i) Δ Dirichlet Laplacian on $D = [0,1]^d$, $x_0 \in H = L_2(D)$,

(ii) multiplication operators $B(t,x)h = G(t,x) \cdot h$ for
$G : [0,T] \times H \to H$ satisfying Lipschitz/Hölder conditions,

(iii) Brownian motion

$$W(t) = \sum_{\boldsymbol{i} \in \mathbb{N}^d} |\boldsymbol{i}|_2^{-\gamma/2} \cdot \beta_{\boldsymbol{i}}(t) \cdot h_{\boldsymbol{i}}$$

with eigenfunctions $h_{\boldsymbol{i}}$ of Δ, independent scalar Bms $(\beta_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{N}^d}$,
either $\gamma = 0$ and $d = 1$ or $\gamma > d \in \mathbb{N}$.

Task: Approximation of $X(t)$ w.r.t. $\|\cdot\|_H$ for all $t \in [0,T]$.

Example

An equation with additive noise

$$
\begin{aligned}
dX(t) &= \Delta X(t)\,dt + B\,dW(t),\\
X(0) &= 0,
\end{aligned}
$$

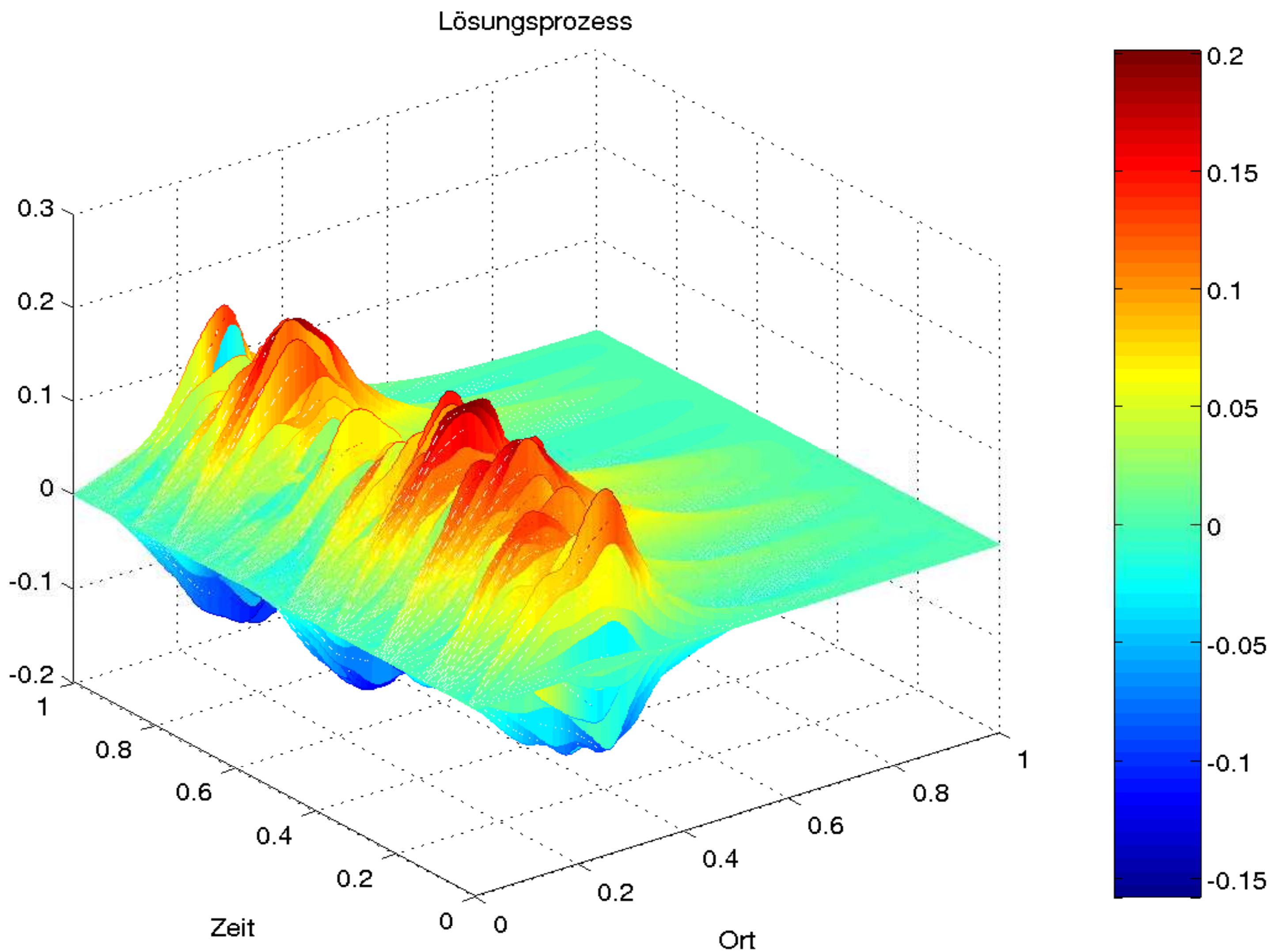
where $d = 1$, $\gamma = 1.5$ and

$$B(h) = g \cdot h$$

for

$$g(u) = \begin{cases} u & \text{if } 0 \le u \le 1/2 \\ 0 & \text{if } 1/2 < u \le 1. \end{cases}$$

