A Semigroup Approach for Weak Approximations with an Application to Infinite Activity Lévy Driven SDEs

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Abstract

Weak approximations have been developed to calculate the expectation value of functionals of stochastic differential equations, and various numerical discretization schemes (Euler, Milshtein) have been studied by many authors. Nevertheless high order schemes were not available in general. We present a decomposition method applicable to jump driven SDE's.

Setting & Goals

Ideas from semigroup operators

The algebraic structure

First example: Coordinate processes

General framework

Weak approximation result

Combination of "coordinates"

Examples: Diffusion, Levy driven SDE (Example: Tempered stable case)

Setting

$$X_t(x) = x + \int_0^t \tilde{V}_0(X_{s-}(x))ds + \int_0^t V(X_{s-}(x))dB_s + \int_0^t h(X_{s-}(x))dY_s. \tag{1}$$

with C_b^∞ coefficients
$\tilde{V}_0 : \mathbf{R}^N \to \mathbf{R}^N$, $V = (V_1, \dots, V_d)$, $h : \mathbf{R}^N \to \mathbf{R}^N \otimes \mathbf{R}^d$
B_t is a d-dim. BM and Y_t is an d-dim. Lévy with triplet $(b, 0, \nu)$ satisfying the condition

$$\int_{\mathbf{R}_0^d} (1 \wedge |y|^p)\nu(dy) < \infty.$$

for any $p \in \mathbb{N}$.

Goal

Our purpose is to find discretization schemes $(X_t^{(n)}(x))_{t=0,T/n,\dots,T}$ for given $T > 0$ such that

$$|E[f(X_T(x))] - E[f(X_T^{(n)}(x))]| \leq \frac{C(T, f, x)}{n^m}.$$

Remarks

1. Proof through a "semigroup type approach"
2. Proof for Asmussen-Rosinski type approach
3. Limiting the number of jumps (why?)
4. Ideas come from Kusuoka type approximations

Set-up for the proof of weak approximation

Define:

$$P_t f(x) = E[f(X_t(x))]$$

$Q_t \equiv Q_t^n$: operator such that the semigroup property is satisfied in $\{kT/n; k = 0, \ldots, n\}$.
$Q_t \approx P_t$ in the sense that $(P_t - Q_t)f(x) = \mathcal{O}(t^{m+1})$. Then the idea of the proof is

$$P_T f(x) - (Q_{T/n})^n f(x) = \sum_{k=0}^{n-1} (Q_{T/n})^k (P_{T/n} - Q_{T/n}) P_{T - \frac{k+1}{n} T} f(x).$$

Simulation (stochastic characterization): Let $M = M_t(x)$ s.t. $Q_t f(x) = E[f(M_t(x))]$. Then

$$Q_T f(x) = (Q_{T/n})^{n} f(x) = E[f(M^1_{T/n} \circ \cdots \circ M^n_{T/n}(x))]$$

Euler-Maruyama scheme:
$M_t(x) := x + \tilde{V}_0(x)t + V(x)B_t + h(x)Y_t$

The algebraic structure

$$P_t = e^{tL} = \sum_{k=0}^{m} \frac{t^k}{k!} L^k + \mathcal{O}(t^{m+1})$$

Note that $L = \sum_{i=1}^{d+1} L_i$.

$$e^{tL_i} = \sum_{k=0}^{m} \frac{t^k}{k!} L_i^k + \mathcal{O}(t^{m+1})$$

Goal: Approximate e^{tL}, through a combination of L_i's s.t.

$$e^{tL} - \sum_{j=1}^{k} \xi_j e^{t_{1,j} A_{1,j}} \cdots e^{t_{\ell_j,j} A_{\ell_j,j}} = \mathcal{O}(t^{m+1})$$

with some $t_{i,j} > 0$, $A_{i,j} \in \{L_0, L_1, \ldots, L_{d+1}\}$ and weights $\{\xi_j\} \subset [0,1]$ with $\sum_{j=1}^{k} \xi_j = 1$. This will correspond to an m-th order discretization scheme.

First example: Coordinate processes

Define the coordinate processes $X_{i,t}(x)$, $i = 0, ..., d+1$, solutions of

$$X_{0,t}(x) = x + \int_0^t V_0(X_{0,s}(x))ds$$
$$X_{i,t}(x) = x + \int_0^t V_i(X_{i,s}(x)) \circ dB_s^i \quad 1 \leq i \leq d$$
$$X_{d+1,t}(x) = x + \int_0^t h(X_{d+1,s-}(x))dY_s.$$

Define

$$Q_{i,t}f(x) := E[f(X_{i,t}(x))]$$

whose generators are

$$L_0 f(x) := (V_0 f)(x), \quad L_i f(x) := \frac{1}{2}(V_i^2 f)(x), \ 1 \leq i \leq d$$
$$L_{d+1} f(x) := \nabla f(x) h(x) b + \int (f(x + h(x)y) - f(x) - \nabla f(x) h(x) \tau(y)) \nu(dy)$$

How does the algebraic argument work?

