Numerical Methods for SDEs driven by Fractional Brownian Motion: Exact Convergence Rates

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Joint work with

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SDEs driven by FBM

(SDE) $$dX_t = a(X_t)dt + \sum_{j=1}^{m} \sigma^{(j)}(X_t)dB_t^{(j)}, \qquad t \geq 0$$
$$X_0 = x_0 \in \mathbb{R}^d$$

where

- $a : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ drift vector
- $\sigma = (\sigma^{(1)}, \ldots, \sigma^{(m)})$ with $\sigma^{(j)} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ diffusion matrix
- $B = (B^{(1)}, \ldots, B^{(m)})'$ m-dimensional **fractional Brownian motion** with Hurst parameter $H \in (0, 1)$, i.e.
  $B^{(1)}, \ldots, B^{(m)}$ independent centered Gaussian processes with continuous sample paths and
  $$\mathrm{E}\,|B_t^{(j)} - B_s^{(j)}|^2 = |t - s|^{2H}, \quad s, t \geq 0$$

SDEs driven by FBM

$$\text{(SDE)} \qquad dX_t = a(X_t)dt + \sum_{j=1}^{m} \sigma^{(j)}(X_t)dB_t^{(j)}, \qquad t \geq 0$$
$$X_0 = x_0 \in \mathbb{R}^d$$

Here: **Pathwise approach** (Coutin, Qian (2002))

Existence and uniqueness under standard assumptions on a and σ for $H > 1/4$ using rough paths theory (Lyons 1994, 1998)

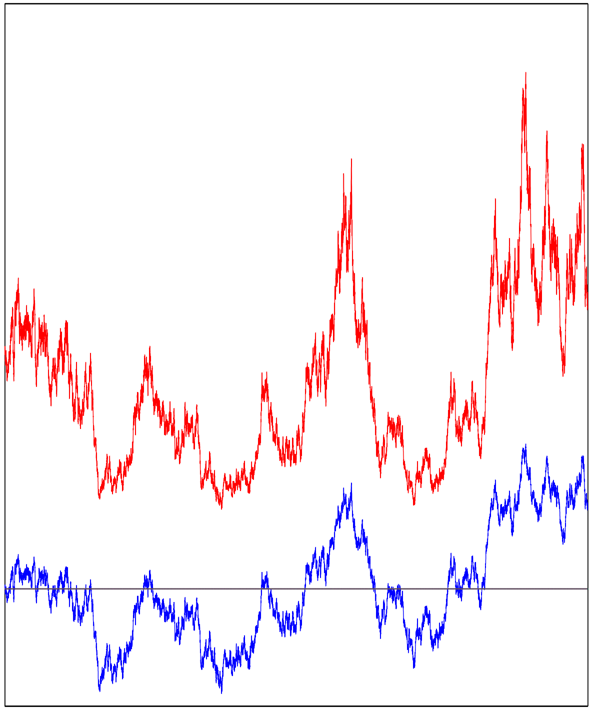
Lin (1995); Klingenhöfer, Zähle (1999); Mikosch, Norvaiša (2000); Nualart, Răşcanu (2002); Errami, Russo (2003); Gubinelli (2004); Zähle (2005); Hu, Nualart (2006); Nourdin, Simon (2007); ...