A trajectory of the solution process X



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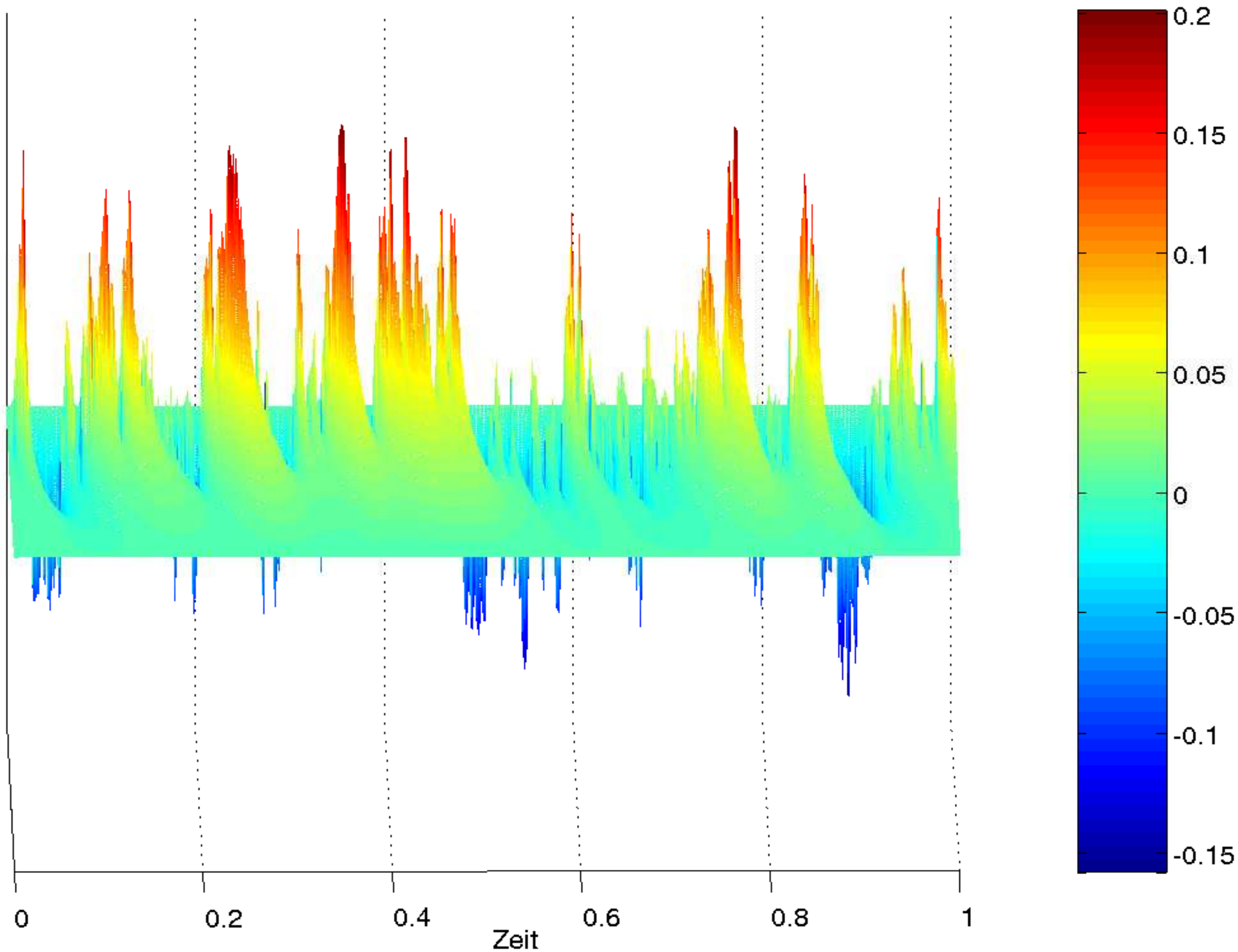


Class $\mathcal{A}$ of algorithms (for any fixed equation):

- choose a finite set $\boldsymbol{I} \subset \mathbb{N}^d$ (scalar Brownian motions $\beta_{\boldsymbol{i}}$),
- choose $n_{\boldsymbol{i}} \in \mathbb{N}$ for $\boldsymbol{i} \in \boldsymbol{I}$ (number of eval's of $\beta_{\boldsymbol{i}}$),
- choose knots $0 < t_{1,\boldsymbol{i}} < \cdots < t_{n_{\boldsymbol{i}},\boldsymbol{i}} \leq T$ for $\boldsymbol{i} \in \boldsymbol{I}$,
- choose $\phi : [0,T] \times \mathbb{R}^n \to H$ with $n = \sum_{\boldsymbol{i} \in \boldsymbol{I}} n_{\boldsymbol{i}}$.