For simplicity let $d + 1 = 2$ then

$$
\begin{aligned}
e^{tL} &= I + tL + \frac{t^2}{2}L^2 + O(t^3) \\
e^{\frac{t}{2}L_1}e^{\frac{t}{2}L_2} &\approx (I + tL_1 + \frac{t^2}{2}L_1^2 + \dots)(I + tL_2 + \frac{t^2}{2}L_2^2 + \dots) \\
&= I + tL + \frac{t^2}{2}\left(L_2^2 + L_1^2 + L_1L_2\right) + O(t^3)
\end{aligned}
$$

then

$$
\begin{aligned}
e^{tL} - e^{\frac{t}{2}L_1}e^{\frac{t}{2}L_2} &= O(t^2) \\
e^{tL} - \frac{1}{2}e^{\frac{t}{2}L_1}e^{\frac{t}{2}L_2} - \frac{1}{2}e^{\frac{t}{2}L_2}e^{\frac{t}{2}L_1} &= O(t^3)
\end{aligned}
$$

finally one needs to obtain a stochastic representation for $\frac{1}{2}e^{\frac{t}{2}L_1}e^{\frac{t}{2}L_2} + \frac{1}{2}e^{\frac{t}{2}L_2}e^{\frac{t}{2}L_1}$.

Examples of schemes:
Ninomiya-Victoir (a):

$$\frac{1}{2}e^{\frac{t}{2}L_0}e^{tL_1}\cdots e^{tL_{d+1}}e^{\frac{t}{2}L_0}+\frac{1}{2}e^{\frac{t}{2}L_0}e^{tL_{d+1}}\cdots e^{tL_1}e^{\frac{t}{2}L_0}$$

Ninomiya-Victoir (b):

$$\frac{1}{2}e^{tL_0}e^{tL_1}\cdots e^{tL_{d+1}}+\frac{1}{2}e^{tL_{d+1}}\cdots e^{tL_1}e^{tL_0}$$

Splitting method:

$$e^{\frac{t}{2}L_0}\cdots e^{\frac{t}{2}L_d}e^{tL_{d+1}}e^{\frac{t}{2}L_d}\cdots e^{\frac{t}{2}L_0}$$

So the idea is

$$P_t f = e^{tL}f \approx \sum_{j=1}^{k}\xi_j e^{t_{1,j}A_{1,j}}\cdots e^{t_{\ell_j,j}A_{\ell_j,j}}f$$

$$\approx \sum_{j=1}^{k}\xi_j E\left[f\left(M_1(t_{1,j},M_2(t_{2,j},(....,M_l(t_{l,j},\cdot))...))\right)\right]$$

General framework

Assumptions $\mathcal{M}$

- If $f \in C_p$ with $p \geq 2$, then $Q_t f \in C_p$ and

$$\sup_{t\in[0,T]} \|Q_t f\|_{C_p} \leq K\|f\|_{C_p}$$

for $K > 0$ independent of n. Futhermore, we assume $0 \leq Q_t f(x) \leq Q_t g(x)$ whenever $0 \leq f \leq g$.

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- For $f_p(x) := |x|^{2p}$ ($p \in \mathbf{N}$),

$$Q_t f_p(x) \leq (1 + Kt) f_p(x) + K' t$$

  for $K = K(T, p)$, $K' = K'(T, p) > 0$.

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  for $K = K(T, p)$, $K' = K'(T, p) > 0$.
- For $m \in \mathbf{N}$, $\delta_m : [0, T] \to \mathbf{R}_+$ denotes a decreasing function s.t.

$$\limsup_{t \to 0+} \frac{\delta_m(t)}{t^{m-1}} = 0.$$

  Usually, we have $\delta_m(t) = t^m$.

Main hypothesis $\mathcal{R}(m, \delta_m)$

For $p \geq 2$, there exists $q = q(m,p) \geq p$ and linear operators $e_k : C_p^{2k} \to C_{p+2k}$ $(k = 0, 1, \dots, m)$ s.t.

(A): For every $f \in C_p^{2(m'+1)}$ with $1 \leq m' \leq m$, the operator Q_t satisfies

$$Q_t f(x) = \sum_{k=0}^{m'} (e_k f)(x) t^k + (\mathrm{Err}_t^{(m')} f)(x), \ \ t \in [0, T], \qquad (2)$$

where $\mathrm{Err}_t^{(m')} f \in C_q$, and satisfies the following condition:

(B): If $f \in C_p^{m''}$ with $m'' \geq 2k$, then $e_k f \in C_{p+2k}^{m''-2k}$ and there exists a constant constant $K = K(T, m) > 0$ such that

$$\|e_k f\|_{C_{p+2k}^{m''-2k}} \leq K \|f\|_{C_p^{m''}} \ \ k = 0, 1, \dots, m. \qquad (3)$$

Furthermore if $f \in C_p^{m''}$ with $m'' \geq 2m' + 2$,

$$\|\mathrm{Err}_t^{(m')} f\|_{C_q} \leq \begin{cases} K t^{m'+1} \|f\|_{C_p^{m''}} & \text{if } m' < m \\ K t \delta_m(t) \|f\|_{C_p^{m''}} & \text{if } m' = m \end{cases}$$

for all $0 \leq t \leq T$.