$H = 1/2$: classical Stratonovich SDE

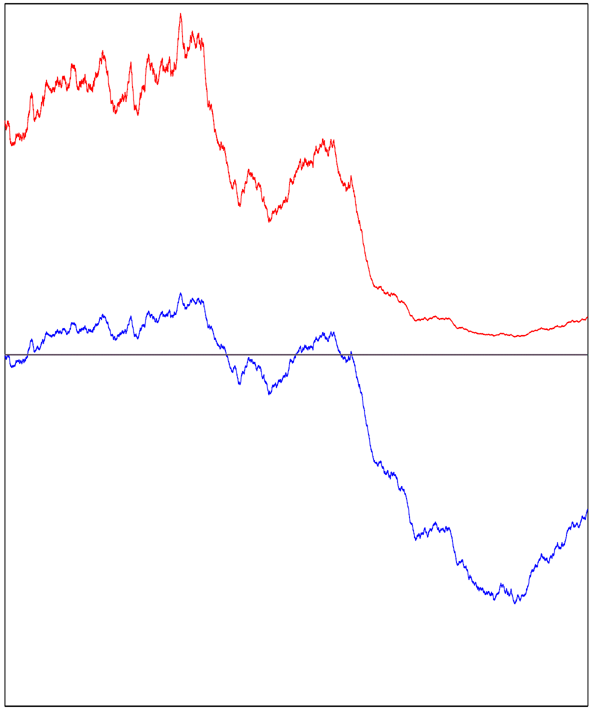
Other approach: Wick-Itô resp. Skorohod integral equation

Biagini et al. (2002); Mishura (2004); Nourdin, Tudor (2006);...

$$dX_t = -0.5X_t dt + 2X_t dB_t, \quad X_0 = 1$$



$H = 0.5$



$H = 0.7$

Blue: $(B_t)_{t\in[0,1]}$ Red: $(X_t)_{t\in[0,1]}$

Outline of the Talk

Problem: Approximation of X_1 by

$$\phi_n(B_{1/n}, B_{2/n}, \ldots, B_1)$$

with $\phi_n : \mathbb{R}^{m,n} \longrightarrow \mathbb{R}^d$, measurable

Mean square error: $e(\phi_n) = \left(\mathrm{E}\,\|X_1 - \phi_n(B_{1/n}, B_{2/n}, \ldots, B_1)\|^2\right)^{1/2}$

In the following:

- simple ϕ_n
- optimal ϕ_n

for

- one-dimensional case ($m = d = 1$) for $H > 1/2$
- fractional Lévy area ($m = 2, d = 3$, non-commutative noise) for $H > 1/4$

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Remark

Exact simulation of fractional noise (Coeurjolly (2000)):

- non-equidistant: Choleski method, cost $\mathcal{O}(n^2)$
- equidistant: Methods of Wood-Chan and Davis-Harte, cost $\mathcal{O}(n\log(n))$

(Input: n iid $\mathcal{N}(0,1)$ random numbers)

One-dimensional Case

(SDE) $$dX_t = a(X_t)dt + \sigma(X_t)dB_t, \qquad t \in [0,1]$$
$$X_0 = x_0 \in \mathbb{R}$$

Here: **Doss-Sussmann** representation

$$X_t = \varphi(A_t, B_t), \qquad t \in [0,1]$$

where

- $\frac{\partial \varphi}{\partial y}(x,y) = \sigma(\varphi(x,y)), \quad \varphi(x,0) = x$
- $A_t' = \exp(-\int_0^{B_t} \sigma'(\varphi(A_t,s))\,ds)\,a(\varphi(A_t,B_t)), \quad A_0 = x_0$

Assumptions:

(A1) $H > 1/2$

(A2) $a \in C^2, \sigma \in C^3$ with bounded derivatives

(A3) a, σ bounded, σ elliptic, i.e. $\inf_{x \in \mathbb{R}} |\sigma(x)| > 0$

Euler Scheme

Notation: $\Delta = \frac{1}{n}$, $t_k = k \cdot \Delta$, $\Delta_k B = B_{t_{k+1}} - B_{t_k}$

$$\overline{X}_0^{(n)} = x_0,$$
$$\overline{X}_{t_{k+1}}^{(n)} = \overline{X}_{t_k}^{(n)} + a(\overline{X}_{t_k}^{(n)})\Delta + \sigma(\overline{X}_{t_k}^{(n)})\Delta_k B, \qquad k = 0, \ldots, n-1$$

Theorem N, Nourdin (2007)

$$e(\overline{X}_1^{(n)}) = \frac{1}{2}\left(\mathrm{E}\left|\int_0^1 \sigma'(X_s) D_s X_1 \, ds\right|^2\right)^{1/2} \cdot n^{-2H+1} + o(n^{-2H+1})$$

with $(D_s X_1)_{s \in [0,1]}$ Malliavin derivative of X_1

Remark Under (A1), (A2):

$$n^{2H-1} \cdot (X_1 - \overline{X}_1^{(n)}) \xrightarrow{a.s.} \frac{1}{2}\int_0^1 \sigma'(X_s) D_s X_1 \, ds$$

Asymptotic error distribution (N, Nourdin (2007))

Euler Scheme

How to analyze the error of the Euler scheme?

