The convergence rate for Euler approximations of solutions of SDEs driven by fractional Brownian motion and approximation schemes for SDEs in Hilbert space

Yuliya Mishura Georgiy Shevchenko

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1 The rate of convergence for Euler approximations of solutions of stochastic differential equations driven by fractional Brownian motion

Recall that $B = (B_t)_{t \geq 0}$ is called fractional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ with Hurst parameter $H \in (\frac{1}{2}, 1)$ if B is a centered Gaussian process with stationary increments and covariance $R_H(t,s) = E(B_t B_s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$.

Numerical solution via time discretization of SDEs driven by Brownian motion has long history. Concerning numerical solution of SDEs driven by fBm, we mention first the paper [Greksch and Anh (1998)], where equations with modified fBm that represents a special semimartingale are studied (recall that fBm itself is not a semimartingale). Papers [Nourdin (2005), Nourdin and Neunkirch (2007)] study Euler approximations for homogeneous one-dimensional SDEs with bounded coefficients having bounded derivatives up to third order, driven by fBm, and prove that error of approximation is a.s. equivalent to $\delta^{2H-1}\xi_t$, and the process ξ_t is given explicitly. These papers also discuss Crank–Nicholson and Milstein schemes for SDEs driven by fBm. To our knowledge, there are no papers concerned with the rate of weak convergence for Euler approximations of fBm-driven SDEs.

We consider the stochastic differential equation on R^d

$$X_t^i = X_0^i + \sum_{j=1}^{m} \int_0^t \sigma^{ij}(s, X_s) dB_s^j + \int_0^t b^i(s, X_s) ds, \quad i = 1, ..., d, \quad t \in [0, T] \tag{1.1}$$

where the processes $B^i, i = 1, ..., m$ are fractional Brownian motions with Hurst parameter H, X_0 is a d-dimensional random variable, the coefficients $\sigma^{i\,j}, b^i : \Omega \times [0, T] \times R^d \to R$ are measurable functions.

The integral in the right-hand side of (1.1) can be understood in the pathwise sense defined in [Zähle (1998), Nualart and Răşcanu (2000)] or in Wick–Skorohod sense [Alòs and Nualart (2002)]. We treat the pathwise case first. We remind that the pathwise integral w.r.t. a one-dimensional fBm B can be defined as

$$\int_a^b f dB = \int_a^b \left(D_{a+}^{\alpha} f\right)(s) \left(D_{b-}^{1-\alpha} B_{b-}\right)(s) ds,$$

where

$$\left(D_{a+}^{\alpha} f\right)(s) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(s)}{(s-a)^{\alpha}} + \alpha \int_a^s \frac{f(s) - f(u)}{(s-u)^{\alpha+1}} du \right] 1\!\!\mathrm{I}_{(a,b)}(s)$$

and

$$\left(D_{b-}^{1-\alpha}B_{b-}\right)(s) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)}\left[\frac{B_{b-}(s)}{(b-s)^{1-\alpha}} + (1-\alpha)\int_s^b \frac{B_{b-}(s)-B_{b-}(u)}{(u-s)^{2-\alpha}}du\right]\mathbb{1}_{(a,b)}(s)$$

are fractional derivatives of corresponding orders,

$$B_{b-}(s) = \left(B_s - B_b\right)\mathbb{1}_{(a,b)}(s).$$

The integral exists for any $\alpha \in (1-H, \nu)$ if, for example, $f \in C^{\nu}(a,b)$ with $\nu + H > 1$. Moreover, in this case pathwise integral admits an estimate

$$\left|\int_a^b f dB\right| \leqq C_0(\omega)\left[\int_a^b \frac{|f(s)|}{(s-a)^{\alpha}}ds + \int_a^b\int_a^s \frac{|f(s)-f(u)|}{(s-u)^{\alpha+1}}du\,ds\right], \tag{1.2}$$

where $C_0(\omega) = C \cdot \sup_{a<s<b}\left|D_{b-}^{1-\alpha}B_{b-}(s)\right| < \infty$ a.s.

Denote $\sigma = (\sigma^{ij})_{d\times m}, b = (b^i)_{d\times 1}$ and for a matrix $A = (a^{ij})_{d\times m}$, and a vector $y = (y^i)_{d\times 1}$ denote $|A| = \sum_{i,j}|a^{ij}|, |y| = \sum_i |y^i|$.

We suppose that the coefficients satisfy the following assumptions

(A) $\sigma(t,x)$ is differentiable in x and there exist such $M>0, 1-H<\beta\leq 1, \frac{1}{H}-1<\kappa\leq 1$ and for any $N>0$ there exists such $M_N>0$ that

1) $|\sigma(t,x)-\sigma(t,y)|\leq M|x-y|, x,y\in R^d, t\in[0,T]$;

2) $|\partial_{x_i}\sigma(t,x)-\partial_{x_i}\sigma(t,y)|\leq M_N|x-y|^\kappa, |x|,|y|\leq N, t\in[0,T]$;

3) $|\sigma(t,x)-\sigma(s,x)|+|\partial_{x_i}\sigma(t,x)-\partial_{x_i}\sigma(s,x)|\leq M|t-s|^\beta, x\in R^d, t,s\in[0,T]$.

(B) 1) for any $N>0$ there exists $L_N>0$ such that

$$|b(t,x)-b(t,y)|\leq L_N|x-y|,\ |x|,|y|\leq N,\ t\in[0,T];$$

2) $|b(t,x)|\leq L(1+|x|)$.

As it was stated in [Nualart and Răşcanu (2000)], under conditions (A)–(B) the equation (1.1) has the unique solution $\{X_t, t\in[0,T]\}$, and for a.a. $\omega\in\Omega$ this solution belongs to $C^{H-\rho}[0,T]$ for any $0<\rho<H$.

Now, let $t \in [0, T], \delta = \frac{T}{N}, \tau_n = \frac{nT}{N} = n\delta, n = 0, ..., N$. Consider discrete Euler approximations of solution of equation (1.1),

$$\widetilde{Y}^{i,\delta}_{\tau_{n+1}} = \widetilde{Y}^{i,\delta}_{\tau_n} + b^i(\tau_n, \widetilde{Y}^{\delta}_{\tau_n})\delta + \sum_{j=1}^{m} \sigma^{ij}(\tau_n, \widetilde{Y}^{\delta}_{\tau_n})\Delta B^j_{\tau_n}, \quad \widetilde{Y}^{i,\delta}_0 = X^i_0,$$

and corresponding continuous interpolations

$$Y^{i,\delta}_t = \widetilde{Y}^{i,\delta}_{\tau_n} + b^i(\tau_n, \widetilde{Y}^{\delta}_{\tau_n})(t - \tau_n) + \sum_{j=1}^{m} \sigma^{ij}(\tau_n, \widetilde{Y}^{\delta}_{\tau_n})(B^j_t - B^j_{\tau_n}), \quad t \in [\tau_n, \tau_{n+1}]. \tag{1.3}$$

Continuous interpolations satisfy the equation

$$Y^{i,\delta}_t = X^i_0 + \int_0^t b^i(t_u, Y^{\delta}_{t_u})du + \sum_{j=1}^{m} \int_0^t \sigma^{i\,j}(t_u, Y^{\delta}_{t_u})dB^j_u, \tag{1.4}$$

where $t_u = \tau_{n_u}, n_u = \max\{n : \tau_n \leq u\}$.

For simplicity we denote the vector of solutions as $X_t = (X^i_t)_{i=1,\dots,d}$, vector of continuous approximations as $Y^{\delta}_t = (Y^{\delta,i}_t)_{i=1,\dots,d}$. Throughout the talk, C denotes a generic constant, whose value is not important and may change from line to line, and we write $C(\cdot)$, if the dependence on some parameters is crucial.

1.1 Some properties of Euler approximations for solutions of pathwise equations

In this section we consider growth and Hölder properties of the approximation process $\left\{Y_t^\delta, t \in [0,T]\right\}$. We need some additional notations. Denote $\varphi_{u,v} := \left|Y_{t_u}^\delta - Y_v^\delta\right|(u-v)^{-\alpha-1}$ for $0 < v < t_u < T$, $0 < \alpha < 1$, $X_t^* := \sup_{0 \leqq s \leqq t} |X_s|$, $Y_t^{\delta,*} := \sup_{0 \leqq s \leqq t} \left|Y_s^\delta\right|$. Further, for any $0 < \rho < H$ there exists such $C = C(\omega, \rho)$ that for any $0 < v < u$

$$|B_u - B_v| \leqq C(\omega,\rho)(u-v)^{H-\rho}. \tag{1.5}$$

We shall use the following statement [Nualart and Răşcanu (2000), Lemma 7.6]

Proposition 1.1. *Let $0 < \alpha < 1$, $a, b > 0$, $x : \mathbb{R}_+ \to R_+$ be a continuous function such that for each t*

$$x_t \leqq a + bt^\alpha \int_0^t (t-s)^{-\alpha} s^{-\alpha} x_s ds.$$

Then $x_t \leqq a c_\alpha \exp\left\{d_\alpha t b^{1/(1-\alpha)}\right\}$, where $c_\alpha = 4e^2 \frac{\Gamma(1-\alpha)}{1-\alpha}$, $d_\alpha = 2\big(\Gamma(1-\alpha)\big)^{1/(1-\alpha)}$, $\Gamma(\cdot)$ is Euler's Gamma function.

Lemma 1.2. *There exists such* $C = C_\alpha > 0$ *that for any* $s \in [0, T]$, $s \neq t_s$ *and* $\delta \leqq 1$, $\alpha \in (0, 1)$ *it holds*

$$J := \int_0^{t_s} (s-u)^{-\alpha-1} \int_u^{t_u} (v - t_v)^{-\alpha} dv\, du \leqq C\delta^{-\alpha}.$$

Proof. Evidently,

$$J = \int_0^{t_s} (v - t_v)^{-\alpha} \int_0^{v} (s-u)^{-\alpha-1} du\, dv \leqq \alpha^{-1} \int_0^{t_s} (v - t_v)^{-\alpha} (s-v)^{-\alpha} dv.$$

Let $t_s = n\delta$ for some $0 < n \leqq N$. Then

$$\int_0^{t_s} (v - t_v)^{-\alpha} (s-v)^{-\alpha} dv = \sum_{k=0}^{n-2} \int_{\tau_k}^{\tau_{k+1}} + \int_{(n-1)\delta}^{(2n-1)\delta/2} + \int_{(2n-1)\delta/2}^{n\delta}.$$

We estimate the integrals individually:

$$\int_{\tau_k}^{\tau_{k+1}} \leqq (s-\tau_{k+1})^{-\alpha} \int_{\tau_k}^{\tau_{k+1}} (v-t_v)^{-\alpha} dv \leqq (1-\alpha)^{-1}(s-\tau_{k+1})^{-\alpha}\delta^{1-\alpha},$$

$$\int_{(n-1)\delta}^{(2n-1)\delta/2} \leqq (\delta/2)^{-\alpha} \int_{(n-1)\delta}^{(2n-1)\delta/2} (v-t_v)^{-\alpha} dv \leqq C\delta^{1-2\alpha},$$

$$\int_{(2n-1)\delta/2}^{n\delta} \leqq (\delta/2)^{-\alpha} \int_{(2n-1)\delta/2}^{n\delta} (s-v)^{-\alpha} dv \leqq C\delta^{1-2\alpha}.$$

Therefore

$$\begin{aligned} J &\leqq C\delta^{1-2\alpha} + \delta^{-\alpha} \sum_{k=0}^{n-2} (s-\tau_{k+1})^{-\alpha}\delta \leqq C\delta^{1-2\alpha} + \delta^{-\alpha} \int_0^{n\delta} (s-v)^{-\alpha} dv \\ &\leqq C\delta^{1-2\alpha} + C\delta^{-\alpha} \leqq C\delta^{-\alpha}. \end{aligned}$$

$\square$

Theorem 1.3. *(i) Let the conditions* (A)–(B) *hold and*

(C) 1) $\quad |\sigma(t,x)| \leq C(1+|x|)$.

Then for any $\varepsilon > 0$ *and* $0 < \rho < H$ *there exists* $\delta_0 > 0$ *and* $\Omega_{\varepsilon,\delta_0,\rho} \subset \Omega$ *such that* $P(\Omega_{\varepsilon,\delta_0,\rho}) > 1-\varepsilon$ *and for any* $\omega \in \Omega_{\varepsilon,\delta_0,\rho}$, $\delta < \delta_0$ *one has* $\left|Y_t^\delta\right| \leqq C(\omega)$, $\left|Y_{t_s}^\delta - Y_{t_r}^\delta\right| \leqq C(\omega)(t_s - t_r)^{H-\rho}$, $0 \leqq r < s \leqq T$.

(ii) If, instead of (A), 2) *and* (C) *we assume that* b *and* σ *are bounded functions, then* $\left|Y_t^\delta\right| \leqq C(\omega)$, $\left|Y_s^\delta - Y_r^\delta\right| \leqq C(\omega)(s-r)^{H-\rho}$, $0 \leqq r < s \leqq T$.

In both cases $C(\omega)$ *does not depend on* δ.

Proof. We can assume that $\delta \leqq 1$. It follows immediately from (A), 1) and 3) and (1.4) that for any $\alpha \in (1-H, \beta \wedge 1/2)$

$$
\begin{aligned}
|Y_t^{i,\delta}| &\leqq \left|X_0^i\right| + \int_0^t \left|b^i(t_u, Y_{t_u}^\delta)\right| du + \sum_{j=1}^m \left|\int_0^t \sigma^{i\,j}(t_u, Y_{t_u}^\delta) dB_u^H\right| \\
&\leqq \left|X_0^i\right| + L\int_0^t \Big(1 + \left|Y_{t_u}^\delta\right|\Big) du + C_0(\omega) \sum_{j=1}^m \int_0^t \left|\sigma^{i\,j}(t_u, Y_{t_u}^\delta)\right| u^{-\alpha} du \\
&\quad + C_0(\omega) \sum_{j=1}^m \int_0^t \int_0^r \left|\sigma^{i\,j}(t_r, Y_{t_r}^\delta) - \sigma^{i\,j}(t_u, Y_{t_u}^\delta)\right| (r-u)^{-\alpha-1} du\, dr \\
&\leqq \left|X_0^i\right| + \Big(C_0(\omega)\frac{T}{1-\alpha} + LT\Big) + \big(C_0(\omega) + CT^\alpha\big) \int_0^t \left|Y_{t_u}^\delta\right| u^{-\alpha} du \\
&\quad + MC_0(\omega) \int_0^t \int_0^{t_r} \Big((t_r - t_u)^\beta + \left|Y_{t_r}^\delta - Y_u^\delta\right| + \left|Y_u^\delta - Y_{t_u}^\delta\right|\Big)(r-u)^{-\alpha-1} du\, dr.
\end{aligned}
\tag{1.6}
$$

(We use here the equality $t_r = t_u$ for $t_r \leqq u < r$.) Denote $C_1(\omega) := m\big(C_0(\omega)\frac{T^{1-\alpha}}{1-\alpha} + LT\big) + |X_0|$, $C_2(\omega) := m(C_0(\omega) + CT^\alpha)$. Further, note that

$t_r - t_u \leqq r - u + \delta$. Also, it follows from representations (1.3) that for any $\rho \in (0, H)$

$$
\begin{aligned}
\left|Y_u^\delta - Y_{t_u}^\delta\right| &\leqq L\Big(1 + \left|Y_{t_u}^\delta\right|\Big)(u - t_u) + C \cdot C(\omega, \rho)\Big(1 + \left|Y_{t_u}^\delta\right|\Big)(u - t_u)^{H-\rho} \\
&\leqq C_3(\omega)\Big(1 + \left|Y_{t_u}^\delta\right|\Big)(u - t_u)^{H-\rho},
\end{aligned}
\tag{1.7}
$$

where $C_3(\omega) = LT^{1-H-\rho} + C \cdot C(\omega, \rho)$.