Weak approximation result

(C): For every $0 \leq k \leq m$ and $j \geq 2k+2$, if $f \in C_p^{1,j}([0,T] \times \mathbf{R}^N)$, then $e_k f \in C_{p+2k}^{1,j-2k}([0,T] \times \mathbf{R}^N)$.
Define $J_{\leq m}(Q_t) = \sum_{k=0}^{m} (e_k f)(x) t^k$

Theorem

Assume $(\mathcal{M})$ and $\mathcal{R}(m, \delta_m)$ for P_t and Q_t with $J_{\leq m}(P_t - Q_t) = 0$. Then for any $f \in C_p^{2(m+1)}$, there exists a constant $K = K(T,x) > 0$ such that

$$\left| P_T f(x) - (Q_{T/n})^n f(x) \right| \leq K \delta_m \left(\frac{T}{n} \right) \| f \|_{C_p^{2(m+1)}}. \tag{4}$$

Theorem

Assume $(\mathcal{M})$ and $\mathcal{R}(m+1, \delta_{m+1})$ for Q_t with $J_{\leq m}(P_t - Q_t) = 0$. Then for each $f \in C_p^{2(m+3)}$, we have

$$P_T f(x) - (Q_{T/n})^n f(x) = \frac{K}{n^m} + \mathcal{O}\left(\left(\frac{T}{n} \right)^{m+1} \vee \delta_{m+1}\left(\frac{T}{n} \right) \right) \tag{5}$$

Properties for algebraic construction

Lemma

Let $Q_t^{Y^1}$ and $Q_t^{Y^2}$ associated with independent processes Y_t^1, Y_t^2 and let $Q_t^{Y^1} Q_t^{Y^2}$ be the composite operator associated with the process $(Y^2 \circ Y^1)_t(x) = Y_t^2(Y_t^1(x))$. Then
(i) If $(\mathcal{M})$ holds for $Q_t^{Y^1}$, $Q_t^{Y^2}$, then it also holds for $Q_t^{Y^1} Q_t^{Y^2}$.
(ii) If $\mathcal{R}(m, \delta_m)$ holds for $Q_t^{Y^1}$, $Q_t^{Y^2}$, then it also holds for $Q_t^{Y^1} Q_t^{Y^2}$.

Combination of "coordinates"

Theorem

Assume $(\mathcal{M})$ and $\mathcal{R}(2,\delta_2)$ are satisfied for $Q_t^{\bar{X}_i}$ $(i = 0, 1, \dots, d+1)$ associated with indep. processes $\bar{X}_0, \dots, \bar{X}_{d+1}$ with $J_{\le 2}(Q_{i,t} - Q_t^{\bar{X}_i}) = 0$. Then all the following operators satisfy $(\mathcal{M})$ and $\mathcal{R}(2,\delta_2)$:

N-V(a) $Q_t^{(a)} = \frac{1}{2} Q_{t/2}^{\bar{X}_0} \prod_{i=1}^{d+1} Q_t^{\bar{X}_i} Q_{t/2}^{\bar{X}_0} + \frac{1}{2} Q_{t/2}^{\bar{X}_0} \prod_{i=1}^{d+1} Q_t^{\bar{X}_{d+2-i}} Q_{t/2}^{\bar{X}_0}$

N-V(b) $Q_t^{(b)} = \frac{1}{2} \prod_{i=0}^{d+1} Q_t^{\bar{X}_i} + \frac{1}{2} \prod_{i=0}^{d+1} Q_t^{\bar{X}_{d+1-i}}$

Splitting $Q_t^{(sp)} = Q_{t/2}^{\bar{X}_0} \cdots Q_{t/2}^{\bar{X}_d} Q_t^{\bar{X}_{d+1}} Q_{t/2}^{\bar{X}'_d} \cdots Q_{t/2}^{\bar{X}'_0}$

where $(\bar{X}'_0, \dots, \bar{X}'_d)$ is a further indep. copy of $(\bar{X}_0, \dots, \bar{X}_d)$. Moreover, we have $J_{\le 2}(Q_t^{(a)}) = J_{\le 2}(Q_t^{(b)}) = J_{\le 2}(Q_t^{(sp)}) = \sum_{k=0}^{2} \frac{t^k}{k!} L^k$. In particular, the above schemes define a second order approximation scheme.

