# Discretization of a stable SPDE in space and time

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**Starting point:** Consider the SPDE

$$\mathrm{d}U_t = \Delta U_t \mathrm{d}t + f(U_t)\mathrm{d}t + \mathrm{d}W_t, \qquad U_0 = u_0$$

on $H = L^2(\mathcal{O})$ where $\mathcal{O}$ is a bounded region in $\mathbb{R}^n$ with sufficiently smooth boundary and

- $f : H \to H$ a nonlinear operator which is Lipschitz continuous:

$$|f(u) - f(v)| \leq L|u - v|$$

- $\Delta \in L(V, V')$ with DBC.

Thereby: $V = H_0^1(\mathcal{O})$. This leads to a Gelfand triplet:

$$V \subseteq H \equiv H' \subseteq V'.$$

**Random dynamical systems:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(X, \|\cdot\|_X)$ a normed space and $\theta = (\theta_t)_{t\in\mathbb{R}}$ a metric dynamical system.

A measurable mapping

$$\varphi : (\mathbb{R}^+ \times \Omega \times X, \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X)) \to (X, \mathcal{B}(X))$$

is called a *random dynamical system* if it satisfies the cocycle property, i.e.

$$\varphi(0, \omega, x_0) = x_0$$
$$\varphi(t + \tau, \omega, x_0) = \varphi(t, \theta_\tau \omega, \varphi(\tau, \omega, x_0))$$

for all $t, \tau \geq 0$, $\omega \in \Omega$ and $x_0 \in X$.

**Random fixed points:** A measurable random variable

$$X^* : (\Omega, \mathcal{F}) \to (X, \mathcal{B}(X))$$

is called a *random fixed point* if

$$\varphi(t, \omega, X^*(\omega)) = X^*(\theta_t \omega)$$

holds for all $t > 0$.

## Finite differences:

**Definition:** An *external approximation* of a space $(X, \|\cdot\|_X)$ consists of

(i) a normed space $(F, \|\cdot\|_F)$ and an isomorphism $\psi$ of $X$ into $F$.

(ii) a family $\{X_h, p_h, r_h\}_{h \in \mathscr{H}}$, in which for each $h \in \mathscr{H}$:

- $(X_h, \|\cdot\|_h)$ is a normed space
- $p_h$ is a linear continous mapping of $X_h$ into $F$
- $r_h$ is a mapping of $X$ into $X_h$.

$$
\begin{tikzcd}
X \arrow[rr, "\psi"] \arrow[dr, "r_h"'] & & F \\
& X_h \arrow[ur, "p_h"'] &
\end{tikzcd}
$$

For $\mathscr{H}$ we use

$$\mathscr{H} = \prod_{i=1}^{n} ]0, h_{i,0}] \qquad \text{where} \quad 0 < h_i \leq h_{i,0}.$$

$h_i$ is the mesh in $x_i$ direction. For the characterization of an external approximation we need two more defintions:

**Definition:** The prolongations operators are said to be stable if their norms

$$\|p_h\| = \sup_{\|u_h\|_h = 1} \|p_h u_h\|_F$$

can be majorized independently of h.

**Definition:** An external approximation is said to be stable if the following two conditions are fulfilled:

1. compatibility condition:

$$\lim_{h \to 0} \|p_h r_h u - \psi u\|_F = 0 \qquad \text{for all} \quad u \in X.$$

2. synchronization condition:

   For each sequence $(u_h) \subseteq X_h$ with

$$p_h u_h \xrightarrow[h \to 0]{} \phi \quad \text{in } F$$

   we have

$$\phi \in \psi X.$$

Now we introduce the approximation of $V$.

Let $F = L^2(\mathcal{O})^{n+1}$ and

$$\psi : V \to F, \quad u \mapsto (u, D_1 u, D_2 u, \ldots, D_n u).$$

The approximated space $V_h$ is given by

$$V_h = \Big\{ u_h \Big| u_h(x) = \sum_{P \in \mathring{\mathcal{O}}_h^1} u_h(P) 1_{\sigma_h(P)}(x), \quad u_h(P) \in \mathbb{R}^n \Big\}$$

and the prolongation resp. restriction operators by

$$\begin{aligned} p_h u_h &= (u_h, \delta_1 u_h, \delta_2 u_h, \ldots, \delta_n u_h) \\ (r_h u)(P) &= \frac{1}{m(\sigma_h(P))} \int_{\sigma_h(P)} u(x)\,\mathrm{d}x \qquad \forall P \in \mathring{\mathcal{O}}_h^1. \end{aligned}$$

**Theorem:** The preceding external approximation of $V$ is stable and convergent. Moreover, for all $h$, $V_h$ is finite dimensional, a subspace of $L^2(\mathcal{O})$ and a Hilbert space with the scalar product

$$((u_h, v_h))_h = \sum_{i=0}^{n} (\delta_i u_h, \delta_i v_h).$$

The next theorem is a discrete analogon of the Poincaré inequality and enables us to equip $V_h$ with another scalar product:

**Theorem:** Let $u_h \in V_h$. Then

$$|u_h| \leq d_{\mathcal{O}} |\delta_i u_h|.$$

Therefore $((\cdot, \cdot))_h$ may start at $i = 1$.