Direct method not possible, since isometry for integrals with respect to fBm involves Malliavin derivative of the integrand: equations / inequalities not closable!

Use Doss-Sussmann representation:

(1) transform (SDE) to (RODE) $A_t = \varphi(X_t, -B_t)$

(2) transform $\overline{X}^{(n)}$ to corresponding approximation $\overline{A}^{(n)}$, i.e.

$$\overline{A}^{(n)}_{t_k} = \varphi(\overline{X}^{(n)}_{t_k}, -B_{t_k}), \quad k = 0, 1, \ldots, n$$

(3) analyze $A_1 - \overline{A}^{(n)}_1$

(4) transform back

Optimal Approximation

Recall:

- Approximation of X_1 by $\phi_n(B_{1/n}, B_{2/n}, \ldots, B_1)$
- Mean square error: $e(\phi_n) = \left(\mathrm{E}\,\|X_1 - \phi_n(B_{1/n}, B_{2/n}, \ldots, B_1)\|^2\right)^{1/2}$

Optimal ϕ_n? Clearly:

$$\phi_n^*(x_1, x_2, \ldots, x_n) = \mathrm{E}(X_1 \mid B_{1/n} = x_1, B_{2/n} = x_2, \ldots, B_1 = x_n)$$

Rate of convergence of ϕ_n^* ?

Remark

Known for $H = 1/2$:

$$e(\phi_n^*) = \frac{1}{\sqrt{12}} \left(\int_0^1 \mathrm{E}\,|\mathcal{Y}_t|^2 \, dt \right)^{1/2} \cdot n^{-1} + o(n^{-1})$$

where $(\mathcal{Y}_t)_{t\in[0,1]}$ random weight function

Cameron, Clark (1980); Newton (1986,...); Cambanis, Hu (1996); Müller-Gronbach (2004)

Optimal Approximation

Theorem N (2008)

(1) If $a'\sigma - a\sigma' = 0$, then

$$e(\phi_n^*) = 0$$

(2) Let

$$\mathcal{Y}_t = (a'\sigma - a\sigma')(X_t) \exp(\int_t^1 a'(X_\tau)\, d\tau + \int_t^1 \sigma'(X_\tau)\, dB_\tau), \;\; t \in [0,1]$$

If

$$\text{(ND)} \qquad \int_0^1 |\, \mathrm{E}\, \mathcal{Y}_t|\, dt > 0,$$

then

$$e(\phi_n^*) \;\geq\; C(a, \sigma, x_0, H) \cdot n^{-H-1/2}$$

Remarks

- $\mathcal{Y}$: time derivative (in mean square sense) of Malliavin derivative $(D_t X_1)_{t\in[0,1]}$
- Minimal error for non-equidistant discretizations: In general also
  $$\inf\{e(\,\mathrm{E}(\,X_1 \,|\, B_{t_1}, \ldots, B_{t_n}\,)\,) :\; t_1, \ldots, t_n \in [0,1]\} \;\geq\; C \cdot n^{-H-1/2}$$

Optimal Approximation

Proof

(1) Doss-Sussmann representation

$$a'\sigma - a\sigma' = 0 \implies a/\sigma = \text{const.} \implies A_t' = \sigma(A_t)\frac{a(x_0)}{\sigma(x_0)},\ A_0 = x_0$$