Moreover, for $\beta > \alpha$

$$
\begin{aligned}
P_t &:= \int_0^t \int_0^{t_r} (t_r - t_u)^\beta (r - u)^{-\alpha-1} du\, dr \leqq \int_0^t \int_0^{t_r} \Big((r - u)^\beta + \delta^\beta\Big)(r - u)^{-\alpha-1} du\, dr \\
&\leqq (\beta - \alpha)^{-1} \int_0^t r^{\beta-\alpha} dr + \alpha^{-1}\delta^\beta \int_0^t (r - t_r)^{-\alpha} dr,
\end{aligned}
$$

and for any $k \geqq 0$ and any power $\pi > -1$

$$
\int_{\tau_k}^{\tau_{k+1}} (r - t_r)^\pi dr = \int_{\tau_k}^{\tau_{k+1}} (r - \tau_k)^\pi dr = C_1 \delta^{\pi+1} \text{ with } C_1 = (\pi + 1)^{-1},
$$

whence

$$
\int_0^t (r - t_r)^{-\alpha} dr \leqq \int_0^T (r - t_r)^{-\alpha} dr = C_1 N \delta^{1-\alpha} = C_1 \delta^{-\alpha}. \tag{1.8}
$$

Therefore

$$P_t \leqq C_1 T^{\beta-\alpha+1} + \alpha^{-1} C_1 \delta^{\beta-\alpha} \leqq C_1 T^{\beta-\alpha+1} + \alpha^{-1} C_1 =: C_2. \tag{1.9}$$

Estimate now

$$Q_t := \int_0^t \int_0^{t_r} \left|Y_u^\delta - Y_{t_u}^\delta\right| (r-u)^{-\alpha-1} du\, dr,$$

using (1.7) and (1.8):

$$\begin{aligned} Q_t &\leqq \left(1 + Y_t^{\delta,*}\right) \int_0^t \int_0^{t_r} (u-t_u)^{H-\rho} (r-u)^{-\alpha-1} du\, dr \\ &\leqq C_3(\omega)\left(1+Y_t^{\delta,*}\right)\delta^{H-\rho}\alpha^{-1} \int_0^t (r-t_r)^{-\alpha} dr \leqq C_4(\omega)\left(1+Y_t^{\delta,*}\right)\delta^{H-\alpha-\rho}, \end{aligned} \tag{1.10}$$

with $C_4(\omega) = C_3(\omega)\alpha^{-1} \cdot C_1$. Note that $Y_t^{\delta,*} := \sup_{0 \leqq s \leqq t} \left|Y_s^\delta\right| < \infty$ for any $t \in [0,T]$ a.s. Substituting (1.9) and (1.10) into (1.6), we obtain that

$$\begin{aligned} \left|Y_t^\delta\right| &\leqq C_5(\omega) + C_2(\omega) \int_0^t \left|Y_{t_u}^\delta\right| u^{-\alpha} du + C_4(\omega)\left(1+Y_t^{\delta,*}\right)\delta^{H-\alpha-\rho} \\ &\quad + C_6(\omega) \int_0^t \int_0^{t_r} \varphi_{r,u} du\, dr \end{aligned} \tag{1.11}$$

with $C_5(\omega) = C_3(\omega) + MC_0(\omega)C_2$, $C_6(\omega) = MC_0(\omega)$. To simplify the notations, in what follows we remove subscripts from $C(\omega)$, writing $C(\omega)$ for all constants depending on ω.

So we can write

$$Y_t^{\delta,*} \leqq C(\omega)\Big(1 + Y_t^{\delta,*}\delta^{H-\alpha-\rho} + \int_0^t \left|Y_{t_u}^{\delta}\right| u^{-\alpha} du + \int_0^t \int_0^{t_r} \varphi_{r,u} du\, dr\Big). \tag{1.12}$$

In turn, we can estimate $\int_0^{t_s} \varphi_{s,u} du$. At first, similarly to the previous estimates,

$$\begin{aligned}
\left|Y_{t_s}^{\delta} - Y_u^{\delta}\right| \leqq C(\omega)\Big[&\int_u^{t_s} \Big(1 + \left|Y_{t_v}^{\delta}\right|\Big) dv + \int_u^{t_s} \Big(1 + \left|Y_{t_v}^{\delta}\right|\Big)(v-u)^{-\alpha} dv \\
&+ \int_u^{t_s} \int_u^{t_v} \left|\sigma(t_v, Y_{t_v}^{\delta}) - \sigma(t_z, Y_{t_z}^{\delta})\right| (v-z)^{-\alpha-1} dz\, dv\Big] \\
\leqq C(\omega)\Big[&(t_s-u)^{1-\alpha} + \int_u^{t_s} \left|Y_{t_v}^{\delta}\right| (v-u)^{-\alpha} dv + \delta^{\beta} \int_u^{t_s} (v-t_v)^{-\alpha} dv \\
&+ \int_u^{t_s} \int_u^{t_v} \varphi_{v,z} dz\, dv + \int_u^{t_s} \int_u^{t_v} \left|Y_z^{\delta} - Y_{t_z}^{\delta}\right| (v-z)^{-\alpha-1} dz\, dv\Big];
\end{aligned} \tag{1.13}$$

multiplying by $(s-u)^{-\alpha-1}$ and integrating over $[0,t_s]$, we obtain that

$$\int_0^{t_s} \varphi_{s,u} du \leqq C(\omega) \sum_{i=1}^{5} Q_s^i, \tag{1.14}$$

where

$$Q_s^1 := \int_0^{t_s} (t_s-u)^{1-\alpha}(s-u)^{-\alpha-1} du \leqq \int_0^{t_s} (s-u)^{-2\alpha} du \leqq C; \tag{1.15}$$

$$\begin{aligned} Q_s^2 &:= \int_0^{t_s} (s-u)^{-\alpha-1} \int_u^{t_s} \left|Y_{t_v}^\delta\right| (v-u)^{-\alpha} dv \\ &= \int_0^{t_s} \left|Y_{t_v}^\delta\right| \int_0^{v} (v-u)^{-\alpha}(s-u)^{-\alpha-1} du\, dv \leqq C_0 \int_0^{t_s} \left|Y_{t_v}^\delta\right| (s-v)^{-2\alpha} dv, \end{aligned} \tag{1.16}$$

where $C_0 = \int_0^\infty (1+y)^{-\alpha-1} y^{-\alpha} dy$; according to Lemma 1.2

$$\begin{aligned} Q_s^3 &:= \delta^\beta \int_0^{t_s} (s-u)^{-\alpha-1} \int_u^{t_s} (v-t_v)^{-\alpha} dv\, du \\ &\leqq C\delta^\beta \delta^{-\alpha} \leqq C. \end{aligned} \tag{1.17}$$

Further, using estimates (1.7), we can conclude that

$$
\begin{aligned}
Q_s^4 &:= \int_0^{t_s} (s-u)^{-\alpha-1} \int_u^{t_s} \int_u^{t_v} \varphi_{v,z} dz\, dv\, du \\
&\leqq \int_0^{t_s} \int_0^{t_v} \int_0^{z\wedge v} \varphi_{v,z}(s-u)^{-\alpha-1} du\, dz\, dv \leqq C \int_0^{t_s} (s-v)^{-\alpha} \int_0^{t_v} \varphi_{v,z} dz\, dv.
\end{aligned}
\tag{1.18}
$$

At last, using estimates (7) and Lemma 1.2, we can conclude that.

$$
\begin{aligned}
Q_s^5 &:= \int_0^{t_s} (s-u)^{-\alpha-1} \int_u^{t_s} \int_u^{t_v} \left|Y_z^\delta - Y_{t_z}^\delta\right| (v-z)^{-\alpha-1} dz\, dv\, du \\
&\leqq C(\omega) \int_0^{t_s} (s-u)^{-\alpha-1} \int_u^{t_s} \int_u^{t_v} (v-z)^{-\alpha-1} dz\, dv\, du \cdot \delta^{H-\rho}\Big(1 + \left|Y_{t_s}^{\delta,*}\right|\Big) \\
&\leqq C(\omega)\Big(1 + \left|Y_{t_s}^{\delta,*}\right|\Big)\delta^{H-\rho-\alpha}.
\end{aligned}
\tag{1.19}
$$

Now, denote $\psi_s := Y_s^{\delta,*} + \int_0^{t_s} \varphi_{s,u} du$. Note that the integrals Q_s^i are finite for $s = k\delta$, i.e. for any $s \in [0, T]$, including $s = t_s$. Then it follows from (1.12) and (1.14)–(1.19) that

$$
\psi_t \leqq C(\omega)\Big(1 + Y_t^{\delta,*}\delta^{H-\alpha-\rho} + \int_0^t \big((t-v)^{-2\alpha} + v^{-\alpha}\big)\psi_v dv\Big).
$$

Let $\varepsilon > 0$ be fixed. Note that all constants $C(\omega)$ are finite a.s. and independent of δ. Thus, we can choose $\delta_0 > 0$ and $\Omega_{\varepsilon,\delta_0,\rho}$ such that $C(\omega)\delta_0^{H-\alpha-\rho} \leqq 1/2$ on $\Omega_{\varepsilon,\delta_0,\rho}$ and $P(\Omega_{\varepsilon,\delta_0,\rho}) > 1-\varepsilon$. Then for any $\omega \in \Omega_{\varepsilon,\delta_0,\rho}$

$$\psi_t \leqq C(\omega) + \frac{1}{2}\psi_t + C(\omega)\int_0^t \big((t-v)^{-2\alpha} + v^{-\alpha}\big)\psi_v dv,$$

whence

$$\psi_t \leqq C(\omega)\Big(1 + t^{2\alpha}\int_0^t (t-v)^{-2\alpha} v^{-2\alpha}\psi_v dv\Big),$$

and it follows immediately from the last equation and Proposition 1.1 that $\psi_t \leqq C(\omega)$ whence, in particular, $\left|Y_t^\delta\right| \leqq C(\omega)$, $t \in [0,T]$, and $\int_0^{t_s} \varphi_u du \leqq C(\omega)$. Moreover, from (1.13) with $u = t_r$, $r \leqq s$, taking into account that $\int_{t_r}^{t_s} (v - t_v)^{-\alpha} dv \leqq \delta^{-\alpha}(t_s - t_r)$, we obtain the estimate

$$\begin{aligned}\left|Y_{t_s}^\delta - Y_{t_r}^\delta\right| \leqq C(\omega)\Big(&(t_s-t_r)^{1-\alpha} + \delta^{\beta-\alpha}(t_s - t_r) + (t_s - t_r)\\ &+ \delta^{H-\rho}\int_{t_r}^{t_s} (v-t_v)^{-\alpha} dv\Big) \leqq C(\omega)(t_s-t_r)^{1-\alpha},\end{aligned}$$

and the statement (i) is proved. (ii) Let $|b(t,x)| \leqq b$, $|\sigma(t,x)| \leqq \sigma$. Then it is very easy to see that the estimate (1.11) will take a form

$$\left|Y_t^\delta\right| \leqq C(\omega)\Big(1+\int_0^t\int_0^{t_r}\varphi_{r,u}du\,dr\Big),$$

(1.13) will perform to

$$\left|Y_{t_s}^\delta - Y_u^\delta\right| \leqq C(\omega)\Big((t_s-u)^{1-\alpha}+(\delta^\beta+\delta^{H-\rho})\int_u^{t_s}(v-t_v)^{-\alpha}dv$$
$$+\int_u^{t_s}\int_u^{t_v}\varphi_{v,z}dz\,dv\Big)$$

and instead of (1.14)–(1.19) we obtain

$$\int_0^{t_s}\varphi_{s,u}du \leqq C(\omega)\Big(1+\int_0^{t_s}(s-v)^{-\alpha}\int_0^{t_v}\varphi_{v,z}dz\,dv\Big),$$

whence the proof easily follows. $\square$

1.2 The estimates of rate of convergence for Euler approximations of the solutions of pathwise equations

Now we establish the estimates of the rate of convergence of our approximations (1.4). We establish even more: an estimate of convergence rate for the norm of the difference $X_t - Y_t^\delta$ in some Besov space, similarly to the result of Theorem 1. Denote $\Delta_{u,s}(X, Y^\delta) := \left|X_s - Y_s^\delta - X_u + Y_u^\delta\right|$ and assume for technical simplicity that $L_N = L$, $M_N = M$ in (A) and (B).

Theorem 1.4. *Let the conditions* (A)–(C) *hold and also*

(D) 1) $|b(t,x) - b(s,x)| \leq C|t-s|^\gamma$, $C > 0$, $2H - 1 < \gamma \leq 1$*;*

2) *the exponent* β *from* (A) 3) *satisfies* $\beta > H$*.*

Then: (i) for any $\varepsilon > 0$ *and any* $\rho > 0$ *sufficiently small there exists* $\delta_0 > 0$ *and* $\Omega_{\varepsilon,\delta_0,\rho}$ *such that* $P(\Omega_{\varepsilon,\delta_0,\rho}) > 1 - \varepsilon$ *and for any* $\omega \in \Omega_{\varepsilon,\delta_0,\rho}$, $\delta < \delta_0$

$$U_\delta := \sup_{0 \leqq s \leqq T} \left(\left|X_s - Y_s^\delta\right| + \int_0^{t_s} \left|\Delta_{u,s}(X, Y^\delta)\right| (s-u)^{-\alpha-1} du \right) \leqq C(\omega) \cdot \delta^{2H-1-\rho},$$

where $C(\omega)$ *does not depend on* δ *and* ε *(but depends on* ρ*);*

(ii) if, in addition, the coefficients b *and* σ *are bounded, then for any* $\rho \in (0, 2H-1)$ *there exists* $C(\omega) < \infty$ *a.s. such that* $U_\delta \leqq C(\omega)\delta^{2H-1-\rho}$, $C(\omega)$ *does not depend on* δ*.*

Proof. (i) Denote $Z_t^\delta := \sup_{0 \leqq s \leqq t} \left| X_s - Y_s^\delta \right|$. Then

$$
\begin{aligned}
Z_t^\delta &:= \sup_{0\le s\le t} |X_s - Y_s^\delta| \le \sup_{0\le s\le t} \int_0^s |b(u, X_u) - b(t_u, Y_{t_u}^\delta)| du \\
&+ \sup_{0\le s\le t} \sum_{i,j=1}^m |\int_0^s (\sigma^{ij}(u, X_u) - \sigma^{ij}(t_u, Y_{t_u}^\delta)) dB_u^i| \le \int_0^t |b(u, X_u) - b(u, Y_u^\delta)| du \\
&+ \int_0^t |b(u, Y_u^\delta) - b(t_u, Y_u^\delta)| du + \int_0^t |b(t_u, Y_u^\delta) - b(t_u, Y_{t_u}^\delta)| du \\
&+ \sup_{0\le s\le t} \sum_{i,j=1}^m |\int_0^s (\sigma^{i\,j}(u, X_u) - \sigma^{i\,j}(u, Y_u^\delta)) dB_u^i| \\
&+ \sup_{0\le s\le t} \sum_{i,j=1}^m |\int_0^s (\sigma^{i\,j}(u, Y_u^\delta) - \sigma^{ij}(t_u, Y_u^\delta)) dB_u^i| \\
&+ \sup_{0\le s\le t} \sum_{i,j=1}^m |\int_0^s (\sigma^{i\,j}(t_u, Y_u^\delta) - \sigma^{i\,j}(t_u, Y_{t_u}^\delta)) dB_u^i| =: \sum_{k=1}^6 I_k.
\end{aligned}
\tag{1.20}
$$

Now we estimate separately all these terms. Evidently,

$$I_1 \leqq L \int_0^t Z_u^\delta du. \tag{1.21}$$

Condition (D) 1) implies that for $\delta \leqq 1$

$$I_2 \leqq C \int_0^t |u - t_u|^\gamma \, du \leqq C\delta^\gamma \leqq C\delta^{2H-1}. \tag{1.22}$$

As it follow from Theorem 2.2, for any $\varepsilon > 0$ and any $\rho \in (0, H)$ there exists $\delta_0 > 0$ and $\Omega_{\varepsilon,\delta_0,\rho} \subset \Omega$ such that $P(\Omega_{\varepsilon,\delta_0,\rho}) > 1 - \varepsilon$ and $C(\omega)$ independent of ε and δ such that for any $\omega \in \Omega_{\varepsilon,\delta_0,\rho}$ it holds $\left|Y_t^\delta - Y_s^\delta\right| \leqq C(\omega)\, |t - s|^{H-\rho}$. In what follows we assume that $\delta < \delta_0 < 1$. Therefore

$$I_3 \leqq L \cdot C(\omega)\delta^{H-\rho} \cdot t \leqq C(\omega)\delta^{H-\rho}, \omega \in \Omega_{\varepsilon,\delta_0,\rho}. \tag{1.23}$$