In this way, approximation

$$\widehat{X}(t) = \phi(t; \ldots, \beta_{\boldsymbol{i}}(t_{\ell,\boldsymbol{i}}), \ldots), \qquad t \in [0,T].$$

Error and **cost**

$$e^2(\widehat{X}) = \mathrm{E}\left(\int_0^T \left\|X(t) - \widehat{X}(t)\right\|_H^2 dt\right), \qquad \mathrm{cost}(\widehat{X}) = n.$$

Minimal error

$$e(n) = \inf\left\{e(\widehat{X}) : \mathrm{cost}(\widehat{X}) \leq n\right\}.$$

II. Results and Remarks

1. Asymptotics of the minimal error
2. Asymptotically optimal algorithms
3. Uniform vs. non-uniform time discretization

Asymptotics of the Minimal Error

Theorem *Müller-Gronbach, R (2007)*

For every non-deterministic equation

$$e(n) \succeq n^{-\alpha^*},$$

where

$$\alpha^* = \frac{\min(\gamma - d, d) + 2}{2(d+2)}$$

if $\gamma = 0$ and $d = 1$ or $\gamma \in \,]d, \infty[\setminus \{2d\}$ and $d \in \mathbb{N}$.

Lower bounds are sharp, i.e., $e(n) \asymp n^{-\alpha^*}$, if

(i) $G(t, x) = G(t)$ and $G : [0, T] \to H$ is smooth, or

(ii) $G(t, x) = g \circ x$ and $g : \mathbb{R} \to \mathbb{R}$ is smooth.

Construction of Almost Optimal Algorithms

Mild solution of

$$\boxed{\begin{aligned} dX(t) &= \Delta X(t)\,dt + B(X(t),t)\,dW(t), \\ X(0) &= x_0 \end{aligned}}$$

given by $X(t) = \sum_{\boldsymbol{j}\in\mathbb{N}^d} Y_{\boldsymbol{j}}(t)\cdot h_{\boldsymbol{j}}$, where

$$\boxed{\left.\begin{aligned} dY_{\boldsymbol{j}}(t) &= -\mu_{\boldsymbol{j}} Y_{\boldsymbol{j}}\,dt + \sum_{\boldsymbol{i}\in\mathbb{N}^d} b_{\boldsymbol{i},\boldsymbol{j}}(X(t),t)\,d\beta_{\boldsymbol{i}}(t), \\ Y_{\boldsymbol{j}}(0) &= \langle x_0, h_{\boldsymbol{j}}\rangle_H \end{aligned}\right\}\ \boldsymbol{j}\in\mathbb{N}^d}$$

with $\mu_{\boldsymbol{j}} = \pi^2\cdot|\boldsymbol{j}|_2^2$ and

$$b_{\boldsymbol{i},\boldsymbol{j}}(x,t) = |\boldsymbol{i}|_2^{-\gamma/2}\cdot\left\langle B(x,t)h_{\boldsymbol{i}}, h_{\boldsymbol{j}}\right\rangle_H.$$

$X(t) = \sum_{\boldsymbol{j} \in \mathbb{N}^d} Y_{\boldsymbol{j}}(t) \cdot h_{\boldsymbol{j}}$, where

$$\boxed{\left.\begin{aligned} dY_{\boldsymbol{j}}(t) &= -\mu_{\boldsymbol{j}} Y_{\boldsymbol{j}}\, dt + \sum_{\boldsymbol{i} \in \mathbb{N}^d} b_{\boldsymbol{i},\boldsymbol{j}}(X(t), t)\, d\beta_{\boldsymbol{i}}(t), \\ Y_{\boldsymbol{j}}(0) &= \langle x_0, h_{\boldsymbol{j}} \rangle_H \end{aligned}\right\} \boldsymbol{j} \in \mathbb{N}^d}$$

Ito-Galerkin-Approximation

Reduction to a finite-dimensional system of SDEs by

1) Space discretization of X: finite set $\boldsymbol{J} \subset \mathbb{N}^d$,

2) Space discretization of W: finite set $\boldsymbol{I} \subset \mathbb{N}^d$.

Approximation of the finite-dimensional system by using

3) Time discretization of the Brownian motion $(\beta_{\boldsymbol{i}})_{\boldsymbol{i} \in \boldsymbol{I}}$,

4) Appropriate strong approximation scheme.

Choose $n \in \mathbb{N}$.