Thus $X_1 = \varphi(A_1, B_1) = \tilde{\varphi}(B_1)$

(2) Linearization via fractional Wiener chaos decomposition

$$X_1 = \mathrm{E}\, X_1 + \int_0^1 \mathrm{E}\, D_t X_1\, dB_t + \dots$$

$$\phi_n^*(B_{1/n}, \dots, B_1) = \mathrm{E}\, \phi_n^*(B_{1/n}, \dots, B_1) + \sum_{i=1}^n a_i B_{i/n} + \dots$$

Thus (by orthogonality of the decomposition)

$$e(\phi_n^*) \geq (\mathrm{E}\, | \int_0^1 \mathrm{E}\, D_t X_1\, dB_t - \sum_{i=1}^n a_i B_{i/n} |^2)^{1/2}$$

Linearized problem:

Embeddings of reproducing kernel Hilbert spaces; results of Stein (1995), Ritter (2000)

An Optimal Implementable Method

Notation: $\Delta = 1/n$, $t_k = k \cdot \Delta$, $\Delta_k B = B_{t_{k+1}} - B_{t_k}$

$$
\begin{aligned}
\widehat{X}_0^{(n)} &= x_0 \\
\widehat{X}_{t_{k+1}}^{(n)} &= \widehat{X}_{t_k}^{(n)} + a(\widehat{X}_{t_k}^{(n)})\Delta + \sigma(\widehat{X}_{t_k}^{(n)})\Delta_k B + \frac{1}{2}\sigma\sigma'(\widehat{X}_{t_k}^{(n)})(\Delta_k B)^2 \\
&\quad + \frac{1}{2}(a'\sigma + a\sigma')(\widehat{X}_{t_k}^{(n)})\Delta_k B\Delta + \frac{1}{2}aa'(\widehat{X}_{t_k}^{(n)})\Delta^2 \\
&\quad + \frac{1}{6}(\sigma^2\sigma'' + \sigma(\sigma')^2)(\widehat{X}_{t_k}^{(n)})(\Delta_k B)^3, \qquad k = 0, \ldots, n-1
\end{aligned}
$$

Remark

For $H = 1/2$:

(i) McShane's method (1974) for Stratonovich SDEs

(ii) Müller-Gronbach (2004): truncated Wagner-Platen method for Itô SDEs

An Optimal Implementable Method

Theorem N (2008)

$$e(\widehat{X}_1^{(n)}) \;=\; \sqrt{|\zeta(-2H)|}\left(\int_0^1 \mathrm{E}\,|\mathcal{Y}_t|^2\,dt\right)^{1/2} \cdot n^{-H-1/2} + o(n^{-H-1/2})$$

where

$$\mathcal{Y}_t = (a'\sigma - a\sigma')(X_t)\,\exp(\textstyle\int_t^1 a'(X_\tau)\,d\tau + \int_t^1 \sigma'(X_\tau)\,dB_\tau),\;\; t \in [0,1]$$

Corollary If

$$\text{(ND)} \qquad \int_0^1 |\,\mathrm{E}\,\mathcal{Y}_t|\,dt > 0,$$

then $\widehat{X}^{(n)}$ optimal, i.e. same convergence rate as conditional expectation!

Multi-dimensional Case

(SDE) $$dX_t = a(X_t)dt + \sum_{j=1}^{m} \sigma^{(j)}(X_t)dB_t^{(j)}, \qquad t \geq 0$$

$$X_0 = x_0 \in \mathbb{R}^d$$

Commutative Noise:

$$\mathcal{D}^{(i)}\sigma^{(j)} = \mathcal{D}^{(j)}\sigma^{(i)}, \qquad i,j = 1,\ldots,m$$

where

$$\mathcal{D}^{(i)} = \sum_{k=1}^{d} \sigma_k^{(i)} \frac{\partial}{\partial x_k}$$