Now we go on with I_4. It follows from (1.2) that for $1-H<\alpha<1/2$

$$\begin{aligned} I_4 \leqq C(\omega) \sum_{i,j=1}^{m} \Big[\int_0^t \left|\sigma^{i\,j}(u,X_u)-\sigma^{i\,j}(u,Y^\delta_{t_u})\right| u^{-\alpha} du \\ + \int_0^t \int_0^r \left|\sigma^{i\,j}(r,X_r)-\sigma^{i\,j}(u,X_u)-\sigma^{i\,j}(r,Y^\delta_r)+\sigma^{i\,j}(u,Y^\delta_u)\right| \\ \times (r-u)^{-\alpha-1} du\, dr \Big] =: I_7 + I_8. \end{aligned} \tag{1.24}$$

Evidently,

$$I_7 \leqq C(\omega) \int_0^t Z^\delta_u u^{-\alpha} du. \tag{1.25}$$

According to [Nualart and Răşcanu (2000), Lemma 7.1], under condition (A)

$$\begin{aligned} |\sigma(t_1,x_1)-\sigma(t_2,x_2)-\sigma(t_1,x_3)+\sigma(t_2,x_4)| \leqq M\,|x_1-x_2-x_3+x_4| \\ + M\,|x_1-x_3| \Big(|t_2-t_1|^\beta + |x_1-x_2|^\kappa + |x_3-x_4|^\kappa \Big). \end{aligned} \tag{1.26}$$

Therefore, $I_8 \leqq \sum_{k=9}^{12} I_k$, where

$$I_9 = C(\omega) \int_0^t \int_0^r \left| X_r - Y_r^\delta \right| (r-u)^{\beta-\alpha-1} du\, dr,$$

$$I_{10} = C(\omega) \int_0^t \int_0^r \left| X_r - Y_r^\delta \right| \left| X_r - X_u \right|^\kappa (r-u)^{-\alpha-1} du\, dr,$$

$$I_{11} = C(\omega) \int_0^t \int_0^r \left| X_r - Y_r^\delta \right| \left| Y_r^\delta - Y_u^\delta \right|^\kappa (r-u)^{-\alpha-1} du\, dr,$$

$$I_{12} = C(\omega) \int_0^t \int_0^r \Delta_{u,r}(X, Y^\delta)(r-u)^{-\alpha-1} du\, dr.$$

Taking into account that $\beta > H > \alpha$, we obtain that

$$I_9 \leqq C(\omega) \int_0^t Z_u^\delta du. \tag{1.27}$$

As it follows from [Nualart and Răşcanu (2000), Theorem 2.1], under assumptions (A) and (B) for any $0 < \rho < H$ there exists such constant $C(\omega)$ that

$$\sup_{0 \leqq t \leqq T} |X_t| \leqq C(\omega), \quad \sup_{0 \leqq s \leqq t \leqq T} |X_t - X_s| \leqq C(\omega) \left| t-s \right|^{H-\rho}. \tag{1.28}$$

Moreover, we can choose $\rho > 0$ and $\alpha > 1 - H$ such that $\kappa(H - \rho) > \alpha$ and

$H - \rho > 2H - 1$, because $\kappa H > 1 - H$. In this case

$$I_{10} \leqq C(\omega) \int_0^t Z_r^\delta \int_0^r (r-u)^{\kappa(H-\rho)-\alpha-1} du\, dr \leqq C(\omega) \int_0^T Z_r^\delta dr. \tag{1.29}$$

It follows from Theorem 2.2 that on $\Omega_{\varepsilon,\delta_0,\rho}$ the same estimate holds for I_{11}.

Now estimate I_5.

$$\begin{aligned} I_5 &\leqq C(\omega) \int_0^t \left|\sigma(u, Y_u^\delta) - \sigma(t_u, Y_u^\delta)\right| u^{-\alpha} du \\ &+ C(\omega) \int_0^t \int_0^r \left|\sigma(r, Y_r^\delta) - \sigma(t_r, Y_r^\delta) - \sigma(u, Y_u^\delta) + \sigma(t_u, Y_u^\delta)\right| (r-u)^{-\alpha-1} du\, dr \\ &=: I_{13} + I_{14}. \end{aligned}$$

Obviously,

$$I_{13} \leqq C(\omega)\delta^{\beta}, \tag{1.30}$$

$$\begin{aligned} I_{14} &\leqq C(\omega)\Big(\int_0^t \int_0^{t_r} + \int_0^t \int_{t_r}^r\Big) \left|...\right| (r-u)^{-\alpha-1} du\, dr \\ &\leqq C(\omega) \int_0^t \int_0^{t_r} \delta^{\beta}(r-u)^{-\alpha-1} du\, dr + \int_0^t \int_{t_r}^r \big((r-u)^{\beta} + (r-u)^{H-\rho}\big) du\, dr \\ &\leqq C(\omega)\big(\delta^{\beta-\alpha} + \delta^{H-\rho-\alpha}\big). \end{aligned} \tag{1.31}$$

Similarly,

$$\begin{aligned} I_6 &\leqq C(\omega) \int_0^t \left|\sigma(t_u, Y_u^{\delta}) - \sigma(t_u, Y_{t_u}^{\delta})\right| u^{-\alpha} du \\ &+ C(\omega) \int_0^t \int_0^r \left|\sigma(t_r, Y_r^{\delta}) - \sigma(t_r, Y_{t_r}^{\delta}) - \sigma(t_u, Y_u^{\delta}) + \sigma(t_u, Y_{t_u}^{\delta})\right| \\ &\qquad \times (r-u)^{-\alpha-1} du\, dr =: I_{15} + I_{16}. \end{aligned} \tag{1.32}$$

Here

$$I_{15} \leqq C(\omega) \int_0^t \delta^{H-\rho} u^{-\alpha} du \leqq C(\omega) \delta^{H-\rho}; \tag{1.33}$$

$$I_{16} \leqq C(\omega) \int_0^t \int_0^r \delta^{H-\rho} (r-u)^{-\alpha-1} du\, dr \leqq C(\omega) \delta^{H-\rho-\alpha}. \tag{1.34}$$

Substituting (1.21)–(1.34) into (1.20), we obtain that on $\Omega_{\varepsilon,\delta_0,\rho}$

$$Z_t^\delta \leqq C(\omega) \Big(\int_0^t Z_r^\delta r^{-\alpha} dr + \delta^{H-\rho-\alpha} + \delta^{H-\rho} + \int_0^t \theta_r dr \Big), \tag{1.35}$$

where $\theta_r = \int_0^r \Delta_{r,u}(X, Y^\delta)(r-u)^{-\alpha-1} du$. Recall that $H - \rho > 2H - 1$, therefore

$$Z_t^\delta \leqq C(\omega) \Big(\int_0^t \big(Z_r^\delta r^{-\alpha} + \theta_r \big) dr + \delta^{2H-1-\rho} \Big).$$

We now estimate θ_r. Evidently, for $t > u$

$$\Delta_{t,u}(X, Y^\delta) \leqq \int_u^t \big| b(s, X_s) - b(t_s, Y_{t_s}^\delta) \big| \, ds + \sum_{i,j=1}^m \Big| \int_u^t \big(\sigma^{i\,j}(s, X_s) - \sigma^{i\,j}(t_s, Y_{t_s}^\delta) \big) dB_s^i \Big| .$$

Therefore, using inequality (1.2), we obtain that $\theta_t \leqq \sum_{k=1}^9 J_k$, where

$$J_1 = \int_0^t \int_u^t \left| b(s, X_s) - b(s, Y_s^\delta) \right| ds (t-u)^{-\alpha-1} du,$$

$$J_2 = \int_0^t \int_u^t \left| b(s, Y_s^\delta) - b(t_s, Y_s^\delta) \right| ds (t-u)^{-\alpha-1} du,$$

$$J_3 = \int_0^t \int_u^t \left| b(t_s, Y_s^\delta) - b(t_s, Y_{t_s}^\delta) \right| ds (t-u)^{-\alpha-1} du,$$

$$J_4 = C(\omega) \int_0^t \int_u^t \left| \sigma(s, X_s) - \sigma(s, Y_s^\delta) \right| (s-u)^{-\alpha} ds (t-u)^{-\alpha-1} du,$$

$$J_5 = C(\omega) \int_0^t \int_u^t \left| \sigma(s, Y_s^\delta) - \sigma(t_s, Y_s^\delta) \right| (s-u)^{-\alpha} ds (t-u)^{-\alpha-1} du,$$

$$J_6 = C(\omega) \int_0^t \int_u^t \left| \sigma(t_s, Y_s^\delta) - \sigma(t_s, Y_{t_s}^\delta) \right| (s-u)^{-\alpha} ds (t-u)^{-\alpha-1} du,$$

$$\begin{aligned} J_7 = C(\omega) \int_0^t \int_u^t \int_u^r & \left| \sigma(r, X_r) - \sigma(r, Y_r^\delta) - \sigma(v, X_v) + \sigma(v, Y_v^\delta) \right| \\ & \times (r-v)^{-\alpha-1} dv \, dr (t-u)^{-\alpha-1} du, \end{aligned}$$

$$\begin{aligned} J_8 = C(\omega) \int_0^t \int_u^t \int_u^r & \left| \sigma(r, Y_r^\delta) - \sigma(t_r, Y_r^\delta) - \sigma(v, Y_v^\delta) + \sigma(t_v, Y_v^\delta) \right| \\ & \times (r-v)^{-\alpha-1} dv \, dr (t-u)^{-\alpha-1} du, \end{aligned}$$

$$\begin{aligned} J_9 = C(\omega) \int_0^t \int_u^t \int_u^r & \left| \sigma(t_r, Y_r^\delta) - \sigma(t_r, Y_{t_r}^\delta) - \sigma(t_v, Y_v^\delta) + \sigma(t_v, Y_{t_v}^\delta) \right| \\ & \times (r-v)^{-\alpha-1} dv \, dr (t-u)^{-\alpha-1} du. \end{aligned}$$

It is clear that

$$J_1 \leqq C \int_0^t Z_s^\delta \int_0^s (t-u)^{-\alpha-1} du\, ds, \quad J_2 \leqq C\delta^\gamma, \quad J_3 \leqq C(\omega)\delta^{H-\rho}.$$

Further,

$$J_4 \leqq C \int_0^t Z_s^\delta \int_0^s (s-u)^{-\alpha}(t-u)^{-\alpha-1} du\, ds.$$

As we noted before, the inner integral $\int_0^s (s-u)^{-\alpha}(t-u)^{-\alpha-1}du \leqq C_0(t-s)^{-2\alpha}$, $C_0 = \int_0^\infty (1+y)^{-\alpha-1}y^{-\alpha}dy$. Therefore $J_4 \leqq C\int_0^t (t-s)^{-2\alpha} Z_s^\delta ds$. Similarly to J_2, $J_5 \leqq C(\omega)\delta^\gamma$, and similarly to J_3, $J_6 \leqq C(\omega) \leqq C(\omega)\delta^{H-\rho}$. Further,

$$J_8 \leqq C(\omega)\delta^\beta \int_0^t \int_u^t \int_u^r (r-v)^{-\alpha-1} dv\, dr (t-u)^{-\alpha-1} du \leqq C(\omega)\delta^\beta;$$

similarly $J_9 \leqq C(\omega)\delta^{H-\rho}$. Now we apply to J_7 the inequality (1.26) and obtain the following estimate of the integrand:

$$\begin{aligned} &\left|\sigma(r,X_r) - \sigma(r,Y_r^\delta) - \sigma(v,X_v) + \sigma(v,Y_v^\delta)\right| \leqq M\Big[\Delta_{r,v}(X,Y^\delta) \\ &+ \left|X_r - Y_r^\delta\right|(r-v)^\beta + \left|X_r - Y_r^\delta\right|\left|X_r - X_v\right|^\kappa + \left|X_r - Y_r^\delta\right|\left|Y_r^\delta - Y_v^\delta\right|^\kappa\Big]. \end{aligned} \tag{1.36}$$

According to this, we write $J_7 \leqq \sum_{k=10}^{13} J_k$, where, in turn,

$$\begin{aligned}
J_{10} &= C(\omega) \int_0^t \int_u^t \int_u^r \Delta_{r,v}(X, Y^\delta)(r-v)^{-\alpha-1} dv\, dr (t-u)^{-\alpha-1} du \\
&= C(\omega) \int_0^t \int_0^r \int_0^v (t-u)^{-\alpha-1} du \Delta_{r,v}(X, Y^\delta)(r-v)^{-\alpha-1} dr\, dv \\
&\leqq C(\omega) \int_0^t (t-r)^{-\alpha} \theta_r dr; \\
J_{11} &= C(\omega) \int_0^t \int_u^t \int_u^r \left| X_r - Y_r^\delta \right| (r-v)^{\beta-\alpha-1} dv\, dr (t-u)^{-\alpha-1} du \\
&\leqq C(\omega) \int_0^t Z_r^\delta \int_0^r (t-u)^{-\alpha-1} \Big(\int_u^r (r-v)^{\beta-\alpha-1} dv \Big) du\, dr \\
&\leqq C(\omega) \int_0^t (t-r)^{-\alpha} Z_r dr,
\end{aligned}$$

$$J_{12} = C(\omega) \int_0^t \int_u^t \int_u^r \left| X_r - Y_r^\delta \right| |X_r - X_v|^\kappa (r-v)^{-\alpha-1} dv\, dr (t-u)^{-\alpha-1} du$$
$$\leqq C(\omega) \int_0^t \int_0^r \int_u^r Z_r^\delta (r-v)^{\kappa(H-\rho)-\alpha-1} dv (t-u)^{-\alpha-1} du\, dr \leqq C(\omega) \int_0^t Z_r^\delta (t-r)^{-\alpha} dr,$$

and $J_{13} \leqq C(\omega) \int_0^t Z_r^\delta (t-r)^{-\alpha} dr$ is obtained the same way. Summing up these estimates, we obtain that

$$J_7 \leqq C(\omega) \int_0^t (t-r)^{-\alpha} \big(Z_r^\delta + \theta_r \big) dr,$$

whence

$$\theta_t \leqq C(\omega) \Big(\int_0^t (t-r)^{-2\alpha} \big(Z_r^\delta + \theta_r \big) dr + \delta^{H-\rho} + \delta^\gamma \Big). \tag{1.37}$$

Coupling together (1.35) and (1.37), and taking into account that $H - \rho > 2H - 1$, $\gamma > 2H - 1$, we obtain

$$\begin{aligned} Z_t^\delta + \theta_t &\leqq C(\omega) \Big(\delta^{2H-1} + \int_0^t \big((t-r)^{-2\alpha} + r^{-\alpha} \big) \big(Z_r^\delta + \theta_r \big) dr \Big) \\ &\leqq C(\omega) \Big(\delta^{2H-1} + t^{2\alpha} \int_0^t (t-r)^{-2\alpha} r^{-2\alpha} \big(Z_r^\delta + \theta_r \big) dr \Big) \end{aligned} \tag{1.38}$$

The proof now follows immediately from (1.38) and Proposition 2.1. The statement (ii) is obvious. □

Remark 1.5. In [Nourdin and Neunkirch (2007)] it is proved that $\left| X_t - Y_t^\delta \right| \delta^{1-2H}$ almost surely converges to some stochastic process ξ_t, which means that the estimate of the rate of convergence in Theorem 1.4 is sharp.