1) Space discretization of X given by

- a hyperbolic cross

$$\boldsymbol{J}_n = \left\{ \boldsymbol{j} \in \mathbb{N}^d : \prod_{\ell=1}^d j_\ell \leq n^{\frac{d}{d+2}} \cdot (\ln n)^{\frac{d(d-1)}{\min(\gamma-d,d)+2}} \right\},$$

if $G(t,x) = G(t)$,

- by a ball

$$\boldsymbol{J}_n = \left\{ \boldsymbol{j} \in \mathbb{N}^d : |\boldsymbol{i}|_2 \leq n^{\alpha^*} \right\}$$

if $G(t,x) = g \circ x$.

2) Space discretization of W given by a ball

$$\boldsymbol{I}_n = \left\{ \boldsymbol{i} \in \mathbb{N}^d : |\boldsymbol{i}|_2 \leq n^{1/(d+2)} \right\}.$$

3) Non-uniform time discretization of $(\beta_{\boldsymbol{i}})_{\boldsymbol{i} \in \boldsymbol{I}_n}$ with step-sizes

$$T/n_{\boldsymbol{i}} \asymp |\boldsymbol{i}|_2^{\gamma/2}, \qquad \boldsymbol{i} \in \boldsymbol{I}_n.$$

4) Drift-implicit Euler scheme $\widehat{X}_n^*$ for $\widetilde{X}(t) = \sum_{\boldsymbol{j} \in \boldsymbol{J}_n} \widetilde{Y}_{\boldsymbol{j}}(t) \cdot h_{\boldsymbol{j}}$ with

$$\boxed{\left.\begin{aligned} d\widetilde{Y}_{\boldsymbol{j}}(t) &= -\mu_{\boldsymbol{j}} \widetilde{Y}_{\boldsymbol{j}}\, dt + \sum_{\boldsymbol{i} \in \boldsymbol{I}_n} b_{\boldsymbol{i},\boldsymbol{j}}(\widetilde{X}(t), t)\, d\beta_{\boldsymbol{i}}(t), \\ \widetilde{Y}_{\boldsymbol{j}}(0) &= \langle x_0, h_{\boldsymbol{j}} \rangle_H \end{aligned}\right\} \; \boldsymbol{j} \in \boldsymbol{J}_n}$$

Theorem *Müller-Gronbach, R (2007)*

$$e(\widehat{X}_n^*) \preceq n^{-\alpha^*} \qquad \text{and} \qquad \operatorname{cost}(\widehat{X}_n^*) \preceq n.$$

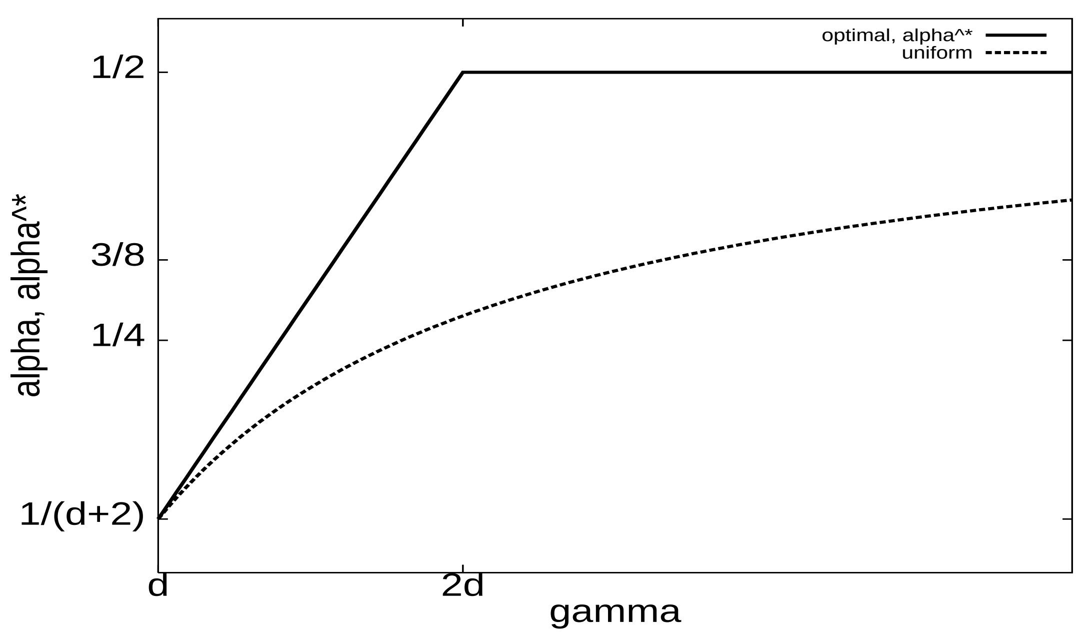
Uniform vs. Non-uniform Time Discretization

Uniform time discretization: Choose $\boldsymbol{I} \subset \mathbb{N}^d$ and $n \in \mathbb{N}$.