Theorem

Let $m = 1$ or 2. Assume $(\mathcal{M})$ and $\mathcal{R}(2m, t^{2m})$ for $Q_t^{[i]}$ $(i = 1, \ldots, \ell)$. Furthermore, we assume

(1) $J_{\leq 2m}\left(P_t - \sum_{i=1}^{\ell} \xi_i Q_t^{[i]}\right) = 0$ *for some real numbers* $\{\xi_i\}_{i=1,\ldots,\ell}$ *with* $\sum_{i=1}^{l} \xi_i = 1$

(2) *There exists a constant* $q = q(m,p) > 0$ *such that for every* $f \in C_p^{m'}$ *with* $m' \geq 2(m+1)$, $(P_t - Q_t^{[i]})f \in C_q^{m'-2(m+1)}$ *and*

$$\sup_{t\in[0,T]} \|(P_t - Q_t^{[i]})f\|_{C_p^{m'-2(m+1)}} \leq C_T \|f\|_{C_q^{m'}} t^{m+1}.$$

Then we have for any $f \in C_p^{4(m+1)}$,

$$\left| P_T f(x) - \sum_{i=1}^{\ell} \xi_i (Q_{T/n}^{[i]})^n f(x) \right| \leq \frac{C(T,f,x)}{n^{2m}}.$$

Note that $\sum_{i=1}^{\ell} \xi_i Q_t^{[i]}$ does not satisfy the semigroup property or the monotonic property.

Example

Example: The following modified Ninomiya-Victoir scheme

$$\frac{1}{2}\Big(e^{\frac{T}{2n}L_0}\prod_{i=1}^{d+1}e^{\frac{T}{n}L_i}e^{\frac{T}{2n}L_0}\Big)^n+\frac{1}{2}\Big(e^{\frac{T}{2n}L_0}\prod_{i=1}^{d+1}e^{\frac{T}{n}L_{d+2-i}}e^{\frac{T}{2n}L_0}\Big)^n$$

is also of order 2.

Example

Fujiwara gives a proof of a similar version of the above theorem and some examples of 4th and 6th order. We introduce the examples of 4th order:

$$\frac{4}{3}\left(\frac{1}{2}\Big(\prod_{i=0}^{d+1}e^{\frac{t}{2}L_i}\Big)^2+\frac{1}{2}\Big(\prod_{i=0}^{d+1}e^{\frac{t}{2}L_{d+1-i}}\Big)^2\right)-\frac{1}{3}\left(\frac{1}{2}\prod_{i=0}^{d+1}e^{tL_i}+\frac{1}{2}\prod_{i=0}^{d+1}e^{tL_{d+1-i}}\right)$$

Example (Diffusion coordinate)

Theorem

Let $V : \mathbf{R}^N \to \mathbf{R}^N \in C_b^\infty$*. The exponential map is defined as* $\exp(V)x = z_1(x)$ *where* z *satisfies the ode*

$$\frac{dz_t(x)}{dt} = V(z_t(x)),\ z_0(x) = x. \tag{6}$$

Lemma

For $i = 0, 1, ..., d$*, the sde*

$$X_{i,t}(x) = x + \int_0^t V_i(X_{i,s}(x)) \circ dB_s^i \tag{7}$$

has a unique solution given by

$$X_{i,t}(x) = \exp(B_t^i V_i)x.$$

Proposition

Let $f \in C_p^{m+1}$. Then we have for $i = 0, 1, \ldots, d$,

$$f\left(\exp(tV_i)x\right) = \sum_{k=0}^{m} \frac{t^k}{k!} V_i^k f(x) + \int_0^t \frac{(t-u)^m}{m!} V_i^{m+1} f(\exp(uV_i)x) du$$

$$\left| \int_0^t \frac{(t-u)^m}{m!} V_i^{m+1} f(\exp(uV_i)x) du \right| \leq C_m \|f\|_{C_p^{m+1}} e^{K|t|} (1 + |x|^{p+m+1}) t^{m+1}.$$

Based on this result, we define the approximation to the solution of the coordinate equation as follows

$$b_m^j(t, V)x = \sum_{k=0}^{m} \frac{t^k}{k!} (V^k e_j)(x), \ j = 1, ..., N.$$

Define $\bar{X}_{i,t}(x) = b_{2m+1}(B_t^i, V_i)x$ for $i = 0, ..., d$. Then we have the following approximation result.