1.3 Approximation of quasilinear Skorohod-type equations

Here we assume that our probability space is the white noise space $(\Omega, \mathcal{F}, P) = (S'(\mathbb{R}), \mathcal{B}(S'(\mathbb{R})), \mu)$, $\diamond$ is the Wick product, $B_t^0 = \langle \omega, 1\!\!\mathrm{I}_{[0,t]} \rangle$ is Brownian motion, $W^0 = \dot{B}^0$ is the white noise (see [Holden et al. (1996)] for definitions). Next, in order to introduce an fBm with Hurst parameter $H > 1/2$ on this space, we define for $f : [0,T] \to \mathbb{R}$ the fractional integral operator

$$Mf(x) = K \int_x^T (s-x)^{H-3/2} f(s)\, ds,$$

where K is some special constant, and set $M_t(x) = M 1\!\!\mathrm{I}_{[0,t]}(x)$. We also define for $f, g : [0,T] \to \mathbb{R}$ the scalar product and the norm

$$\langle f, g \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T f(t) g(s) \, |t-s|^{2H-2}\, dt\, ds, \quad \|f\|^2_{\mathcal{H}} = \langle f, f \rangle_{\mathcal{H}}.$$

The process

$$B_t = \langle M_t, \omega \rangle, \quad t \in [0,T]$$

is the fBm with Hurst parameter H. Let also $W = \dot{B}$ be the fractional white noise. Detailed description of the white noise theory can be found in [Elliott and van der Hoek (2003)], [Hu and Øksendal (2003)].

Consider quasilinear Skorohod-type equation driven by fractional white noise

$$X(t) = X_0 + \int_0^t b(s, X(s), \omega)\, ds + \int_0^t \sigma(s) X(s) \diamond W(s)\, ds \tag{1.39}$$

with non-random initial condition X_0. Suppose that coefficients b and σ satisfy the following:

(E) 1) The linear growth condition and Lipschitz condition on b:

$$|b(t, x, \omega)| \leqq C(1 + |x|), \quad |b(t, x, \omega) - b(t, y, \omega)| \leqq C\,|x - y|\,;$$

2) "Smoothness" of b w.r.t. ω: for any $t \in [0, T]$ and for $h \in L^1(\mathbb{R})$

$$|b(t, x, \omega + h) - b(t, x, \omega)| \leqq C(1 + |x|) \int_{\mathbb{R}} |h(s)|\, ds.$$

3) Hölder continuity of b w.r.t. t or order H with constant that grows linearly in x:

$$|b(t, x, \omega) - b(s, x, \omega)| \leqq C(1 + |x|)\, |t - s|^H\,;$$

4) Hölder continuity of σ w.r.t. t or order H:

$$|\sigma(t) - \sigma(s)| \leqq C\,|t - s|^H\,.$$

Remark 1.6. The condition (E) 2) is true if, for example, the coefficient b has stochastic derivative growing at most linearly in x. It is obviously true if b is non-random.

Define for $t \in [0,T]$ $\sigma_t(s) = \sigma(s)1\!\mathrm{I}_{[0,t]}(s)$ and denote

$$J_\sigma(t) = \exp^\diamond\left\{-\int_0^t \sigma(s)dB_s\right\} = \exp\left\{-\int_{\mathbb{R}} M\sigma_t(s)dB^0(s) - \frac{1}{2}\left\|\sigma_t\right\|_{\mathcal{H}}^2 ds\right\}$$

the fractional Wick exponent. It follows from [Mishura (2003), Theorem 2] that under assumptions (E) equation (1.39) has the unique solution that belongs to all L^p and can be represented in the form

$$X(t) = J_\sigma(t) \diamond Z(t),$$

where the process $Z(t)$ solves (ordinary) differential equation

$$Z(t) = X_0 + \int_0^t J_\sigma(s)b(s, J_\sigma^{-1}(s)Z(s), \omega + M\sigma_s)\, ds. \tag{1.40}$$

This gives the following idea of constructing time-discrete approximations of the solution of (1.39). Take the uniform partitioning $\{\tau_n = n\delta,\ n = 1, \ldots, N\}$ of the segment $[0, T]$ and define first the approximations of Z in a recursive way:

$$\begin{aligned} &\widetilde{Z}(0) = X_0, \\ &\widetilde{Z}(\tau_{n+1}) = \widetilde{Z}(\tau_n) + \widetilde{J}(\tau_n) b(\tau_n, \widetilde{J}^{-1}(\tau_n)\widetilde{Z}(\tau_n), \omega + M\widetilde{\sigma}_n)\delta, \end{aligned} \tag{1.41}$$

where

$$\begin{aligned} &\widetilde{J}(t) = \exp\left\{-\int_0^t \widetilde{\sigma}(s) dB_s - \frac{1}{2}\left\|\widetilde{\sigma}1\!\!\mathrm{I}_{[0,t]}\right\|_{\mathcal{H}}^2\right\}, \\ &\widetilde{\sigma}(s) = \sigma(t_s),\ \widetilde{\sigma}_n = \widetilde{\sigma}1\!\!\mathrm{I}_{[0,\tau_n]}. \end{aligned}$$

Note that both $\|\widetilde{\sigma}_n\|_{\mathcal{H}}$ and $M\widetilde{\sigma}_n$ are easily computable as finite sums of elementary integrals. Further, we interpolate continuously by

$$\widetilde{Z}(t) = X_0 + \int_0^t \widetilde{J}(t_s) b(t_s, \widetilde{J}^{-1}(t_s)\widetilde{Z}(t_s), \omega + M\widetilde{\sigma}_{n_s})\, ds, \tag{1.42}$$

where $n_s = \max\{n : \tau_n \leqq s\}$, and set

$$\widetilde{X}(t) = T_{-M\widetilde{\sigma}1\!\!\mathrm{I}_{[0,t]}} \widetilde{J}^{-1}(t)\widetilde{Z}(t), \tag{1.43}$$

where for $h \in S'(\mathbb{R})$ T_h is the shift operator, $T_h F(\omega) = F(\omega + h)$.

Lemma 1.7. *Under the assumption* (E) 1) *the following estimate is true*

$$\left|e^{\alpha_1} b(t, e^{-\alpha_1} x, \omega) - e^{\alpha_2} b(t, e^{-\alpha_2} x, \omega)\right| \leqq C(1 + e^{\alpha_1} + e^{\alpha_2} + |x|)\, |\alpha_1 - \alpha_2|\,.$$

Proof. Write

$$\begin{aligned}&\left|e^{\alpha_1} b(t, e^{-\alpha_1} x, \omega) - e^{\alpha_2} b(t, e^{-\alpha_2} x, \omega)\right| \\ &\leqq \left|e^{\alpha_1} b(t, e^{-\alpha_1} x, \omega) - e^{\alpha_1} b(t, e^{-\alpha_2} x, \omega)\right| + \left|e^{\alpha_1} b(t, e^{-\alpha_2} x, \omega) - e^{\alpha_2} b(t, e^{-\alpha_2} x, \omega)\right|\end{aligned}$$

and apply (E) 1). □

Lemma 1.8. *Let ξ_1 and ξ_2 be jointly Gaussian variables. Then for $q \geqq 1$*

$$\mathsf{E}\left|e^{\xi_1} - e^{\xi_2}\right|^{2q} \leqq C(L, q) \left(\mathsf{E}\,(\xi_1 - \xi_2)^2\right)^q,$$

where $L = \max\left\{\mathsf{E}\,\xi_1^2, \mathsf{E}\,\xi_2^2\right\}$.

Proof. By Lagrange theorem, Cauchy–Schwartz inequality and Gaussian property,

$$\mathsf{E}\left|e^{\xi_1} - e^{\xi_2}\right|^{2q} \leqq \left(\mathsf{E}\, e^{4q\xi_1} + e^{4q\xi_2}\, \mathsf{E}\,|\xi_1 - \xi_2|^{4q}\right)^{1/2} \leqq C(L)C(q) \left(\mathsf{E}\,(\xi_1 - \xi_2)^2\right)^q,$$

as required. □

Our first result is about convergence of $\widetilde{Z}$ to Z.

Theorem 1.9. *Under conditions* (E) *for any* $p \geqq 1$ *the following estimate holds:*

$$\mathsf{E}\left|Z(t) - \widetilde{Z}(t)\right|^{2p} \leqq C(p)\delta^{2pH}. \tag{1.44}$$

Proof. Firstly, we remind that $Z(t)$ belongs to all L^q and $\mathsf{E}\,|Z(t)|^q \leqq C(q)$. Therefore equation (1.40) together with the condition (E) 2) gives $\mathsf{E}\,|Z(t) - Z(s)|^q \leqq C(q)\,|t-s|^q$. Equation (1.41) and the condition (E) 1) allow to write

$$\left|\widetilde{Z}(\tau_{n+1})\right| \leqq (1 + C\delta)\left|\widetilde{Z}(\tau_n)\right| + C\delta\widetilde{J}(\tau_n) \leqq e^{C\delta}\left|\widetilde{Z}(\tau_n)\right| + C\delta\widetilde{J}(\tau_n).$$

This gives an estimate

$$\left|\widetilde{Z}(\tau_n)\right| \leqq C\sum_{k=0}^{N-1}\widetilde{J}(\tau_k)\delta.$$

Then for any $q \geqq 1$ by the Jensen inequality,

$$\left|\widetilde{Z}(\tau_n)\right|^q \leqq C(q)\sum_{k=0}^{N-1}\widetilde{J}^q(\tau_k)\delta,$$

Taking expectations, we get

$$\mathsf{E}\left|\widetilde{Z}(\tau_n)\right|^q \leqq C(q)\sum_{k=0}^{N-1}\mathsf{E}\,\widetilde{J}^q(\tau_k)\delta.$$

Using that each $\widetilde{J}$ is exponent of Gaussian variable and σ is bounded on $[0,T]$, we obtain

$$\mathsf{E}\left|\widetilde{Z}(\tau_n)\right|^q \leqq C(q)\sum_{k=0}^{N-1}\delta = C(q).$$

This through (1.42) and (E) 1) implies $\mathsf{E}\left|\widetilde{Z}(t)\right|^q \leqq C(q)$.

Now write

$$\left|Z(t)-\widetilde{Z}(t)\right| \leqq I_1+I_2+I_3+I_4+I_5,$$

where

$$I_1 = \left| \int_0^t \widetilde{J}(t_s)\big(b(t_s, \widetilde{J}^{-1}(t_s)Z(t_s), \omega + M\widetilde{\sigma}_{n_s}) \right.$$

$$\left. - b(t_s, \widetilde{J}^{-1}(t_s)\widetilde{Z}(t_s), \omega + M\widetilde{\sigma}_{n_s})\big)\, ds \right|,$$

$$I_2 = \left| \int_0^t \big(\widetilde{J}(t_s)b(t_s, \widetilde{J}^{-1}(t_s)Z(t_s), \omega + M\widetilde{\sigma}_{n_s}) \right.$$

$$\left. - J_\sigma(s)b(t_s, J_\sigma^{-1}(s)Z(t_s), \omega + M\widetilde{\sigma}_{n_s})\big)\, ds \right|,$$

$$I_3 = \left| \int_0^t J_\sigma(s)\big(b(s, J_\sigma^{-1}(s)Z(t_s), \omega + M\widetilde{\sigma}_{n_s}) \right.$$

$$\left. - b(t_s, J_\sigma^{-1}(s)Z(t_s), \omega + M\widetilde{\sigma}_{n_s})\big)\, ds \right|,$$

$$I_4 = \left| \int_0^t J_\sigma(s)\big(b(s, J_\sigma^{-1}(s)Z(t_s), \omega + M\widetilde{\sigma}_{n_s}) - b(s, J_\sigma^{-1}(s)Z(t_s), \omega + M\sigma_s)\big)\, ds \right|,$$

$$I_5 = \left| \int_0^t J_\sigma(s)\big(b(s, J_\sigma^{-1}(s)Z(s), \omega + M\sigma_s) - b(s, J_\sigma^{-1}(s)Z(t_s), \omega + M\sigma_s)\big)\, ds \right|.$$

We first estimate using Lemma 1.7

$$
\begin{aligned}
I_2 \leqq C \int_0^t \big(1 + J_\sigma(s) + \widetilde{J}(t_s) + |Z(t_s)|\,\big) \Bigg(\bigg| \int_0^s \big(\sigma(u) - \widetilde{\sigma}(u)\big)\, dB_u \bigg| \\
+ \big|\sigma(t_s)\big(B_s - B(t_s)\big)\big| + \frac{1}{2} \Big| \|\sigma_s\|^2_{\mathcal{H}} - \|\widetilde{\sigma}_{n_s}\|^2_{\mathcal{H}} \Big| \Bigg)\, ds \\
\leqq C \int_0^t \big(1 + J_\sigma(s) + \widetilde{J}(t_s) + |Z(t_s)|\,\big) \\
\cdot \Bigg(\bigg| \int_0^s \big(\sigma(u) - \widetilde{\sigma}(u)\big)\, dB_u \bigg| + |B_s - B_{t_s}| + \delta^H \Bigg)\, ds,
\end{aligned}
$$

where the inequality $\Big| \|\sigma_s\|^2_{\mathcal{H}} - \|\widetilde{\sigma}_{n_s}\|^2_{\mathcal{H}} \Big| < C\delta^H$ is due to E 4) and boundedness of σ on $[0, T]$. Applying Cauchy–Schwartz inequality, we arrive to

$$
\begin{aligned}
I_2 \leqq & C \left(\int_0^T \big(1 + J^2_\sigma(s) + \widetilde{J}^2(t_s) + Z^2(t_s)\big) ds \right)^{1/2} \\
& \cdot \left(\int_0^T \left(\left(\int_0^s \big(\sigma(u) - \widetilde{\sigma}(u)\big)\, dB_u \right)^2 + (B_t - B_{t_s})^2 + \delta^{2H} \right) ds \right)^{1/2}.
\end{aligned}
$$

Further, from (E) 3)

$$I_3 \leqq C \int_0^T (J_\sigma(s) + |Z(s)|) \, ds \delta^H,$$

from (E) 2)

$$I_3 \leqq C \int_0^T (J_\sigma(s) + |Z(s)|) \, ds \delta^H.$$

Condition (E) 1) allows to estimate

$$I_1 \leqq C \int_0^t \left| Z(t_s) - \widetilde{Z}(t_s) \right| \, ds,$$

$$I_5 \leqq C \int_0^t |Z(s) - Z(t_s)| \, ds.$$

Summing up these estimates yields

$$\left|Z(t)-\widetilde{Z}(t)\right| \leqq C\left(\int_0^T \big(1+J_\sigma^2(s)+\widetilde{J}^2(t_s)+Z^2(t_s)\big)ds\right)^{1/2}$$

$$\cdot\left(\delta^{2H}+\int_0^T\left(\left(\int_0^s \big(\sigma(u)-\widetilde{\sigma}(u)\big)\,dB_u\right)^2+(B_t-B_{t_s}^2)\right)ds\right)^{1/2}$$

$$+\,C\int_0^T \left|Z(t_s)-\widetilde{Z}(t_s)\right|ds+C\int_0^t |Z(s)-Z(t_s)|\;ds.$$

Then, using (discrete) Gronwall inequality, we get

$$\left|Z(t)-\widetilde{Z}(t)\right| \leqq C\left(\int_0^T \big(1+J_\sigma^2(s)+\widetilde{J}^2(t_s)+Z^2(t_s)\big)ds\right)^{1/2}$$

$$\cdot\left(\delta^{2H}+\int_0^T\left(\left(\int_0^s \big(\sigma(u)-\widetilde{\sigma}(u)\big)\,dB_u\right)^2+(B_t-B_{t_s}^2)\right)ds\right)^{1/2}$$

$$+\,C\int_0^t |Z(s)-Z(t_s)|\;ds.$$

Then we raise this to the $2p$th power and use Jensen's inequality. The last term will be bounded by

$C(p)\delta^{2p}$, in the first one we apply Cauchy–Schwartz inequality for expectations, Jensen's inequality and use uniform boundedness of moments for Z, J_σ and $\widetilde{J}$ (for J_σ and $\widetilde{J}$ it follows from the fact that the both are exponents of some Gaussian variables with bonded variance) to get

$$\mathsf{E}\left|Z(t)-\widetilde{Z}(t)\right|^{2p} \leqq C(p)\Bigg(\delta^{2pH}+\Big(\mathsf{E}\Big[\Big|\int_0^T\big(\sigma(u)-\widetilde{\sigma}(u)\big)\,dB_u\Big|^{4p}\Big]\Big)^{1/2} + \Big(\mathsf{E}\left|B_t-B_{t_s}\right|^{4p}\Big)^{1/2}\Bigg).$$

Using again that $\mathsf{E}\left|\cdot\right|^{4p}=C(p)(\mathsf{E}\,(\cdot)^2)^{2p}$ for Gaussian variables, we get

$$\begin{aligned}\mathsf{E}\left|Z(t)-\widetilde{Z}(t)\right|^{2p} &\leqq C(p)\Bigg(\delta^{2pH}+\Big(\mathsf{E}\Big[\Big|\int_0^T\big(\sigma(u)-\widetilde{\sigma}(u)\big)\,dB_u\Big|^{2}\Big]\Big)^{p} + \Big(\mathsf{E}\left|B_t-B_{t_s}\right|^{2}\Big)^{p}\Bigg)\\ &\leqq C(p)\big(\delta^{2pH}+\|\sigma-\widetilde{\sigma}\|_{\mathcal{H}}^{2p}\big)\leqq C(p)\delta^{2pH},\end{aligned}$$

the last is due to (E) 4). This is the desired result. $\square$

Now we are ready to state the main result of this section.