Evaluate $\beta_{\boldsymbol{i}}$ with $\boldsymbol{i} \in \boldsymbol{I}$ equidistantly with common step-size T/n.

Example For $dX(t) = \Delta X(t)\, dt + dW(t)$,

$$\inf\{e(\widehat{X}) : \widehat{X} \text{ based on uniform discr.}, \operatorname{cost}(\widehat{X}) \le n\} \succeq n^{-\frac{\gamma-d+2}{2(\gamma+2)}}.$$



Remarks

- Strong approximation of stochastic heat equations is computationally hard, since

$$\inf\{e(\widehat{X}_n) : \mathrm{cost}(\widehat{X}_n) \leq n\} \asymp n^{-\alpha^*}$$

for

$$\alpha^* \in \left]1/(d+2), 1/2\right].$$

- For sufficiently smooth noise, i.e., $\gamma > 2d$, we have $\alpha^* = 1/2$ as for finite-dimensional systems of SDEs.

- Optimal order of convergence only via non-uniform time discretization.

- Similar results (for particular equations) for the error

$$e^2(\widehat{X}) = \mathrm{E}\big(\|X(T) - \widehat{X}(T)\|_H^2\big),$$

at a fixed time instance T.

III. Quadrature for Stochastic Heat Equations

Compute

$$S(f) = \mathrm{E}(f(V)) = \int_{\mathfrak{V}} f \, dP_V,$$

where

$$V = X \quad \text{and} \quad f : \mathfrak{V} \to \mathbb{R} \quad \text{for} \quad \mathfrak{V} = L_2([0,T], H)$$

or

$$V = X(T) \quad \text{and} \quad f : \mathfrak{V} \to \mathbb{R} \quad \text{for} \quad \mathfrak{V} = H.$$

Randomized algorithm to compute $S(f) = \mathrm{E}(f(V))$ for $f \in F$

$$\widehat{S} : F \times \Omega \to \mathbb{R}, \quad \widehat{S}(f, \omega) = \varphi(f(V_1(\omega)), \ldots, f(V_\nu(\omega)))$$

with evaluation of f at random knots

$$V_1(\omega), \ldots, V_\nu(\omega) \in \bigcup_{m=1}^{\infty} \mathfrak{V}_m$$

with a fixed scale of finite dimensional subspaces

$$\mathfrak{V}_1 \subset \mathfrak{V}_2 \subset \ldots \subset \mathfrak{V}.$$

Cost and Error of $\widehat{S}$

$$\mathrm{cost}(\widehat{S}) = E\Big(\sum_{i=1}^{\nu} \inf\{\dim \mathfrak{V}_m : V_i \in \mathfrak{V}_m\} + \#\,\text{arith.op.} + \#\,\text{rand.numbers}\Big),$$

$$e(\widehat{S}) = \sup_{f \in F} e(\widehat{S}(f)) = \sup_{f \in F} \big(E|S(f) - \widehat{S}(f)|^2\big)^{1/2}.$$

Multilevel Monte Carlo Algorithms

Assumptions

- F is the class of Lip(1)-functionals on $\mathfrak{V}$,

$$|f(v) - f(w)| \leq \|v - w\|, \qquad v, w \in \mathfrak{V}.$$

- $(\widehat{V}_n)_n$ is a sequence of approximations of V such that

$$\left(\mathrm{E}\,\|\widehat{V}_n - V\|^2\right)^{1/2} \preceq n^{-\alpha} \quad \text{with } \alpha \in \left]0, 1/2\right],$$
$$\mathrm{cost}(f(\widehat{V}_n), f(\widehat{V}_m)) \preceq \max(n, m). \tag{C(α)}$$

Standard Monte-Carlo with k independent replications of $\widehat{V}_n$

$$\widehat{S}_{k,n}(f) = \frac{1}{k} \sum_{i=1}^{k} f(\widehat{V}_n^{(i)}).$$

Multilevel Monte-Carlo

- see *Heinrich (1998), Heinrich, Sindambiwe (1999)* for integral equations, parametric integration,
- see *Giles (2006), Giles, Higham, Mao (2007), Avikainen (2007), Creutzig, Dereich, Gronbach, R (2007)* for SDEs.