Proposition

(i) For every $p \geq 1$,

$$\|X_{i,t}(x) - \bar{X}_{i,t}(x)\|_{L^p} \leq C(p, m, T)(1 + |x|^{2(m+1)})t^{m+1}.$$

(ii) Let $f \in C_p^1$. Then we have

$$E[|f(X_{i,t}(x)) - f(\bar{X}_{i,t}(x))|] \leq C(m, T)\|f\|_{C_p^1}(1 + |x|^{p+2(m+1)})t^{m+1}.$$

As a result of this proposition we can see that $\mathcal{R}(m, t^m)$ holds for the operators associated with $b_m(t, V_0)x$ and $b_{2m+1}(B_t^i, V_i)x$, $1 \leq i \leq d$. Indeed, we have for $m' \leq m$,

$$\begin{aligned} E[f(\bar{X}_{i,t}(x))] &= Q_{i,t}f(x) + E[f(\bar{X}_{i,t}(x)) - f(X_{i,t}(x))] \\ &= \sum_{k=0}^{m'} \frac{t^k}{k!} L_i^k f(x) + (E_t^{m'} f)(x) \end{aligned}$$

where $(E_t^{m'} f)(x)$

where

$$(E_t^{m'} f)(x) := (\mathrm{Err}_t^{(m')} f)(x) + E[f(\bar{X}_{i,t}(x)) - f(X_{i,t}(x))]$$

and $(\mathrm{Err}_t^{(m')} f)(x)$ is defined through a previous proposition using L_i and Q_i instead of L and P. Furthermore, using (ii), we have that the error term $E_t^{m'}$ satisfies (B) in assumption $\mathcal{R}(m, t^m)$.

where

$$(E_t^{m'} f)(x) := (\mathrm{Err}_t^{(m')} f)(x) + E[f(\bar{X}_{i,t}(x)) - f(X_{i,t}(x))]$$

and $(\mathrm{Err}_t^{(m')} f)(x)$ is defined through a previous proposition using L_i and Q_i instead of L and P. Furthermore, using (ii), we have that the error term $E_t^{m'}$ satisfies (B) in assumption $\mathcal{R}(m, t^m)$.

Proposition Assume that $(V_i^k e_j)$ $(2 \leq k \leq m,\ 0 \leq i \leq d,\ 1 \leq j \leq N)$ satisfies the linear growth condition then $(\mathcal{M})$ holds for $\bar{X}_{i,t}(x)$, $i = 0, \ldots, d$.

Theorem

Assume that $(V_i^k e_j)$ $(2 \leq k \leq m,\ 0 \leq i \leq d,\ 1 \leq j \leq N)$ satisfies the linear growth condition. Let $\bar{X}_{i,t}(x)$ be defined by

$$\bar{X}_{i,t}(x) = b_{2m+1}(B_t^i, V_i)x = \sum_{k=0}^{2m+1} \frac{1}{k!}(V_i^k I)(x) \int_{0<t_1<\cdots<t_k<t} 1 \circ dB_{t_1}^i \cdots \circ dB_{t_k}^i .$$

Denote by $Q_t^{\bar{X}_i}$ the semigroup associated with $\bar{X}_{i,t}(x)$. Then $Q_t^{\bar{X}_i}$ satisfies $(\mathcal{M})$ and $\mathcal{R}(m, t^m)$. Furthermore $J_{\leq m}(Q_{i,t} - Q_t^{\bar{X}_i}) = 0$.

Runge-Kutta methods:

We say here that c_m is an s-stage explicit Runge-Kutta method of order m for the ODE (6) if it can be written in the form

$$c_m(t,V)x = x + t\sum_{i=1}^{s}\beta_i k_i(t,V)x$$

where $k_i(t,V)x$ defined inductively by

$$\begin{aligned} k_1(t,V)x &= V(x), \\ k_i(t,V)x &= V\Big(x + t\sum_{j=1}^{i-1}\alpha_{i,j}k_j(t,V)x\Big),\ 2 \le i \le s, \end{aligned}$$

and satisfies

$$|\exp(tV)x - c_m(t,V)x| \le C_m e^{K|t|}(1+|x|^{m+1})|t|^{m+1}$$

for some constants $((\beta_i,\alpha_{i,j})_{1\le j<i\le s})$. Runge-Kutta formulas of order less than or equal to 7 are well known.

Proposition

(i) For every $p \geq 1$,

$$\|X_{i,t}(x) - c_{2m+1}(B^i_t, V_i)x\|_{L^p} \leq C(p,m,T)(1+|x|^{2(m+1)})t^{m+1} \quad (8)$$

(ii) Let $f \in C^1_p$. Then we have

$$E[|f(X_{i,t}(x)) - f(c_{2m+1}(B^i_t, V_i)x)|] \leq C(m,T)\|f\|_{C^1_p}(1+|x|^{2(m+1)})t^{m+1} \quad (9)$$

Next we show that $(\mathcal{M})$ still holds for the Runge-Kutta schemes.
Proposition $(\mathcal{M})$ holds for $c_m(B^i_t, V_i)x$, $i = 0, \ldots, d$.
Consequently, as in the Taylor scheme, $\mathcal{R}(m, t^m)$ and $(\mathcal{M})$ hold for the operators associated with $c_m(t, V_0)x$ and $c_{2m+1}(B^i_t, V_i)x$, $1 \leq i \leq d$.