Theorem 1.10. *Under conditions* (E) *approximations* $\widetilde{X}$ *defined by* (1.43) *converge to the solution* X *of* (1.39) *in the mean-square sense, and moreover*

$$\mathsf{E}\,(X(t)-\widetilde{X}(t))^2 \leqq C\delta^{2H}.$$

Proof. Estimate first for $h \in L^1(\mathbb{R})$

$$\begin{aligned}
&T_h Z(t) - Z(t) \leqq A_1 + A_2 + A_3\\
&A_1 = \int_0^t T_h J_\sigma(s)\Big| b(s,(T_h J_\sigma^{-1})T_h Z(s),\omega+h+M\sigma_s)\\
&\qquad\qquad\qquad - b(s,(T_h J_\sigma^{-1})Z(s),\omega+h+M\sigma_s)\Big|\,ds,\\
&A_2 = \int_0^t T_h J_\sigma(s)\Big| b(s,(T_h J_\sigma^{-1})Z(s),\omega+h+M\sigma_s)\\
&\qquad\qquad\qquad - b(t,(T_h J_\sigma^{-1}(s))Z(s),\omega+M\sigma_s)\Big|\,ds,\\
&A_3 = \int_0^t \Big| T_h J_\sigma(s) b(t,(T_h J_\sigma^{-1}(s))Z(s),\omega+M\sigma_s)\\
&\qquad\qquad\qquad - J_\sigma(s) b(t,J_\sigma^{-1}(s)Z(s),\omega+M\sigma_s)\Big|\,ds.
\end{aligned}$$

The condition (E) 1) gives $A_1 \leqq C \int_0^t \left|T_h Z(s) - Z(s)\right| \, ds$, the condition (E) 2) gives

$$A_2 \leqq C \int_0^T (1 + |Z(s)|) \, ds \int_{\mathbb{R}} |h(s)| \, ds$$

and Lemma 1.7 with boundedness of σ yields

$$\begin{aligned} A_3 &\leqq C \int_0^T (1 + J_\sigma(s) + T_h J(\sigma) + |Z(s)|) \, ds \left| \int_{\mathbb{R}} M\sigma(s) \, h(s) \, ds \right|. \\ &\leqq C \int_0^T (1 + J_\sigma(s) + T_h J(\sigma) + |Z(s)|) \, ds \int_{\mathbb{R}} |h(s)| \, ds. \end{aligned}$$

Applying Gronwall lemma, we get

$$|T_h Z(t) - Z(t)| \leqq C \int_0^T (1 + J_\sigma(s) + T_h J(\sigma) + |Z(s)|) \, ds \int_{\mathbb{R}} |h(s)| \, ds.$$

Raising this inequality to the $2p$ th power, taking expectations and using Jensen inequality and boundedness of moments of Z, J_σ and $T_h J_\sigma$ (the last follows from the Girsanov theorem, Cauchy–Schwartz inequality and assumptions on h), we get

$$\mathsf{E} \left(T_h Z(t) - Z(t)\right)^{2p} \leqq C(p) \left(\int_0^T |h(s)| \, ds \right)^{2p}.$$

Further,

$$
\begin{aligned}
&\mathsf{E}\left(X(t) - \widetilde{X}(t)\right)^2 \leqq 3(A_1 + A_2 + A_3),\\
&A_1 = \mathsf{E}\left(\overline{J}(t) T_{-M\widetilde{\sigma}\mathbb{1}_{[0,t]}}\left(Z(t) - \widetilde{Z}(t)\right)\right)^2,\\
&A_2 = \mathsf{E}\left(\left(J_{-\sigma}(t) - \overline{J}(t)\right) T_{-M\widetilde{\sigma}\mathbb{1}_{[0,t]}} Z(t)\right)^2,\\
&A_3 = \mathsf{E}\left(J_{-\sigma}(t)\left(T_{-M\sigma}(1 - T_{-M(\widetilde{\sigma}\mathbb{1}_{[0,t]} - \sigma_t)}\right) Z(t)\right)^2,
\end{aligned}
$$

where

$$
\begin{aligned}
J_{-\sigma}(t) &= \exp\left\{\int_{\mathbb{R}} M\sigma_t(s)\, dB_s^0 - \frac{1}{2}\left\|\sigma_t\right\|_{\mathcal{H}}^2\right\},\\
\overline{J}(t) &= \exp\left\{\int_{\mathbb{R}} M(\widetilde{\sigma}\mathbb{1}_{[0,t]})(s)\, dB_s^0 - \frac{1}{2}\left\|\widetilde{\sigma}\mathbb{1}_{[0,t]}\right\|_{\mathcal{H}}^2\right\}.
\end{aligned}
$$

Now estimate using Cauchy–Schwartz inequality, Girsanov theorem (which can be applied as σ and

$\widetilde{\sigma}$ are bounded on $[0,T]$) and Theorem 1.9

$$\begin{aligned} A_1 &\leqq \Big(\mathsf{E}\, \overline{J}^4(t)\, \mathsf{E}\, T_{-M\widetilde{\sigma}1\!\!\mathrm{I}_{[0,t]}} \big(Z(t) - \widetilde{Z}(t)\big)^4 \Big)^{1/2}, \\ &\leqq C\Big(\mathsf{E}\, \widetilde{J}(t)\big(Z(t) - \widetilde{Z}(t)\big)^4 \Big)^{1/2} \\ &\leqq C\Big(\mathsf{E}\, \widetilde{J}^2(t)\, \mathsf{E}\, \big(Z(t) - \widetilde{Z}(t)\big)^8 \Big)^{1/4} \leqq C\delta^{2H}. \end{aligned}$$

Similar reasoning and Lemma 1.8 imply

$$A_2 \leqq C\,\mathsf{E} \left(\int_{\mathbb{R}} M(\widetilde{\sigma}1\!\!\mathrm{I}_{[0,t]} - \sigma_t)(s)\, dB^0_s + \frac{1}{2}\big(\|\sigma_t\|^2_{\mathcal{H}} - \|\widetilde{\sigma}1\!\!\mathrm{I}_{[0,t]}\|^2_{\mathcal{H}} \big) \right)^2.$$

Using condition (E) 4), we obtain $A_2 \leqq C\delta^{2H}$. And for A_3, using the above estimate, we get

$$A_3 \leqq \int_0^t \big| M(\widetilde{\sigma}1\!\!\mathrm{I}_{[0,t]} - \sigma_t)(s) \big|\, ds \leqq C\delta^{2H}.$$

This concludes the proof. $\square$

Remark 1.11. It is natural to assume that the coefficient b is expressed in the terms of fBm B rather then in the terms of underlying Brownian motion B^0 (or underlying “Brownian” white noise ω.) This justifies the fact that it is σ not $M\sigma$ what is discretized in (1.41).

Remark 1.12. Similarly to the proof of Theorem 1.10 one can prove that for any $s \geqq 1$

$$\mathsf{E}\left|X(t) - \widetilde{X}(t)\right|^s \leqq \delta^{sH}.$$

The case $s = 2$ is considered in the paper to keep classical "scent" of results.

Remark 1.13. Results of this section can be generalized for random initial condition X_0 in the following form: under conditions (E) and L^p-integrability of the initial condition one has convergence in any L^s for $s < p$ with

$$\mathsf{E}\left|X(t) - \widetilde{X}(t)\right|^s \leqq \delta^{sH}.$$

Proofs need some simple changes: Hölder inequality for appropriate powers instead of Cauchy–Schwartz one.

2 Approximation schemes for stochastic differential equations in Hilbert space

Numerical solution of stochastic differential equations (SDE) has numerous applications. A classical example of application is based on Feynmann–Kac formula, which provides a connection between solution of a parabolic partial differential equation and solution of an SDE. Many equations, which arise in modeling of physical, chemical, biological phenomena, stock prices, involve randomness. This randomness, however, is not always well modeled by the classical white noise — Wiener process. Nevertheless, often, with proper choice of scale or by considering asymptotic behavior of a system, it becomes Gaussian.

The idea to solve an SDE numerically with a method similar to the Euler's method for non-random differential equations originates from [Maruyama (1955)]. Further development of the theory is connected with [Milstein (1974)], where a higher order accuracy scheme was constructed, and [Wagner and Platen (1978)], who proposed a method to construct schemes of arbitrary order via stochastic Taylor expansions. The monographs [Milstein (1988)], [Kloeden and Platen (1992)] contain virtually complete theory of approximation of numerical solution of finite systems of SDE with regular coefficients. It is worth to mention also the paper [Schurz (1999)], which contains close to exhaustive (for the publication date) bibliography on numerical solution of SDE, and a monograph [Kuznetsov (1998)], which, in addition to extensive theory of numerical solution of SDE, sets a new (other than those in [Milstein (1988)], [Kloeden and Platen (1992)]) method of generating of multiple Wiener integrals. We mention also the paper [Kolodii (1997)], where the theorem on convergence of approximations of Itô–Volterra equations is proved (without giving the rate of convergence). Closely related papers are those concerned with numerical treatment of stochastic partial differential equations (SPDE). These are, in particular, papers [Gyöngy and Krylov (2003a)], [Gyöngy and Krylov (2003b)], where the rate of convergence of SPDE by "splitting-up" methods is estimated, [Millet and Sanz-Solé (2000)], who considered approximations of stochastic wave propagation equation, [Gyöngy and Millet (2005)], who considered approximate solution of SPDE with monotone operators, [Du and Zhang (2002)], who made an estimate for the rate of convergence of approximations of linear elliptic and parabolic equations, [Shardlow (2003)], [Pettersson and Signahl (2005)], who considered approximations for stochastic heat equation, and PhD theses [Roman (2000)], which treated Runge–Kutta type schemes for parabolic SPDE.

2.1 Approximation of solutions via Milstein scheme

Let X be separable Hilbert space, $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $(\mathcal{F}_t,\ t \in [0,T])$ be a flow of σ-algebras, $W(t)$ be $\mathcal{F}_t$-adapted cylindrical Wiener process in X.

Consider a stochastic evolution equation

$$X(t) = X_0 + \int_0^t \big(AX(s) + a(s, X(s))\big)\, ds + \int_0^t b(s, X(s)) dW(s). \tag{2.45}$$

Here a and b are measurable from $[0,T] \times X$ to X and $\mathcal{L}_2(X,X)$, the space of Hilbert–Schmidt operators, respectively, $A\colon\ D(A) \to X$ is a linear operator, X_0 is $\mathcal{F}_0$-measurable.

In what follows we omit subscript of the norm $\|\cdot\|_X$, and write simply $\mathcal{L}_2$ for $\mathcal{L}_2(X,X)$ and also $\mathcal{L}$ for $\mathcal{L}(X,X)$, where $\mathcal{L}(X,Y)$ is the space of linear continuous operators from X to Y.

The general approach for Milstein scheme is following. Assume that A generates a strongly continuous semigroup $\{U(t),\ 0 \leqq t \leqq T\}$. The strong solution of (2.45) is also a “mild”solution, i.e.,

$$X(t) = U(t)\, X_0 + \int_0^t U(t-s)\, a(s, X(s))\, ds + \int_0^t U(t-s)\, b(s, X(s))\, dW(s). \tag{2.46}$$

The last equation is a particular case of Itô–Volterra type equation, and being able to solve the last one, we will be ready to apply the obtained results to stochastic evolution equations.

2.1.1 Approximate solution of Itô–Volterra type equations via Milstein scheme

An abstract Itô–Volterra equation is of the form

$$X(t) = m(t) + \int_0^t a(t,s,X(s))\,ds + \int_0^t b(t,s,X(s))\,dW(s), \qquad t \in [0,T], \tag{2.47}$$

where $a\colon\ S \times X \to X$, $b\colon\ S \times X \to \mathcal{L}_2$ are measurable functions ($S = \{(t,s) \in [0,T]^2\colon\ s \leqq t\}$), $m(t)$ is some $\mathcal{F}_t$-adapted continuous square integrable process. As in the case of ordinary SDE, Lipschitz continuity and linear growth conditions

$$\|a(t,s,x) - a(t,s,y)\| + \|b(t,s,x) - b(t,s,y)\|_{\mathcal{L}_2} \leqq C\,\|x-y\|\,, \tag{2.4a}$$

$$\|a(t,s,x)\| + \|b(t,s,x)\|_{\mathcal{L}_2} \leqq C(1+\|x\|) \tag{2.4b}$$

guarantee that continuous pathwise unique solution of equation (2.47) exists in $L_2(\Omega)$, moreover $\sup \mathbf{E}\,\|X(t)\|^2 < \infty$ (see, e.g., [Daletskii and Fomin (1983)]). If the coefficients $a,\ b$ are differentiable in first variable, the derivatives $a_t',\ b_t'$ are of linear growth w.r.t. the last variable, and $m(t)$ has stochastic differential, then, using the stochastic Fubini theorem, we get that the process

$X(t)$ has stochastic differential

$$\begin{aligned} dX(t) &= dm(t) + \Big(a(t,t,X(t)) + \int_0^t a_t'(t,s,X(s))\,ds \\ &\quad + \int_0^t b_t'(t,s,X(s))\,dW(s) \Big) dt + b(t,t,X(t))\,dW(t). \end{aligned}$$

Assume that the Fréchet derivative $\frac{\partial}{\partial x} b = b_x'$ exists in $\mathcal{L}(X, \mathcal{L}_2)$ and is bounded measurable function of its arguments. Now we construct approximations of equation (2.47) via Milstein scheme [Milstein (1988)]. For a given $N \in \mathbf{N}$ put $\delta = T/N$ and let $\tau_n = n\delta$, $n = 0, 1, \ldots, N$, be uniform partitioning of $[0, T]$. Assuming that some approximation $m^\delta(t)$ of the process $m(t)$ is given, we construct approximations successively:

$$\begin{aligned} Y_{n+1}^\delta &= m^\delta(\tau_{n+1}) + \sum_{i=0}^{n} \Big(a(\tau_{n+1}, \tau_i, Y_i^\delta)\,\delta + b(\tau_{n+1}, \tau_i, Y_i^\delta)(W(\tau_{i+1}) - W(\tau_i)) \\ &\quad + \int_{\tau_i}^{\tau_{i+1}} b_x'(\tau_{n+1}, \tau_i, Y_i^\delta)\, b(\tau_i, \tau_i, Y_i^\delta)(W(s) - W(\tau_i))\,dW(s) \Big). \end{aligned} \tag{2.5}$$

Remark 2.1. Note that formula (2.5) involves multiple Wiener integrals, which have the distribution hard to simulate. The natural question arises, whether it is possible to get the same rate of convergence for a scheme which involves only increments of Wiener process? The answer is given by the well-known “Clark–Cameron paradox” (see, e.g., [Clark and Cameron (1980)]): for dimension greater than 1 any approximation scheme based on increments of Wiener process on the intervals of partition has in general the same rate of convergence as Euler’s scheme.

Remark 2.2. Integrals in (2.5) are well defined if

$$\int_{\tau_n}^{\tau_{n+1}} \mathbf{E}\Big\| b'_x(\tau_{n+1}, \tau_i, Y_i^\delta)\, b(\tau_i, \tau_i, Y_i^\delta)(W(s) - W(\tau_n))\Big\|_{\mathcal{L}_2}^2 ds < \infty.$$

From linear growth condition for b and boundedness of b'_x we get that integrand does not exceed $C(1 + \mathbf{E}\|Y_i^\delta\|^2)$. Thus the boundedness of $\mathbf{E}\|Y_i^\delta\|^2$ can be proved by induction in n (the rest of summands in (2.5) are estimated in obvious way with the use of linear growth of a and b).