Multilevel Monte-Carlo Basic idea: For $\boldsymbol{n} \in \mathbb{N}^L$ with $n_1 < \ldots < n_L$

$$\mathrm{E}(f(\widehat{V}_{n_L})) = \sum_{\ell=2}^{L} \mathrm{E}(\underbrace{f(\widehat{V}_{n_\ell}) - f(\widehat{V}_{n_{\ell-1}})}_{\Delta_\ell(f)}) + \mathrm{E}(\underbrace{f(\widehat{V}_{n_1})}_{\Delta_1(f)}).$$

Multilevel Monte-Carlo Basic idea: For $\boldsymbol{n} \in \mathbb{N}^L$ with $n_1 < \ldots < n_L$

$$\mathrm{E}(f(\widehat{V}_{n_L})) = \sum_{\ell=2}^{L} \mathrm{E}(\underbrace{f(\widehat{V}_{n_\ell}) - f(\widehat{V}_{n_{\ell-1}})}_{\Delta_\ell(f)}) + \mathrm{E}(\underbrace{f(\widehat{V}_{n_1})}_{\Delta_1(f)}).$$

Choose $\boldsymbol{k} \in \mathbb{N}^L$ with $k_1 \geq \ldots \geq k_L$ and define

$$\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}}(f) = \sum_{\ell=1}^{L} \frac{1}{k_\ell} \sum_{i=1}^{k_\ell} \Delta_\ell^{(i)}(f)$$

with independent

$$\underbrace{\Delta_1^{(1)}(f), \ldots, \Delta_1^{(k_1)}(f)}_{\text{i.i.d. as } \Delta_1(f)}, \ldots, \underbrace{\Delta_L^{(1)}(f), \ldots, \Delta_L^{(k_L)}(f)}_{\text{i.i.d. as } \Delta_L(f)}.$$

Theorem *Giles (2006)*

For the standard Monte-Carlo method with $k \asymp n^{2\alpha}$

$$e(\widehat{S}_{k,n}) \preceq \left(\mathrm{cost}(\widehat{S}_{k,n})\right)^{-\alpha/(1+2\alpha)}.$$

For $f \in F$ with $\mathrm{Var}(f(X)) > 0$ and $\mathrm{cost}(f(\widehat{X}_n)) \asymp n$

$$e(\widehat{S}_{k,n}(f)) \asymp \left(\mathrm{cost}(\widehat{S}_{k,n})\right)^{-\alpha/(1+2\alpha)}.$$

For the multilevel Monte-Carlo method with $n_\ell = 2^\ell, k_\ell = 2^{L-\ell}$

$$e(\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}}) \preceq \left(\mathrm{cost}(\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}})\right)^{-\alpha} \cdot \left(\log(\mathrm{cost}(\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}}))\right)^{\lfloor 2\alpha \rfloor}.$$

Remark Better bounds if bias estimates are also available.

Application to SPDEs and Numerical Experiments

We consider

- a heat equation and
- a Burgers equation

with

- $d = 1$,
- additive space-time white noise,
- initial value zero,
- and Dirichlet boundary conditions.

Heat equation

$$\frac{\partial}{\partial t}X(t,u) = \frac{\partial^2}{\partial u^2}X(t,u) + \frac{\partial^2}{\partial t \partial u}W(t,u).$$

Heat equation

$$\frac{\partial}{\partial t} X(t,u) = \frac{\partial^2}{\partial u^2} X(t,u) + \frac{\partial^2}{\partial t \partial u} W(t,u).$$

Strong approximations $\widehat{V}_n$ of

$$V = X \qquad \text{or} \qquad V = X(1)$$

with $C(1/6)$ via finite differences, implicit Euler scheme and bilinear interpolation, see *Gyöngy (1999)*. Then

$$e(\widehat{S}_{k,n}) \asymp (\mathrm{cost}(\widehat{S}_{k,n}))^{-1/8},$$

$$e(\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}}) \preceq (\mathrm{cost}(\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}}))^{-1/6}.$$

Heat equation

$$\frac{\partial}{\partial t} X(t,u) = \frac{\partial^2}{\partial u^2} X(t,u) + \frac{\partial^2}{\partial t \partial u} W(t,u).$$

Strong approximations $\widehat{V}_n$ of

$$V = X(1)$$

with $C(1/2)$ by spectral Galerkin with non-uniform time discretization and implicit Euler scheme, see *Müller-Gronbach, R, Wagner (2008)*. Then

$$e(\widehat{S}_{k,n}) \asymp (\mathrm{cost}(\widehat{S}_{k,n}))^{-1/4},$$

$$e(\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}}) \preceq (\mathrm{cost}(\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}}))^{-1/2} \cdot \log(\mathrm{cost}(\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}})).$$

Heat equation

$$\frac{\partial}{\partial t} X(t,u) = \frac{\partial^2}{\partial u^2} X(t,u) + \frac{\partial^2}{\partial t \partial u} W(t,u).$$

Strong approximations $\widehat{V}_n$ of

$$V = X(1)$$

with $C(1/2)$ by spectral Galerkin with non-uniform time discretization and implicit Euler scheme, see *Müller-Gronbach, R, Wagner (2008)*. Then