Example: Compound Poisson

$$Y_t = \sum_{i=1}^{N_t} J_i$$

where (N_t) : Poisson (λ) and (J_i) are i.i.d. $\mathbf{R}^d$-r.v. indep. of (N_t) with $J_i \in \bigcap_{p\geq 1} L^p$.
In this case Y_t is a Lévy process with generator of the form

$$\int_{\mathbf{R}_0^d} (f(x+y) - f(x))\nu(dy)$$

where $\tau \equiv 0$, $b = 0$, $\nu(\mathbf{R}_0^d) = \lambda < \infty$ and $\nu(dy) = \lambda P(J_1 \in dy)$.
Then in this case

$$X_t^{d+1}(x) = x + \int_0^t h(X_{s-}^{d+1}(x))dY_s, \ t \in [0, T] \tag{10}$$

which can be solved explicitly.

Indeed, let $(G_i(x))$ be defined by recursively

$$G_0 = x$$

$$G_i = G_{i-1} + h(G_{i-1})J_i.$$

Then the solution can be written as $X_t^{d+1}(x) = G_{N_t}(x)$. Define for fixed $M \in \mathbf{N}$, the approximation process $\bar{X}_{d+1,t} = G_{N_t \wedge M}(x)$. This approximation requires the simulation of at most M jumps. In fact, the rate of convergence is fast as the following result shows.
Proposition Let $M \in \mathbf{N}$. Then the process $G_{N_t \wedge M}(x)$ satisfies $(\mathcal{M})$ and $\mathcal{R}(M, t^{M-\kappa})$ for arbitrary small $\kappa > 0$. Furthermore $J_{\leq M}(Q_{d+1,t} - Q_t^{\bar{X}_{d+1}}) = 0$.

Remark: Adaptive Weak Approximation of Diffusions with Jumps
E. Mordecki, A. Szepessy, R. Tempone and G. E. Zouraris

Infinite activity approximated by a process with no small jumps

Define for $\varepsilon > 0$ Lévy proc. (Y_t^ε) with Lévy triplet $(b, 0, \nu^\varepsilon)$

$$\nu^\varepsilon(E) := \nu(E \cap \{y : |y| > \varepsilon\}), \; E \in \mathcal{B}(\mathbf{R}_0^d). \tag{11}$$

Consider the approximate coordinate SDE

$$\bar{X}_{d+1,t}(x) = x + \int_0^t h(\bar{X}_{d+1,s-}(x))dY_s^\varepsilon,$$

$$L_{d+1}^{1,\varepsilon} f(x) = \nabla f(x)h(x)b + \int (f(x+h(x)y) - f(x) - \nabla f(x)h(x)\tau(y))\nu^\varepsilon(dy)$$

Now we derive the error estimate for $\bar{X}_{d+1,t}$.

Theorem

Assume that $\sigma^2(\varepsilon) := \int_{|y|\leq\varepsilon} |y|^2 \nu(dy) \leq t^{M+1}$ for $\varepsilon \equiv \varepsilon(t) \in (0,1]$. Then we have that $Q_t^{\bar{X}_{d+1}}$ satisfies $(\mathcal{M})$ and $\mathcal{R}(M, t^M)$. Furthermore $J_{\leq M}(Q_{d+1,t} - Q_t^{\bar{X}_{d+1}}) = 0$.

Asmussen-Rosinski type approximation

Consider the new approximate SDE

$$\bar{X}_{d+1,t}(x) = x + \int_0^t h(\bar{X}_{d+1,s}(x))\Sigma_\varepsilon^{1/2} dW_s + \int_0^t h(\bar{X}_{d+1,s-}(x)) dY_s^\varepsilon$$

where W_t is a new d-dim.BM indep. of B_t and Y_t^ε, and Σ_ε is the symmetric and semi-positive definite $d \times d$ matrix defined as

$$\Sigma_\varepsilon = \int_{|y| \le \varepsilon} yy^* \nu(dy). \tag{12}$$

Since the above SDE is also driven by a jump-diffusion process, we can also simulate it using the second order discretization schemes.