In this case: Doss-Sussmann representation $X_t = \varphi(A_t, B_t)$

SDE with commutative noise “ $\Longleftrightarrow$ “ one-dimensional SDE

Fractional Lévy Area

$$\text{(fLA)} \qquad dX_t = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2}X_t^{(2)} \end{pmatrix} dB_t^{(1)} + \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2}X_t^{(1)} \end{pmatrix} dB_t^{(2)}, \qquad t \in [0,1]$$

$$X_0 = 0$$

Prototype for SDEs with non-commutative noise

$$X_t^{(1)} = B_t^{(1)}, \quad X_t^{(2)} = B_t^{(2)}, \quad X_t^{(3)} = \frac{1}{2}\int_0^t B_s^{(1)}\, dB_s^{(2)} - \frac{1}{2}\int_0^t B_s^{(2)}\, dB_s^{(1)}$$

Approximation of (fLA): quadrature problem

Assumption: $H > 1/4$

Euler Scheme

$$\overline{X}_1^{(3)(n)} = \frac{1}{2}\sum_{i=0}^{n-1} B_{i/n}^{(1)}(B_{(i+1)/n}^{(2)} - B_{i/n}^{(2)}) - \frac{1}{2}\sum_{i=0}^{n-1} B_{i/n}^{(2)}(B_{(i+1)/n}^{(1)} - B_{i/n}^{(1)})$$

Theorem N, Tindel, Unterberger (2008)

$$e(\overline{X}_1^{(n)}) = \begin{cases} \alpha_1(H)\cdot n^{-2H+1/2} + o(n^{-2H+1/2}) & \text{for } H \in (1/4, 3/4) \\ \frac{9}{128}\cdot\sqrt{\log(n)}n^{-1} + o(\sqrt{\log(n)}n^{-1}) & \text{for } H = 3/4 \\ \alpha_2(H)\cdot n^{-1} + o(n^{-1}) & \text{for } H \in (3/4, 1) \end{cases}$$

Remarks

- Euler scheme for general multi-dimensional SDE (Mishura, Shevchenko (2008)):
  pathwise convergence rate $n^{-2H+1+\varepsilon}$ for $H > 1/2$
- No ”diagonal” noise in (fLA)
- Critical point $H = 3/4$: compare with behavior of quadratic variation of fBm
  Guyon, Leon (1989); Istas, Lang (1997); Corcuera et al. (2006);...

Midpoint Scheme

$$\widehat{X}_1^{(3)(2n)} = \frac{1}{2}\sum_{i=0}^{n-1} B^{(1)}_{(i+1/2)/n}(B^{(2)}_{(i+1)/n} - B^{(2)}_{i/n}) - \frac{1}{2}\sum_{i=0}^{n-1} B^{(2)}_{(i+1/2)/n}(B^{(1)}_{(i+1)/n} - B^{(1)}_{i/n})$$

Theorem N, Tindel, Unterberger (2008)

$$e(\widehat{X}_1^{(2n)}) = \alpha_3(H) \cdot n^{-2H+1/2} + o(n^{-2H+1/2})$$

Questions

- Asymptotic error distributions for Euler- and Midpoint scheme?
- Midpoint scheme optimal, i.e. $e(\mathrm{E}(X_1 \mid B_{1/n}, \dots, B_1)) \geq C(H) \cdot n^{-2H+1/2}$ for (fLA)?

  Known for $H = 1/2$: $e(\mathrm{E}(X_1 \mid B_{1/n}, \dots, B_1)) = \frac{1}{\sqrt{8}} \cdot n^{-1/2}$

  Cameron, Clark (1980); ...

Summary

Exact convergence rates in the one-dimensional case for $H > 1/2$

- Euler scheme: n^{-2H+1}
- McShane's method: $n^{-H-1/2}$
- Conditional expectation:

  $n^{-\infty}$ if drift and diffusion commute

  $n^{-H-1/2}$ under a non-degeneracy condition on the Malliavin derivative

Exact convergence rates for the fractional Lévy area

- Euler scheme: $\begin{cases} n^{-2H+1/2} & \text{if} \quad H \in (1/4, 3/4) \\ \sqrt{\log(n)}n^{-1} & \text{if} \quad H = 3/4 \\ n^{-1} & \text{if} \quad H \in (3/4, 1) \end{cases}$
- Midpoint scheme: $n^{-2H+1/2}$
- Conjecture for conditional expectation: $n^{-2H+1/2}$