Note that in this case approximations are not step-by-step, i.e., in order to get the value of the approximation Y_{n+1}^δ at the node τ_{n+1}, we must know not only previous value Y_n^δ but also all preceding values. This phenomenon results not from the choice of the scheme, but rather from the fact that a solution of (2.47) in general has not the Markov property. Therefore, one cannot formulate for Itô–Volterra equations the statement analogous to Milstein’s theorem concerning the relation between global and local rates of convergence, see [Milstein (1988)].

Putting $Y^\delta(\tau_n) = Y_n^\delta$, we make continuous interpolation

$$Y^\delta(t) = m^\delta(t) + \int_0^t a(t, \tau_{n_s}, Y^\delta(\tau_{n_s}))\, ds + \int_0^t b(t, \tau_{n_s}, Y^\delta(\tau_{n_s}))\, dW(s)$$
$$+ \int_0^t b'_x(t, \tau_{\tau_{n_s}}, Y^\delta(\tau_{n_s}))b(\tau_{n_s}, \tau_{n_s}, Y^\delta(\tau_{n_s}))(W(s) - W(\tau_{n_s}))\, dW(s),$$

where $n_s = \max\{n\colon\ \tau_n < s\}$. We list assumptions on the coefficients $a,\ b$ and the process $m(t)$ which will be used in the following to prove the convergence of approximations.

1) Process $m(t)$ admits stochastic differential

$$dm(t) = \alpha(t)\, dt + \beta(t)\, dW(t),$$

coefficients $\alpha(t)$, $\beta(t)$ are $\mathcal{F}_t$-adapted continuous square integrable processes in X and $\mathcal{L}_2$ respectively, and

$$\int_0^T \mathbf{E}\, \|\beta(t)\|_{\mathcal{L}_2}^4\, dt < \infty.$$

2) Assumptions (2.4a) are fulfilled.

3) The functions $a,\ b$ are Lipschitz continuous in s:

$$\begin{aligned}&\|a(t,s,x)-a(t,u,x)\|+\|b(t,s,x)-b(t,u,x)\|_{\mathcal{L}_2}\\&\leqq C(1+\|x\|)\ |s-u|\,.\end{aligned}\tag{2.7a}$$

4) The derivatives $a'_t,\ b'_t$ satisfy the linear growth condition:

$$\|a'_t(t,s,x)\|+\|b'_t(t,s,x)\|_{\mathcal{L}_2}\leqq C(1+\|x\|).$$

5) The derivatives b'_x, a'_x are bounded, and b'_x is Lipschitz continuous in s:

$$\|a'_x(t,s,x)\|_{\mathcal{L}}+\|b'_x(t,s,x)\|_{\mathcal{L}(X,\mathcal{L}_2)}\leqq C,\tag{2.7b}$$

$$\|b'_x(t,s,x)-b'_x(t,u,x)\|_{\mathcal{L}(X,\mathcal{L}_2)}\leqq C\,|s-u|\,.\tag{2.7c}$$

5) Second derivatives $a''_{xx},\ b''_{xx}$ and the function b are bounded:

$$\|a''_{xx}(t,s,x)\|_{\mathcal{L}(X\oplus X,X)}+\|b''_{xx}(t,s,x)\|_{\mathcal{L}(X\oplus X,\mathcal{L}_2)}+\|b(t,s,x)\|_{\mathcal{L}_2}\leqq C.\tag{2.7d}$$

Theorem 2.1. *If the coefficients of equation* (2.47) *satisfy the above conditions*, *and also*

$$\mathbf{E}\left\|m^{\delta}(t)-m(t)\right\|^{2} \leqq C\delta^{2},$$

then the approximations (**??**) *converge to the solution of* (2.47), *moreover*

$$\mathbf{E}\left\|X(t)-Y^{\delta}(t)\right\|^{2} \leqq K\delta^{2}. \tag{2.8}$$

Proof. Put $Z(t)=\mathbf{E}\|X(t)-Y^{\delta}(t)\|^{2}$. We have $Z(t)\leqq 3(\|m^{\delta}(t)-m(t)\|^{2}+A+B)$, where

$$\begin{aligned} A &= \mathbf{E}\left\| \int_{0}^{t} \Big(a\big(t,s,X(s)\big)-a\big(t,\tau_{n_s},Y^{\delta}(\tau_{n_s})\big)\Big)\,ds\right\|^{2}, \\ B &= \mathbf{E}\left\| \int_{0}^{s} \Big(b\big(t,s,X(s)\big)-b\big(t,\tau_{n_s},Y^{\delta}(\tau_{n_s})\big)\Big)\,dW(s)\right\|^{2}. \end{aligned}$$

The plan to estimate both of these integrals is the same: we split integrals into several summands so that integrand of each summand is increment of a function with respect to a single variable; the summands are estimated individually. In that way,

$$
\begin{aligned}
A &\leqq C(A_1 + A_2 + A_3), \\
A_1 &= \mathbf{E}\Big\| \int_0^t \Big(a\big(t, \tau_{n_s},\, X(\tau_{n_s})\big) - a\big(t,\, \tau_{n_s},\, Y^{\delta}(\tau_{n_s})\big)\Big)\, ds \Big\|^2 \\
&\leqq C \int_0^t \mathbf{E}\Big\| a(t, \tau_{n_s},\, X(\tau_{n_s})) - a\big(t, \tau_{n_s},\, Y^{\delta}(\tau_{n_s})\big)\Big\|^2 ds \leqq C \int_0^t Z(\tau_{n_s})\, ds, \\
A_2 &= \mathbf{E}\Big\| \int_0^t \Big(a\big(t, s, X(\tau_{n_s})\big) - a\big(t, \tau_{n_s}, X(\tau_{n_s})\big)\Big)\, ds \Big\|^2 \\
&\leqq C \int_0^t \mathbf{E}\Big\| a\big(t, s, X(\tau_{n_s})\big) - a\big(t, \tau_{n_s}, X(\tau_{n_s})\big)\Big\|^2 ds \\
&\leqq C \int_0^t \mathbf{E}\Big(1 + \big\|X(\tau_{n_s})\big\|^2\Big)\, \delta^2\, ds \leqq C\delta^2, \\
A_3 &= \mathbf{E}\Big\| \int_0^t \Big(a\big(t, s, X(\tau_{n_s})\big) - a\big(t, s, X(s)\big)\Big)\, ds \Big\|^2 .
\end{aligned}
$$

To estimate A_3, we use the Itô formula (see [Greksch and Tudor (1995)]). Indeed, the process $X(t)$ is a sum of continuous process with bounded variation and of a square integrable martingale, therefore

$$
\begin{aligned}
&a\big(t,s,X(s)\big) - a(t,s,X\left(\tau_{n_s}\right)) \\
&\quad = \int_{\tau_{n_s}}^{s} a'_x\big(t,s,X(u)\big)\Bigg(\alpha(u) + a\big(u,u,X(u)\big) + \int_0^u a'_t(u,v,X(v))\,dv \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \int_0^u b'_t(u,v,X(v))\,dW(v)\Bigg)\,du \\
&\qquad + \int_{\tau_{n_s}}^{s} a'_x\big(t,s,X(u)\big)\Big(\beta(u) + b\big(u,u,X(u)\big)\Big)\,dW(u) \\
&\qquad + \int_{\tau_{n_s}}^{s} a''_{xx}\big(t,s,X(u)\big)\Big(\big(\beta(u) + b\big(u,u,X(u)\big)\big)\big(\beta(u) + b\big(u,u,X(u)\big)\big)^{\top}\Big)\,du.
\end{aligned}
$$

Here for $a \in \mathcal{L}(X \oplus X, X),\ b \in \mathcal{L}_2$ we simplified the abbreviation $\big(\{e_k,\ k \geqq 1\}$ is an orthonormal base in X):

$$
a(bb^{\top}) := \sum_{k=1}^{\infty} a(be_k, be_k).
$$

The last means not scalar product, but bilinear form arguments. We estimate the last expression as

$$\left\|a(bb^\top)\right\| \leqq \sum_{k=1}^{\infty} \|a(be_k, be_k)\| \leqq \|a\|_{\mathcal{L}(X\oplus X,X)} \sum_{k=1}^{\infty} \|be_k\|^2 \leqq \|a\|_{\mathcal{L}(X\oplus X,X)} \|b\|_{\mathcal{L}_2}.$$

Now split A_3 into summands, which correspond to the summands in Itô formula, and estimate them individually:

$$\begin{aligned} A_{31} &= \mathbf{E}\left\| \int_0^t \int_{\tau_{n_s}}^s a'_x\big(t,s,X(u)\big)\Big(\alpha(u) + a\big(u,u,X(u)\big)\Big)\,du\,ds\right\|^2 \\ &\leqq C\mathbf{E}\int_0^t (s-\tau_{n_s}) \int_{\tau_{n_s}}^s \left\|a'_x\big(t,s,X(u)\big)\right\|_{\mathcal{L}}^2 \\ &\qquad\qquad \times \Big(\left\|\alpha(u)\right\|^2 + \left\|a\big(u,u,X(u)\big)\right\|\Big)^2\,du\,ds \\ &\leqq C\delta \int_0^t \int_{\tau_{n_s}}^s \mathbf{E}\big(1 + \|X(u)\|^2\big)\,du\,ds \leqq C\delta^2, \end{aligned}$$

$$
\begin{aligned}
A_{32} &= \mathbf{E}\Big\| \int_0^t \int_{\tau_{n_s}}^s a'_x\big(t,s,X(u)\big)\Big(\beta(u)+b\big(u,u,X(u)\big)\,dW(u)\Big)\,ds\Big\|^2 \\
&= \mathbf{E}\Big\| \int_0^t \int_u^{\tau_{n_u}+\delta} a'_x\big(t,s,X(u)\big)\Big(\beta(u)+b\big(u,u,X(u)\big)\Big)\,ds\,dW(u)\Big\|^2 \\
&\leqq C\delta \int_0^t \int_u^{\tau_{n_u}+\delta} \mathbf{E}\big\|a'_x\big(t,s,X(u)\big)\big\|_{\mathcal{L}}^2 \\
&\qquad\qquad \times \Big(\big\|\beta(u)\big\|_{\mathcal{L}_2}^2 + \big\|b(u,u,X(u))\big\|_{\mathcal{L}_2}^2\Big)\,ds\,du \leqq C\delta^2, \\
A_{33} &= \mathbf{E}\Big\| \int_0^t \int_{\tau_{n_s}}^s a''_{xx}(t,s,X(u)) \\
&\qquad \times \Big(\Big(\beta(u)+b\big(u,u,X(u)\big)\Big)\Big(\beta(u)+b\big(u,u,X(u)\big)\Big)^\top\Big)\,du\,ds\Big\|^2 \\
&\leqq C\int_{\tau_{n_s}}^s \|a''_{xx}\|_{\mathcal{L}(X\oplus X,X)}\mathbf{E}\Big(\big\|b(u,u,X(u))\big\|_{\mathcal{L}_2}^4 + \|\beta(u)\|_{\mathcal{L}_2}^4\Big)\,du \leqq C\delta^2,
\end{aligned}
$$

$$
\begin{aligned}
A_{34} &= \mathbf{E}\left\| \int_0^t \int_{\tau_{n_s}}^s a'_x(t,s,X(u)) \int_0^u a'_t(s,v,X(v))\,dv\,du\,ds \right\|^2 \\
&\leqq C\delta \int_0^t \int_{\tau_{n_s}}^s \int_0^u \mathbf{E}\,\|a'_x(t,s,X(s))\|_{\mathcal{L}}^2\,\|a'_t(s,v,X(v))\|^2\,dv\,du\,ds \\
&\leqq C\delta \int_0^t \int_{\tau_{n_s}}^s \int_0^u \mathbf{E}\Big(1+\|X(v)\|^2\Big)\,dv\,du\,ds \leqq C\delta^2, \\
A_{35} &= \mathbf{E}\left\| \int_0^t \int_{\tau_{n_s}}^s a'_x\big(t,s,X(u)\big) \int_0^u b'_t\big(s,v,X(v)\big)\,dW(v)\,du\,ds \right\|^2 \\
&\leqq C\delta \int_0^t \int_{\tau_{n_s}}^s \int_0^u \mathbf{E}\,\|a'_x(t,s,X(s))\|_{\mathcal{L}}^2\,\|b'_t(s,v,X(v))\|_{\mathcal{L}_2}^2\,dv\,du\,ds \\
&\leqq C\delta \int_0^t \int_{\tau_{n_s}}^s \int_0^u \mathbf{E}\Big(1+\|X(v)\|^2\Big)\,dv\,du\,ds \leqq C\delta^2.
\end{aligned}
$$

Quite analogously, except the fact that we do not use the inequality

$$
\mathbf{E}\left\| \int_0^t \cdots ds \right\|^2 \leqq t \int_0^t \mathbf{E}\,\|\cdots\|^2\ ds,
$$

but rather we use the isometric identity

$$\mathbf{E}\left\| \int_0^t \cdots dW(s) \right\|^2 = \int_0^t \mathbf{E} \left\| \cdots \right\|_{\mathcal{L}_2}^2 \, ds,$$

we estimate the term B:

$$\begin{aligned}
B &\leqq C(B_1 + B_2 + B_3 + B_4), \\
B_1 &= \mathbf{E}\left\| \int_0^t \Big(b\big(t, \tau_{n_s}, X(\tau_{n_s})\big) - b\big(t, \tau_{n_s}, Y^\delta(\tau_{n_s})\big) \Big)\, ds \right\|^2 \\
&\leqq C \int_0^t \mathbf{E}\Big\| b\big(t, \tau_{n_s}, X(\tau_{n_s})\big) - b\big(t, \tau_{n_s}, Y^\delta(\tau_{n_s})\big) \Big\|^2 ds \leqq C \int_0^t Z(\tau_{n_s})\, ds, \\
B_2 &= \mathbf{E}\left\| \int_0^t \Big(b\big(t, s, X(\tau_{n_s})\big) - b\big(t, \tau_{n_s}, X(\tau_{n_s})\big) \Big)\, ds \right\|^2 \\
&\leqq C \int_0^t \mathbf{E}\Big\| b\big(t, s, X(\tau_{n_s})\big) - b\big(t, \tau_{n_s}, X(\tau_{n_s})\big) \Big\|^2 ds \\
&\leqq C \int_0^t \mathbf{E}\Big(1 + \|X(\tau_{n_s})\|^2 \Big)\, \delta^2\, ds \leqq C\delta^2,
\end{aligned}$$

$$
\begin{aligned}
B_3 &= \mathbf{E}\bigg\| \int_0^t \Big(b\big(t,s,X(s)\big) - b\big(t,s,X(\tau_{n_s})\big)\Big)\, dW(s) \\
&\qquad\qquad - \int_0^t \int_{\tau_{n_s}}^s b'_x(t,s,X(u))\, b(u,u,X(u))\, dW(u)\, dW(s) \bigg\|^2, \\
B_4 &= \mathbf{E}\bigg\| \int_0^t \int_{\tau_{n_s}}^s \Big(b'_x\big(t,s,X(u)\big)\, b\big(u,u,X(u)\big) \\
&\qquad\qquad\qquad - b'_x\big(t,\tau_{n_s},Y^\delta(\tau_{n_s})\big)\, b\big(\tau_{n_s},\, \tau_{n_s},Y^\delta(\tau_{n_s})\big)\Big)\, dW(u)\, dW(s) \bigg\|^2.
\end{aligned}
$$

The term B_3 is estimated analogously to A_3 (note that the summand involving double Wiener integral, is canceled in this case; it is the summand having worse rate of vanishing). Note that the function $b'_x(t,s,x)\, b(s,s,x)$ is Lipschitz continuous and of linear growth under the assumptions made, thus the term B_4 is estimated the same way as $A,\ B$. Thus, we arrive at the estimate

$$
Z(t) \leqq C\bigg(\delta^2 + \int_0^t Z(\tau_{n_s})\, ds\bigg),
$$

which through Gronwall's lemma leads to $Z(t) \leqq C\delta^2$ with constant independent of δ. Theorem 2.1 is proved.