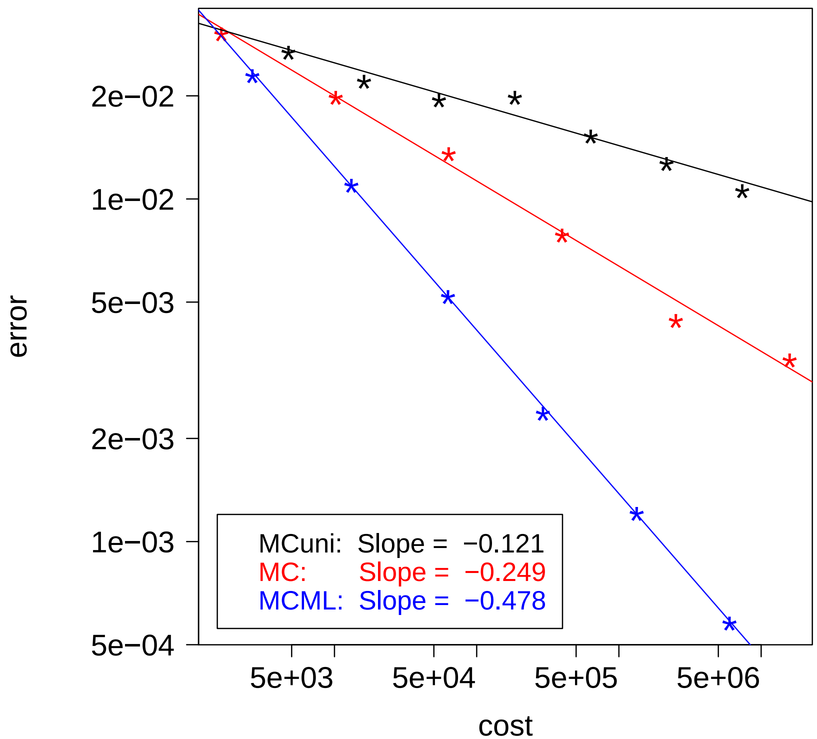
$$e(\widehat{S}_{k,n}) \asymp (\mathrm{cost}(\widehat{S}_{k,n}))^{-1/4},$$
$$e(\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}}) \preceq (\mathrm{cost}(\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}}))^{-1/2} \cdot \log(\mathrm{cost}(\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}})).$$

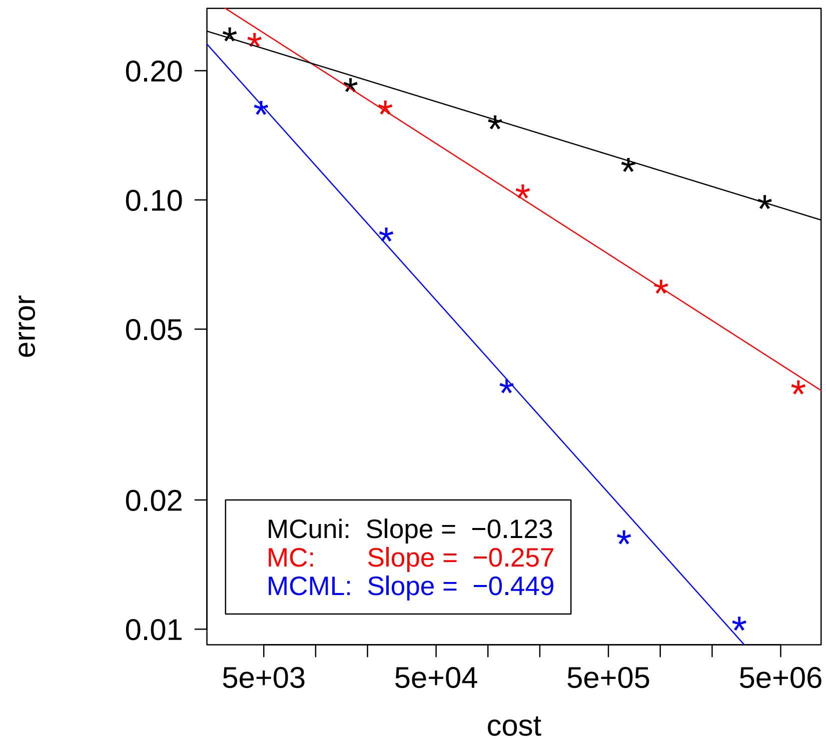
Remark Uniform time-discretization or finite difference approach yields

$$e(\widehat{S}_{k,n}) \asymp (\mathrm{cost}(\widehat{S}_{k,n}))^{-1/8}, \qquad e(\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}}) \preceq (\mathrm{cost}(\widehat{S}^{\mathsf{ML}}_{\boldsymbol{k},\boldsymbol{n}}))^{-1/6}.$$

Numerical results

$$\int_0^1 X(1,u)\,du \qquad\qquad \max_{0\le u\le 1} X(1,u).$$





Burgers equation

$$\frac{\partial}{\partial t} X(t,u) = \frac{\partial^2}{\partial u^2} X(t,u) + X(t,u) \cdot \frac{\partial}{\partial u} X(t,u) + \frac{\partial^2}{\partial t \partial u} W(t,u).$$

Error bound for semi-discretization in space via finite differences, see *Alabert, Gyöngy (2006)*.