**From SPDE to RPDE:** It's very convenient for our approach to rewrite the SPDE as a RPDE. For that we need to introduce the Ornstein-Uhlenbeck-process:

**Definition:** The unique stationary solution of the SPDE

$$\mathrm{d}U_t = \Delta U_t \mathrm{d}t + \mathrm{d}W_t$$

is called Ornstein-Uhlenbeck-process. It's given by

$$\widehat{U}(\theta_t\omega) \equiv \widehat{U}_t(\omega) = \int\limits_{-\infty}^{t} S_\Delta(t-s)\,\mathrm{d}W_s(\omega)$$

where $S_\Delta(t-s)$ denotes the strongly continuous semigroup generated by $\Delta$.

The OU-process is tempered, i.e. the mapping

$$t \mapsto |\widehat{U}(\theta_t\omega)|$$

has subexponential growth.

Subtracting the solution of the OU-process from any solution $U_t$ of the SPDE we see that the difference $X_t = U_t - \widehat{U}_t$ is pathwise differential in time and satisfies pathwise the RPDE

$$\frac{\partial}{\partial t} X_t = \Delta X_t + f(X_t + \widehat{U}_t).$$

**Spatial approximation:** We are now able to approximate the given RPDE by finite differences. For that we define

$$
\begin{aligned}
X_t^h(x) &= \sum_{P \in \mathring{\mathcal{O}}_h^1} X_t^h(P) 1_{\sigma_h(P)}(x), \quad X_t^h(P) \in \mathbb{R}^n \\
\Delta_h &= \sum_{i=1}^{n} \delta_i^2 \quad \Rightarrow \langle -\Delta_h u_h, u_h \rangle \geq d_{\mathcal{O}}^{-2} |u_h|^2 \\
\widehat{U}_t^h(P) &= (r_h \widehat{U}_t)(P) \quad \Rightarrow |\widehat{U}^h(\theta_t \omega)| \leq |\widehat{U}(\theta_t \omega)|
\end{aligned}
$$

and get

$$
\frac{\partial}{\partial t} X_t^h = \Delta_h X_t^h + f(X_t^h + \widehat{U}_t^h), \quad X_0^h = x_0^h. \tag{1}
$$

**Theorem:** Let

$$C := \frac{1}{d_{\mathcal{O}}^2} - L > 0.$$

Then the solution of (1) generates a RDS $\varphi_h$ which has a unique random fixed point.

**Discretization in time:** Implicit Euler scheme:

$$X_{m+1}^h = X_m^h + \left[ \Delta_h X_{m+1}^h + f_h(X_{m+1}^h + \widehat{U}_{(m+1)k}^h) \right] k, \quad X_0^h = x_0^h \tag{2}$$

with

$$f_h(u_h)(x) := \sum_{P \in \mathring{\mathcal{O}}_h^1} 1_{\sigma_h(P)}(x) \frac{1}{m(\sigma_h(P))} \int_{\sigma_h(P)} f(u_h)(y)\, \mathrm{d}y.$$

Then $f_h$ is Lipschitz continuous with constant $L$.

**Theorem:** Let

$$C := \frac{1}{d_{\mathcal{O}}^2} - L > 0.$$

Then the solution of (2) generates a RDS $\varphi_{h,k}$ which has a unique random fixed point.

**Convergence of the spatial discretization:**

**Theorem:** As $h \to 0$ it holds

$$X_h^*(\omega) \to X^*(\omega) \quad \text{in } H.$$

Therefore we get the following diagram:

$$\begin{tikzcd}
\varphi(t,\theta_{-t}\omega,x_0) \arrow[r] & X^*(\omega) \\
\varphi_h(t,\theta_{-t}\omega,x_0^h) \arrow[r] & X_h^*(\omega) \arrow[u, "h\to 0"'] \\
\varphi_{h,k}(nk,\theta_{-nk}\omega,x_0^h) \arrow[r] & X_{h,k}^*(\omega) \arrow[u, dotted, "k\to 0"']
\end{tikzcd}$$

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