2.1.2 Approximations of semilinear evolution equations via Milstein scheme

Now turn back to equation (2.45). As it was already mentioned, its strong solution is also mild one, i.e., it solves the equation

$$X(t) = U(t)\,X_0 + \int_0^t U(t-s)\,a(s,X(s))\,ds + \int_0^t U(t-s)\,b(s,X(s))\,dW(s). \tag{2.9}$$

We impose the following assumptions, which guarantee existence, uniqueness and continuity of the solution of (2.45), (2.9) (the proof can be found in [Greksch and Tudor (1995)]).

(A) Conditions of Lipschitz continuity and linear growth are fulfilled:

$$\begin{aligned} \|a(t,x)\|_X + \|b(t,x)\|_{\mathcal{L}_2} &\leqq C(1+\|x\|_X), \\ \|a(t,x)-a(t,y)\|_X + \|b(t,x)-b(t,y)\|_{\mathcal{L}_2} &\leqq C\,\|x-y\|_X\,, \end{aligned} \tag{2.10a}$$

(B) The operator A generates a strongly continuous operator semigroup $\{U(t), 0 \leqq t \leqq T\}$ on X.

(C) “Smoothness” conditions hold for the coefficients $a,\ b$:

$$\|Aa(t,x)\| + \|Ab(t,x)\|_{\mathcal{L}_2} \leqq C(1+\|x\|). \tag{2.10b}$$

Under assumption that the derivative b'_x is bounded, the Milstein approximations for equation (2.9) can be constructed using Milstein approximations for Itô–Volterra equations:

$$\begin{aligned} Y^\delta_{n+1} &= U(\tau_{n+1})\, Y^\delta_0 \\ &\quad + \sum_{i=0}^{n} U(\tau_{n+1} - \tau_i) \Bigg(a(\tau_i, Y^\delta_i)\, \delta + b(\tau_i, Y^\delta_i)(W(\tau_{i+1}) - W(\tau_i)) \\ &\qquad\qquad + \int_{\tau_i}^{\tau_{i+1}} b'_x(\tau_i, Y^\delta_i)\, b(\tau_i, Y^\delta_i)(W(s) - W(\tau_i))\, dW(s) \Bigg). \end{aligned} \tag{2.11}$$

Using the semigroup property, we can rewrite the last as

$$\begin{aligned} Y^\delta_{n+1} &= U(\delta) \Bigg(Y^\delta_n + a(\tau_n, Y^\delta_n)\, \delta + b(\tau_n, Y^\delta_n)\big(W(\tau_{n+1}) - W(\tau_n)\big) \\ &\qquad + \int_{\tau_n}^{\tau_{n+1}} b'_x(\tau_n, Y^\delta_n)\, b(\tau_n, Y^\delta_n) \\ &\qquad\qquad \times (W(s) - W(\tau_n))\, dW(s) \Bigg), \qquad n \geqq 0. \end{aligned} \tag{2.12}$$

Remark that approximations are step-by step in this case, thanks to the Markov property of the solutions of (2.45).

Interpolate the approximations (2.12) continuously:

$$\begin{aligned} Y^\delta(t) &= U(t)\,Y_0^\delta + \int_0^t U(t-\tau_{n_s})\Big(a\big(\tau_{n_s}, Y^\delta(\tau_{n_s})\big)\,ds + b\big(\tau_{n_s}, Y^\delta(\tau_{n_s})\big)\,dW(s) \\ &\quad + b'_x\big(\tau_{n_s}, Y^\delta(\tau_{n_s})\big)\,b\big(\tau_{n_s}, Y^\delta(\tau_{n_s})\big)\big(W(s)-W(\tau_{n_s})\big)\,dW(s)\Big). \end{aligned} \tag{2.13}$$

Now suppose that the coefficients of equation (2.45) satisfy the assumptions (A)–(C) and the following conditions which supply the convergence of Milstein scheme for ordinary SDE.

1) The functions $a,\ b$ are Lipschitz continuous in t:

$$\|a(t,x)-a(s,x)\| + \|b(t,x)-b(s,x)\|_{\mathcal{L}_2} \leqq C\,|t-s|\,(1+\|x\|). \tag{2.14a}$$

2) The derivatives b'_x, a'_x are bounded, and b'_x is Lischitz continuous in t:

$$\|a'_x(t,x)\|_{\mathcal{L}} + \|b'_x(t,x)\|_{\mathcal{L}(X,\mathcal{L}_2)} \leqq C, \tag{2.14b}$$

$$\|b'_x(t,x)-b'_x(s,x)\|_{\mathcal{L}(X,\mathcal{L}_2)} \leqq C\,|t-s|\,. \tag{2.14c}$$

3) The second derivatives $a''_{xx},\ b''_{xx}$ and the function b are bounded:

$$\|a''_{xx}(t,x)\|_{\mathcal{L}(X\oplus X,X)} + \|b''_{xx}(t,x)\|_{\mathcal{L}(X\oplus X,\mathcal{L}_2)} + \|b(t,x)\|_{\mathcal{L}_2} \leqq C. \tag{2.14d}$$

Assume also that $\mathbf{E}\,\|AX_0\|^2 < \infty$.

Theorem 2.2. *The approximations* (2.13) *converge to the solution of equation* (2.45), *and moreover*

$$\mathbf{E}\left\|X(t) - Y^\delta(t)\right\|^2 \leqq K\delta^2.$$

Proof. It is proved in [Greksch and Tudor (1995)] that under the conditions (A)–(C) equation (2.45) has strong pathwise unique solution $X(t)$, which belongs to $D(A)$ a.s. for a.a. t; also this solution $X(t)$ has stochastic differential

$$dX(t) = \mathbb{1}_{X(t)\in D(A)}\, AX(t)\, dt + a(t, X(t))\, dt + b(t, X(t))\, dW(t),$$

which is used further in Itô formula, and under assumption $\mathbf{E}\,\|AX_0\|^2 < \infty$ it holds

$$\sup_{t\in[0,T]} \mathbf{E}\,\|AX(t)\|^2 < \infty.$$

We have

$$\mathbf{E}\left\|X(t) - Y^\delta(t)\right\|^2 \leqq 2\Big(\mathbf{E}\,\|X(t) - X_1(t)\|^2 + \mathbf{E}\left\|X_1(t) - Y^\delta(t)\right\|^2\Big),$$

where

$$X_1(t) = U(t)\, X_0 + \int_0^t U(t - \tau_{n_s})\Big(a\big(s, X(s)\big)\, ds + b\big(s, X(s)\big)\, dW(s)\Big).$$

The difference $\mathbf{E}\|X_1(t) - Y^\delta(t)\|^2$ is estimated the same way as in Theorem 2.1, except the fact that Itô formula yields another summands, which we estimate:

$$\mathbf{E}\left\| \int_0^t U(t-\tau_{n_s}) \int_{\tau_{n_s}}^s a'_x(s, X(u))\, AX(u)\, du\, ds \right\|^2$$

$$\leqq C\delta \int_0^t \int_{\tau_{n_s}}^s \|U(t-\tau_{n_s})\|^2\, \mathbf{E}\, \|a'_x(s, X(u))\|^2\, \|AX(u)\|^2\, du\, ds \leqq C\delta^2,$$

$$\mathbf{E}\left\| \int_0^t U(t-\tau_{n_s}) \int_{\tau_{n_s}}^s b'_x(s, X(u))\, AX(u)\, du\, dW(s) \right\|^2$$

$$\leqq \delta \int_0^t \int_{\tau_{n_s}}^s \|U(t-\tau_{n_s})\|^2\, \mathbf{E}\, \|b'_x(s, X(u))\|^2_{\mathcal{L}(X, \mathcal{L}_2)}\, \|AX(u)\|^2\, du\, ds \leqq C\delta^2.$$

Further,

$$
\begin{aligned}
&\mathbf{E}\left\|X(t)-X_1(t)\right\|^2 \\
&\quad = \mathbf{E}\bigg\| \int_0^t \big(U(t-s)-U(t-\tau_{n_s})\big)\Big(a\big(s,X(s)\big)\,ds + b\big(s,X(s)\big)\,dW(s)\Big)\bigg\|^2 \\
&\quad = \mathbf{E}\bigg\| \int_0^t \bigg(\int_{\tau_{n_s}}^s U(t-v)\,A\,dv\bigg)\Big(a\big(s,X(s)\big)\,ds + b\big(s,X(s)\big)\,dW(s)\Big)\bigg\|^2 \\
&\quad \leqq C\bigg(\delta \int_0^t \int_{\tau_{n_s}}^s \|U(t-v)\| \\
&\qquad\qquad \times \Big(\mathbf{E}\left\|Aa(s,X(s))\right\|^2 + \mathbf{E}\left\|Ab(s,X(s))\right\|_{\mathcal{L}_2}^2\Big)\,dv\,ds\bigg) \\
&\quad \leqq C\delta \int_0^t \int_{\tau_{n_s}}^s \mathbf{E}(1+\|X(s)\|^2)\,dv\,ds \leqq C\delta^2.
\end{aligned}
$$

Theorem 2.2 is proved.

Remark 2.3. It is not hard to see that the proposed method of approximate solution of equation (2.45) is the well-known (at least for non-random equations) “splitting-up”method. At first we split equation (2.45) into the following ones:

$$dX^1(t) = a(t, X^1(t))\,dt + b(t, X^1(t))\,dW(t), \qquad dX^2(t) = AX^2(t)\,dt.$$

Then consecutively on each intervals of partition we solve approximately the first equation:

$$\begin{aligned} Y^{\delta,1}_{n+1} \quad &= \quad Y^\delta_n + a(\tau_n, Y^\delta_n)\,\delta + b(\tau_n, Y^\delta_n)\big(W(\tau_{n+1}) - W(\tau_n)\big) \\ &\quad + \int_{\tau_n}^{\tau_{n+1}} b'_x(\tau_n, Y^\delta_n)\, b(\tau_n, Y^\delta_n)\big(W(s) - W(\tau_n)\big)\,dW(s), \end{aligned}$$

then the result is plugged as initial condition into the second one:

$$Y^{\delta,2}_{n+1} = U(\delta)\,Y^{\delta,1}_{n+1},$$

which gives approximate solution of (2.45) (the formula for $Y^{\delta,2}_{n+1}$ coincides with (2.12)).

Remark 2.4. Another important observation is that there is no need to solve the equation for X^2 from previous remark explicitly, it is enough to solve it numerically. More precisely, in formula (2.12) one can change the operator $U(\delta)$ for such $\widetilde{U}_\delta$ that

$$\big\|U(\delta) - \widetilde{U}_\delta\big\|_{\mathcal{L}} \leqq C\delta^2$$

with constant independent of δ. Indeed, the last estimate implies

$$\left\| U^n(\delta) - \widetilde{U}_\delta^n \right\|_{\mathcal{L}} \leqq C\delta.$$

Denote the modified approximations $\widetilde{Y}_n^\delta$ and write

$$\begin{aligned} \widetilde{Y}_{n+1}^\delta \quad = \quad & \widetilde{U}_\delta^n Y_0^\delta + \sum_{i=0}^{n} \widetilde{U}_\delta^{n+1-i} \Big(a(\tau_i, Y_i^\delta)\, \delta + b(\tau_i, Y_i^\delta)(W(\tau_{i+1}) - W(\tau_i)) \\ & + \int_{\tau_i}^{\tau_{i+1}} b'_x(\tau_i, Y_i^\delta)\, b(\tau_i, Y_i^\delta)(W(s) - W(\tau_i))\, dW(s) \Big). \end{aligned}$$

Comparing this with (2.11), we obtain

$$\mathbf{E}\left\| Y_{n+1}^\delta - \widetilde{Y}_{n+1}^\delta \right\|^2 \leqq C \left(\delta^2 + \delta \sum_{i=1}^{n} \mathbf{E}\left\| Y_i^\delta - \widetilde{Y}_i^\delta \right\|^2 \right),$$

and, using the discrete version of Gronwall's lemma, we arrive at

$$\mathbf{E}\left\| Y_n^\delta - \widetilde{Y}_n^\delta \right\|^2 \leqq C\delta^2.$$

2.2 Approximation by finite-dimensional processes

As in the previous sections, we start by considering Itô–Volterra equation

$$X(t) = m(t) + \int_0^t a(t,s,X(s))\,ds + \int_0^t b(t,s,X(s))\,dW(s). \tag{2.1}$$

Assume that its coefficients satisfy the linear growth and Lipschitz conditions:

$$\begin{aligned} \|a(t,s,x)\| + \|b(t,s,x)\|_{\mathcal{L}_2} &\leqq C(1+\|x\|), \\ \|a(t,s,x)-a(t,s,y)\| + \|b(t,s,x)-b(t,s,y)\|_{\mathcal{L}_2} &\leqq C\,\|x-y\|\,, \end{aligned} \tag{2.2a}$$

and that there exists an increasing function $h(t)$, $t>0$, such that $h(t)\to 0$ as $t \to 0$ and

$$\|a(t,u,x)-a(s,u,x)\| + \|b(t,u,x)-b(s,u,x)\| \leqq h(t-s)(1+\|x\|). \tag{2.2b}$$

Let $\{e_n,\ n \geqq 1\}$ be an orthonormal base in X, denote $E_n = \mathrm{span}\{e_i,\ i \leqq n\}$, P_n the projection operator to E_n. We construct the finite-dimensional approximation for equation (2.1) in the following way:

$$X_n(t) = P_n m(t) + \int_0^t P_n a(t,s,X_n(s))\,ds + \int_0^t P_n b(t,s,X_n(s))P_n\,dW(s). \tag{2.3}$$

We prove first the convergence $\mathbf{E}\left\|X_n(t)-X(t)\right\|^2 \to 0$, $n\to\infty$, in a more general case. Let X_n be solution of the equation

$$X_n(t) = m_n(t) + \int_0^t a_n(t,s,X_n(s))\,ds + \int_0^t b_n(t,s,X_n(s))\,dW(s),$$

where m_n is mean-square continuous adapted process and the coefficients a_n, b_n satisfy

$$\begin{gathered}\|a_n(t,s,x)\| + \|b_n(t,s,x)\|_{\mathcal{L}_2} \leqq C(1+\|x\|),\\ \|a_n(t,s,x)-a_n(t,s,y)\| + \|b_n(t,s,x)-b_n(t,s,y)\|_{\mathcal{L}_2} \leqq C\,\|x-y\|\end{gathered} \tag{2.4a}$$

also let there exist such an increasing function $h(t)$, $t>0$, that $h(t)\to 0$ as $t\to 0$ and

$$\begin{gathered}\|a_n(t,u,x)-a_n(s,u,x)\| + \|b_n(t,u,x)-b_n(s,u,x)\| \leqq h(t-s)(1+\|x\|),\\ \mathbf{E}\left\|m_n(t)-m_n(s)\right\|^2 \leqq h^2(t-s).\end{gathered} \tag{2.4b}$$

Further, assume that for all $t,s\in[0,T]$ and $x\in X$ as $n\to\infty$

$$\|a(t,s,x)-a_n(t,s,x)\| + \|b(t,s,x)-b_n(t,s,x)\|_{\mathcal{L}_2} + \mathbf{E}\left\|m(t)-m_n(t)\right\|^2 \longrightarrow 0. \tag{2.5}$$