Theorem

Assume that $0 < \varepsilon \equiv \varepsilon(t) \le 1$ *is chosen as to satisfy that* $\int_{|y| \le \varepsilon} |y|^3 \nu(dy) \le t^{M+1}$*. Then we have that* $Q_t^{\bar{X}_{d+1}}$ *satisfies* $(\mathcal{M})$ *and* $\mathcal{R}(M, t^M)$*. Furthermore* $J_{\le M}(Q_{d+1,t} - Q_t^{\bar{X}_{d+1}}) = 0$*.*

Idea of the proof

$$
\begin{aligned}
&Q_{d+1,t}f(x) - Q_t^{\bar{X}_{d+1}}f(x) \\
&= \sum_{k=1}^{M} \frac{t^k}{k!} \left((L_{d+1})^k - \left(L_{d+1}^{1,\varepsilon} \right)^k \right) f(x) \\
&+ \int_0^t \frac{(t-u)^M}{M!} \left(Q_{d+1,u} (L_{d+1})^{M+1} - Q_u^{\bar{X}_{d+1}} \left(L_{d+1}^{1,\varepsilon} \right)^{M+1} \right) f(x) du.
\end{aligned}
$$

It is enough to prove:

$$
|(L_{d+1} - L_{d+1}^{1,\varepsilon}) f(x)| \leq C \|f\|_{C_p^2} (1 + |x|^{p+2}) t^{M+1}.
$$

Change of triplets

$$
\begin{aligned}
(b, 0, \nu), \tau &\Rightarrow (b_\varepsilon, 0, \nu), \tau_\varepsilon \\
(b, 0, \nu^\varepsilon), \tau &\Rightarrow (b_\varepsilon, 0, \nu^\varepsilon), \tau_\varepsilon
\end{aligned}
$$

where $\tau_\varepsilon(y) = y 1_{\{|y| \leq \varepsilon\}}$. Then

$$
\begin{aligned}
&|(L_{d+1} - L^{1,\varepsilon}_{d+1})f(x)| \qquad (13)\\
&\leq \Big| \int \nabla f(x)h(x)(y-\tau_\varepsilon(y))(\nu(dy)-\nu^\varepsilon(dy))\Big| \\
&+ \Big| \int \int_0^1 (1-\theta)\frac{d^2}{d\theta^2} f(x+\theta h(x)y)d\theta(\nu(dy)-\nu^\varepsilon(dy))\Big|.
\end{aligned}
$$

We first obtain that for $\varepsilon > 0$,

$$\int (y-\tau_\varepsilon(y))(\nu(dy)-\nu^\varepsilon(dy)) = 0$$

since the support of the measure $(\nu - \nu^\varepsilon)$ is $\{|y| \leq \varepsilon\}$. Also

$$\Big| \int \int_0^1 \frac{d^2}{d\theta^2} f(x+\theta h(x)y)d\theta(\nu(dy)-\nu^\varepsilon(dy))\Big| \leq C\|f\|_{C^2_p}(1+|x|^{p+2})\sigma^2(\varepsilon)$$

and hence as $\sigma^2(\varepsilon) \leq t^{M+1}$, one obtains that $J_{\leq M}(Q_{d+1,t} - Q_t^{\bar{X}_{d+1}}) = 0$ and that $Q_t^{\bar{X}_{d+1}}$ satisfies $(\mathcal{M})$ and $\mathcal{R}(M, t^M)$.

Example: Other decompositions with at most one jump per interval

$\tau(y) = y1_{|y|<1}$, assume that $\int_{|y|<1} |y|\nu(dy) < \infty$.
Then we decompose the operator

$$L_{d+1} = L^1_{d+1} + L^2_{d+1} + L^3_{d+1}$$

$$L^1_{d+1} f(x) := \nabla f(x) h(x) \left(b - \int_{\varepsilon<|y|\leq 1} \tau(y)\nu(dy) \right)$$

$$L^2_{d+1} f(x) := \int_{|y|\leq\varepsilon} (f(x + h(x)y) - f(x) - \nabla f(x)h(x)\tau(y))\nu(dy)$$

$$L^3_{d+1} f(x) := \int_{\varepsilon<|y|} f(x + h(x)y) - f(x)(dy).$$

The operator L^1_{d+1} can be exactly generated using
$\bar{X}^1_{d+1,t} = x + \left(b - \int_{\varepsilon<|y|\leq 1} \tau(y)\nu(dy) \right) \int_0^t h\left(\bar{X}^1_{d+1,s}\right) ds.$
Therefore we only need to approximate L^2_{d+1} and L^3_{d+1}.

Approximation for L^2_{d+1}. Define the dist. fct. $F_\varepsilon(dy) = \lambda_\varepsilon^{-1} |y|^r 1_{|y|\le\varepsilon}\nu(dy)$ with $\lambda_\varepsilon = \int_{|y|\le\varepsilon} |y|^r \nu(dy) < \infty$. Let $Y_\varepsilon \sim F_\varepsilon$. Define $\bar{X}_t^{2,\varepsilon}(x) = x + h(x)W_t\sqrt{\lambda_\varepsilon}$, where W is a d-dim. BM with cov. matrix given by $\Sigma_{ij} = |Y^\varepsilon|^{-r} Y_i^\varepsilon Y_j^\varepsilon$ which is indep. of everything else.