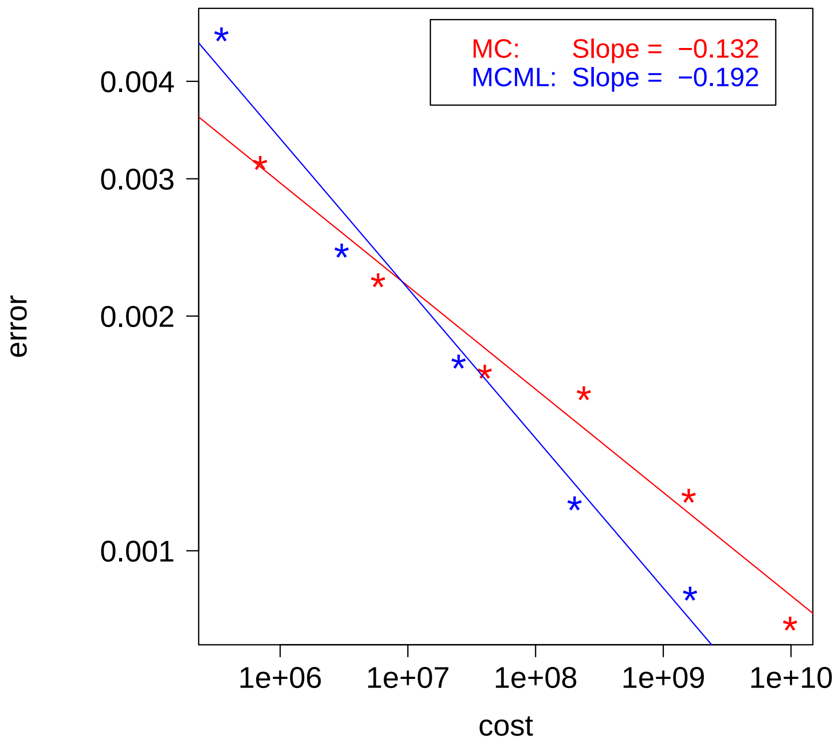
Strong approximations $\widehat{V}_n$ of

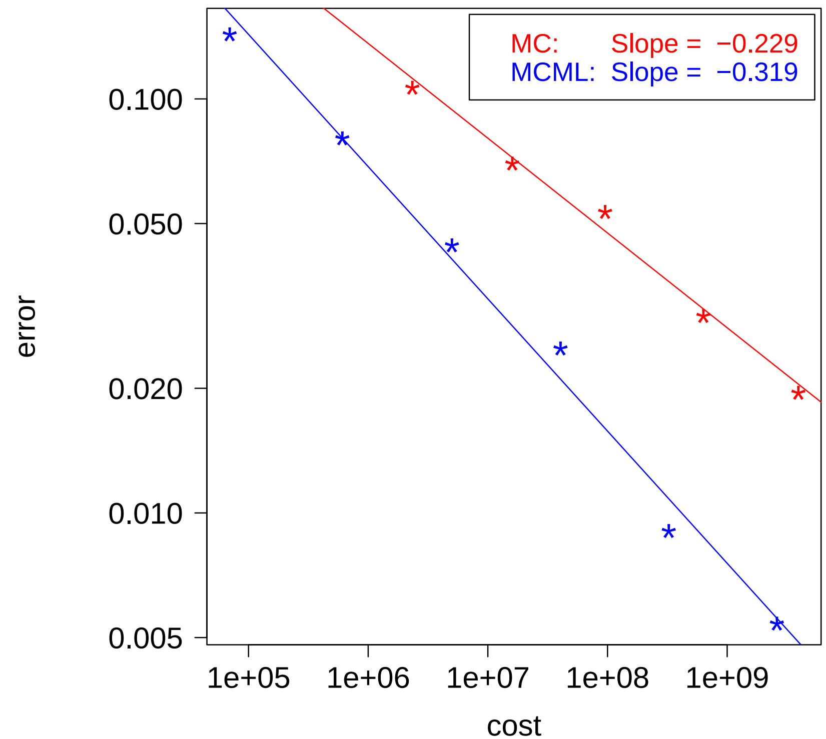
$$V = X(1)$$

via finite difference, implicit Euler scheme, and linear interpolation, supposing that $\alpha = 1/6$.

Numerical results

$$\int_0^1 X(1,u)\,du \qquad\qquad \max_{0\le u\le 1} X(1,u).$$





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