Theorem 2.3. *Under assumptions* (2.2), (2.4), (2.5) *the following uniform on* $[0, T]$ *convergence holds*

$$\mathbf{E}\,\|X(t) - X_n(t)\|^2 \longrightarrow 0, \qquad n \to \infty.$$

Proof. As before, it can be easily shown that $\mathbf{E}\|X(t)\|^2$, $\mathbf{E}\|X_n(t)\|^2$ are bounded in n and t. Without loss of generality we will assume that

$$\mathbf{E}\|m(t) - m(s)\|^2 \leqq h^2(t-s).$$

This immediately implies

$$\mathbf{E}\,\|X_n(t) - X_n(s)\|^2 \leqq Ch_0(t-s), \qquad \mathbf{E}\,\|X(t) - X(s)\|^2 \leqq Ch_0(t-s), \tag{2.6}$$

with $h_0(t) = \max\left\{h^2(t), t\right\}$. For positive integer N put $\delta = T/N$, take uniform partition $\tau_k = k\delta$ of the segment $[0, T]$ and consider the processes

$$\begin{aligned}
X^\delta(t) &= m(t) + \int_0^t a(t, s, X^\delta(\tau_{n_s}))\,ds + \int_0^t b(t, s, X^\delta(\tau_{n_s}))\,dW(s), \\
X_n^\delta(t) &= m_n(t) + \int_0^t a_n(t, s, X_n^\delta(\tau_{n_s}))\,ds + \int_0^t b_n(t, s, X_n^\delta(\tau_{n_s}))\,dW(s),
\end{aligned}$$

where, as before, $n_s = \max\{n\colon\ \tau_n < s\}$. We have

$$Z_n^\delta(t) := \mathbf{E}\left\|X_n^\delta(t) - X_n(t)\right\|^2 \leqq C(A_1 + A_2),$$

$$\begin{aligned}
A_1 &= \mathbf{E}\int_0^t \Bigg(\Big\|a_n\big(t, s, X_n(\tau_{n_s})\big) - a_n\big(t, s, X_n(s)\big)\Big\|^2 \\
&\qquad + \Big\|b_n(t, s, X_n(\tau_{n_s})) - b_n(t, s, X_n(s))\Big\|_{\mathcal{L}_2}^2\Bigg)\, ds \\
&\leqq C\int_0^t \mathbf{E}\big\|X_n(\tau_{n_s}) - X_n(s)\big\|^2\, ds \leqq Ch_0(\delta), \\
A_2 &= \mathbf{E}\int_0^t \Bigg(\Big\|a_n\big(t, s, X_n(\tau_{n_s})\big) - a_n\big(t, s, X_n^\delta(\tau_{n_s})\big)\Big\|^2 \\
&\qquad + \Big\|b_n\big(t, s, X_n(\tau_{n_s})\big) - b_n\big(t, s, X_n^\delta(\tau_{n_s})\big)\Big\|_{\mathcal{L}_2}^2\Bigg)\, ds \\
&\leqq C\int_0^t \mathbf{E}\left\|X_n(\tau_{n_s}) - X_n^\delta(\tau_{n_s})\right\|^2\, ds = C\int_0^t Z_n^\delta(\tau_{n_s})\, ds.
\end{aligned}$$

Consequently,

$$Z_n^\delta(t) \leqq C\Bigg(h_0(\delta) + \int_0^t Z_n^\delta(\tau_{n_s})\, ds\Bigg),$$

whence with the use of Gronwall's lemma we obtain $Z_n^\delta(t) \leqq Ch_0(\delta)$. Note that the constant here depends only on constants from (2.4a), thus it is independent of n, δ. Analogously we get $\mathbf{E}\|X(t) - X^\delta(t)\|^2 \leqq Ch_0(\delta)$. Further,

$$\mathbf{E}\left\|X_n^\delta(\tau_{k+1}) - X^\delta(\tau_{k+1})\right\|^2 \leqq C\mathbf{E}(\|m(t) - m_n(t)\|^2 + B_1 + B_2),$$

$$\begin{aligned}
B_1 &= \mathbf{E}\int_0^{\tau_{k+1}} \Big(\Big\|a\big(t, s, X^\delta(\tau_{n_s})\big) - a_n\big(t, s, X^\delta(\tau_{n_s})\big)\Big\|^2 \\
&\qquad + \Big\|\big(t, s, X(\tau_{n_s})\big) - b_n\big(t, s, X(\tau_{n_s})\big)\Big\|_{\mathcal{L}_2}^2\Big)\, ds, \\
B_2 &= \mathbf{E}\int_0^{\tau_{k+1}} \Big(\Big\|a_n\big(t, s, X^\delta(\tau_{n_s})\big) - a_n\big(t, s, X_n^\delta(\tau_{n_s})\big)\Big\|^2 \\
&\qquad + \Big\|b_n\big(t, s, X^\delta(\tau_{n_s})\big) - b_n\big(t, s, X_n^\delta(\tau_{n_s})\big)\Big\|_{\mathcal{L}_2}^2\Big)\, ds \\
&\leqq C\sum_{i=0}^{k} \mathbf{E}\left\|X^\delta(\tau_i) - X_n^\delta(\tau_i)\right\|^2.
\end{aligned}$$

The term B_1 vanishes as $n \to \infty$ by the dominated convergence theorem (integrable dominant is

$C(1+\|X(\tau_{n_s})\|^2)$). Convergence $B_2 \to 0$ as $n \to \infty$ can be proved by induction in k. Hence, we obtain

$$\limsup_{n\to\infty} \mathbf{E}\,\|X(\tau_k) - X_n(\tau_k)\|^2 \leqq Ch_0(\delta).$$

Mean-square continuity of $X(t)$ and $X_n(t)$ (the estimates (2.6)) imply $\limsup_{n\to\infty} \mathbf{E}\,\|X(t) - X_n(t)\|^2 \leqq Ch_0(\delta)$. Here left-hand side does not depend on δ, while the constant in the right-hand side does not depend on t, thus, passing to limit as $\delta \longrightarrow 0$, we get the desirable result. Theorem 2.1 is proved.

Remark 2.5. Note that in this case standard for this paper argument with the use of the Gronwall lemma does not work. The point is that by use of the Gronwall lemma we aim at getting an estimate like $\mathbf{E}\|X_n(t) - X(t)\|^2 \leqq \sigma_n$, where σ_n is certain vanishing sequence, which depends on initial condition and coefficients ("convergence rate"). But, unfortunately, this dependence is very unclear even if the coefficients depend linearly on x.

Corollary 1. Assume that the conditions (2.2) hold. Then the finite-dimensional approximations $X_n(t)$given by (2.3) converge to the solution $X(t)$ of equation (2.1) in mean-square sense, i.e., uniformly in $t \in [0,T]$

$$\mathbf{E}\big\|X(t) - X_n(t)\big\|^2 \longrightarrow 0, \qquad n \to \infty.$$

Proof. Trivial estimates $\|P_n a\| \leqq \|a\|$ and $\|P_n b P_n\|_{\mathcal{L}_2} \leqq \|b\|_{\mathcal{L}_2}$ imply that the assumptions (2.4) on the coefficients of equation (2.3) are fulfilled. Conditions (2.5) hold evidently. Therefore, we can apply Theorem 2.3. Corollary 2.1 is proved.

Consider particular case, when he coefficients $a,\ b$ are independent of t.

Corollary 2. If the coefficients of the equations

$$\begin{aligned} X(t) &= X_0 + \int_0^t a(s, X(s))\,ds + \int_0^t b(s, X(s))\,dW(s), \\ X_n(t) &= X_0^n + \int_0^t a_n(s, X_n(s))\,ds + \int_0^t b_n(s, X_n(s))\,dW(s) \end{aligned}$$

satisfy Lipschitz and linear growth conditions (2.10a) with common constant and if for $s \in [0, T]$, $x \in X$

$$\|a_n(s,x) - a(s,x)\| + \|b_n(s,x) - b(s,x)\|_{\mathcal{L}_2} + \mathbf{E}\,\|X_0^n - X_0\|^2 \longrightarrow 0, \qquad n \to \infty,$$

then the $\big($uniform in $t \in [0, T]\big)$ convergence takes place

$$\mathbf{E}\,\|X_n(t) - X(t)\|^2 \longrightarrow 0, \qquad n \to \infty.$$

In particular, finite-dimensional approximations

$$Y_n(t) = P_n X_0 + \int_0^t P_n a(s, Y_n(s))\, ds + \int_0^t P_n b(s, Y_n(s))\, P_n\, dW(s)$$

converge in mean-square sense to $X(t)$.

Now consider equation (2.45) and assume that conditions (A)–(C) of subsection 1.2 hold. Assume also that $E_n \subset D(A)$. (This holds, e.g., if A is a differential operator and E_n is set of polynomials.) Then the operator $A_n = P_n A P_n$ (being bounded) also generates strongly continuous semigroup. We define the finite-dimensional approximations of equation (2.45) as solutions of the equations

$$X_n(t) = P_n X_0 + \int_0^t \big(P_n A X(s) + P_n a(s, X(s))\big)\, ds + \int_0^t P_n b(s, X(s))\, P_n dW(s),$$

or

$$X_n(t) = U_n(t)\, X_0 + \int_0^t U_n(t-s) \Big(P_n a(s, X_n(s))\, ds + P_n b(s, X_n(s))\, P_n\, dW(s)\Big), \quad (2.7)$$

where $U_n(t) = e^{A_n t}$. It is clear that due to boundedness of A_n the conditions (A)–(C), which guarantee existence and uniqueness of solution, also hold for this equation. The verification of conditions (2.2) and (2.4a) is of no difficulty. First of the conditions of (2.4b) can be rewritten in the following way:

$$\left\| \int_s^t U_n(v-u) A_n P_n a(u,x)\, dv \right\| + \left\| \int_s^t U_n(v-u)\, A_n P_n b(u,x)\, P_n\, dv \right\|_{\mathcal{L}_2}$$

$$\leqq h(t-s)(1+\|x\|).$$

This is true, for instance, in the case when integrands are of linear growth, that is, when

$$\|U_n(v-u)\, A_n P_n a(u,x)\| + \|U_n(v-u)\, A_n P_n b(u,x)\, P_n\|_{\mathcal{L}_2} \leqq C_n(1+\|x\|).$$

If $AP_n = P_n A$ (e.g., if $\{e_n\}$ are eigenvectors of the operator A), then there are no problems, as the left-hand side expression is equal to

$$\|P_n U(v-u)\, Aa(u,x)\| + \|P_n U(v-u)\, Ab(u,x)\, P_n\|_{\mathcal{L}_2}$$

and can be estimated from above by $C(\|Aa(u,x)\| + \|Ab(u,x)\|_{\mathcal{L}_2})$.

Another way to construct finite-dimensional approximations of (2.45) is as follows. If the operator A is continuous, then by Corollary 2 of Theorem 2.3 no further assumptions are needed for convergence. Thus, if we can construct approximations of the solution of equation (2.45) with unbounded operator by solutions of equations with bounded operators in their right-hand side, then we can construct finite-dimensional approximations for (2.45). The next section is devoted to this problem.

2.3 Approximation by solutions of SDE with bounded coefficients

We will consider approximations of solutions of equation (2.45) by solutions of equations with bounded coefficients. Assume that the coefficients of equation (2.45) satisfy conditions (A)–(C). For $h > 0$ put $A_h = h^{-1}(U(h) - I) \in \mathcal{L}$ and consider the equation

$$X^{(h)}(t) = X_0 + \int_0^t \Big(A_h X^{(h)}(s) + a\big(s, X^{(h)}(s)\big)\Big)\, ds + \int_0^t b(s, X^{(h)}(s))\, dW(s). \quad (2.8)$$

Due to the Lipschitz continuity and linear growth of $a,\ b$ and boundedness of A, there exists unique solution to this equation, which is also a mild solution, i.e., a solution to the equation

$$X^{(h)}(t) = U^{(h)}(t)X_0 + \int_0^t U^{(h)}(t-s)\Big(a(s,\ X^{(h)}(s))\, ds + b(s,\ X^{(h)}(s))\, dW(s)\Big),$$

where $U^{(h)}(t) = e^{A_h t}$. Assume further that $\mathbf{E}\|AX_0\|^2 < \infty$. As it was already mentioned, this implies $\sup_{t\in[0,T]} \|AX(t)\|^2 < \infty$. The following theorem is true.

Theorem 2.4. *If the coefficients of equation* (2.45) *satisfy conditions* (A)–(C), *then the approximations* $X^{(h)}$ *converge to the solution* $X(t)$ *of this equation*, *moreover*

$$\mathbf{E}\left\|X(t) - X^{(h)}(t)\right\|^2 \leqq Ch^{2/3}.$$

Proof. It is known (see [Butzer and Berens (1967)]) that

$$\left\|U(t)\,x - U^{(h)}(t)\,x\right\| \leqq \omega_T(h^{1/3}, x) + Ch^{1/3}\,\|x\| \tag{2.9}$$

with constant independent of h. Here ω_T is the modulus of continuity of the semigroup $U(t)$:

$$\omega_T(\varepsilon, x) = \sup\left\{\,\|U(t)\,x - U(s)\,x\|\,,\ 0 \leqq s \leqq t \leqq T,\ |t-s| < \varepsilon\right\} \leqq C\varepsilon\,\|Ax\|\,.$$

The inequality (2.9) implies in particular that the norms $\|U^{(h)}(t)\|$ are bounded uniformly in h and $t \in [0,T]$. Now estimate $Z^{(h)}(t)$:

$$Z^{(h)}(t) = \mathbf{E}\big\|X(t) - X^{(h)}(t)\big\|^2 \leqq C\big(D_1 + D_2 + B_1 + B_2\big),$$

where

$$
\begin{aligned}
D_1 &= \mathbf{E}\left\|\left(U(t)-U^{(h)}(t)\right)X_0\right\|^2 \leqq C\mathbf{E}(\omega_T^2(h^{1/3},X_0)+h^{2/3}\left\|X_0\right\|^2) \\
&\leqq Ch^{2/3}(\mathbf{E}\left\|AX_0\right\|^2+\mathbf{E}\left\|X_0\right\|^2) \leqq Ch^{2/3}, \\
D_2 &= \mathbf{E}\Bigg\|\int_0^t U^{(h)}(t-s)\Big(\big(a(s,X(s))-a(s,X^{(h)}(s))\big)\,ds \\
&\qquad\qquad + \big(b(s,X(s))-b(s,X^{(h)}(s))\big)\,dW(s)\Big)\Bigg\|^2 \\
&\leqq C\mathbf{E}\int_0^t \Bigg(\left\|a(s,X(s))-a(s,X^{(h)}(s))\right\|^2 \\
&\qquad\qquad + \left\|b(s,X(s))-b(s,X^{(h)}(s))\right\|_{\mathcal{L}_2}^2\Bigg)\,ds \\
&\leqq C\int_0^t \mathbf{E}\left\|X(s)-X^{(h)}(s)\right\|^2\,ds = C\int_0^t Z^{(h)}(s)\,ds, \\
B_1 &= \mathbf{E}\left\|\int_0^t (U(t-s)-U^{(h)}(t-s))\,a(s,X(s))\,ds\right\|^2 \\
&\leqq \mathbf{E}\int_0^t Ch^{2/3}\Big(\left\|a(s,X(s))\right\|^2+\left\|Aa(s,X(s))\right\|^2\Big)\,ds \\
&\leqq Ch^{2/3}\int_0^t \mathbf{E}(1+\left\|X(s)\right\|^2)\,ds \leqq Ch^{2/3},
\end{aligned}
$$

$$
\begin{aligned}
B_2 &= \mathbf{E}\left\| \int_0^t (U(t-s) - U^{(h)}(t-s))\, b(s, X(s))\, dW(s) \right\|^2 \\
&= \mathbf{E} \int_0^t \left\| (U(t-s) - U^{(h)}(t-s))\, b(s, X(s)) \right\|_{\mathcal{L}_2}^2 \, ds \\
&= \mathbf{E} \int_0^t \sum_{n=1}^{\infty} \left\| (U(t-s) - U^{(h)}(t-s))\, b(s, X(s))\, e_n \right\|^2 \, ds \\
&\leqq C\mathbf{E} \int_0^t \sum_{n=1}^{\infty} h^{2/3} \Big(\|b(s, X(s))\, e_n\|^2 + \|Ab(s, X(s))\, e_n\|^2 \Big)\, ds \\
&= Ch^{2/3} \mathbf{E} \int_0^t \Big(\|b(s, X(s))\|_{\mathcal{L}_2}^2 + \|Ab(s, X(s))\|_{\mathcal{L}_2}^2 \Big)\, ds \\
&\leqq Ch^{2/3} \int_0^t \mathbf{E}(1 + \|X(s)\|^2)\, ds \leqq Ch^{2/3}.
\end{aligned}
$$

Thus, we got the estimate

$$
Z^{(h)}(t) \leqq C\left(h^{2/3} + \int_0^t Z^{(h)}(s)\, ds \right),
$$

whence with the use of Gronwall's lemma the statement of the theorem follows.

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