Lemma

(*)1.Assume that $\int_{|y|\le\varepsilon} |y|^3\nu(dy) \le Ct$ *and* $\sup_{\varepsilon\in(0,1]} \int_{|y|\le\varepsilon} |y|^{4-r}\nu(dy) < \infty$ *then*

$$\left|E\left[f(\bar{X}_t^{2,\varepsilon})\right] - f(x) - tL^2_{d+1}f(x)\right| \le \|f\|_{C_p^2}(1+|x|^{p+2})t^2.$$

That is, condition $\mathcal{R}(2,t^2)$ is satisfied.

2. Assume that $\sup_{\varepsilon\in(0,1]} \int_{|y|\le\varepsilon} |y|^{2+\frac{(2-r)(p-2)}{2}}\nu(dy) < \infty$, *then assumption $(\mathcal{M})$ is satisfied with*

$$E\left[\left|\bar{X}_{d+1}^{2,\varepsilon}(x)\right|^p\right] \le (1+Kt)|x|^p + K't$$

for all $p \ge 2$.

The approximation for L^3_{d+1} is defined as follows. Let $G_\varepsilon(dy) = C_\varepsilon^{-1} 1_{|y|>\varepsilon}\nu(dy)$, $C_\varepsilon = \int_{|y|>\varepsilon}\nu(dy)$ and let $Z^\varepsilon \sim G_\varepsilon$ and let S^ε be a Bernoulli r.v. indep. of Z^ε. If $S^\varepsilon = 0$ define $\bar{X}^{3,\varepsilon}_t(x) = x$, otherwise $\bar{X}^{3,\varepsilon}_t(x) = x + h(x)Z^\varepsilon$.

Lemma

(**)1. Assume that $\left|C_\varepsilon^{-1} P\left[S^\varepsilon = 1\right] - t\right| \leq Ct^2$ *then*

$$\left|E\left[f(\bar{X}^{3,\varepsilon}_t)\right] - f(x) - tL^3_{d+1}f(x)\right| \leq Ct^2 \left\|f\right\|_{C^1_p} (1+|x|^{p+1}) \int_{|y|>\varepsilon} |y|\nu(dy)$$

That is, condition $\mathcal{R}(2, t^2)$ *is satisfied.*

2. If $C_\varepsilon^{-1} P\left[S^\varepsilon = 1\right] \leq Ct$ *then assumption* $(\mathcal{M})$ *is satisfied with*

$$E\left[\left|\bar{X}^{3,\varepsilon}_{d+1}(x)\right|^p\right] \leq (1 + Kt)|x|^p + K't$$

for all $p \geq 2$.

Weighted version l (Importance sampling)

Weight $l : \mathbf{R}^d \to \mathbf{R}$. Let $F^l_\varepsilon(dy) = \lambda_\varepsilon l(y) 1_{|y|\le\varepsilon}\nu(dy)$ with $\lambda_\varepsilon^{-1} = \int_{|y|\le\varepsilon} l(y)\nu(dy)$. Let $Y_\varepsilon \sim F_\varepsilon$. Define $\bar{X}^{2,\varepsilon}_t(x) = x + h(x)W_t\sqrt{\lambda_\varepsilon}$, where W is a d-dim. BM with cov. matrix given by $\Sigma_{ij} = l(Y^\varepsilon)^{-1}Y^\varepsilon_i Y^\varepsilon_j$ which is indep. of everything else.

Lemma

1. Assume that $\int_{|y|\le\varepsilon} |y|^3\nu(dy) \le Ct$ *and* $\sup_{\varepsilon\in(0,1]} \int_{|y|\le\varepsilon} |y|^4 l(y)^{-1}\nu(dy) < \infty$ *then*

$$\left| E\left[f(\bar{X}^{2,\varepsilon}_t)\right] - f(x) - tL^2_{d+1}f(x)\right| \le C\left\|f\right\|_{C^2_p}(1+|x|^{p+2})t^2.$$

That is, condition $\mathcal{R}(2, t^2)$ *is satisfied.*
2.Assume that $\sup_{\varepsilon\in(0,1]} \int_{|y|\le\varepsilon} |y|^p l(y)^{-\frac{p-2}{2}}\nu(dy) < \infty$, *then assumption* $(\mathcal{M})$ *is satisfied with*

$$E\left[\left|\bar{X}^{2,\varepsilon}_{d+1}(x)\right|^p\right] \le (1+Kt)|x|^p + K't$$

One can also use localization functions for $|y| > \varepsilon$ as follows. Let $G_{\varepsilon,l}(dy) = C_{\varepsilon,l}^{-1} l(y) 1_{|y|>\varepsilon} \nu(dy)$, $C_{\varepsilon,l} = \int_{|y|>\varepsilon} l(y)\nu(dy)$ and let $Z^{\varepsilon,l} \sim G_{\varepsilon,l}$ and let S^{ε} be a Bernoulli r.v. indep. of $Z^{\varepsilon,l}$.Then consider the following two subcases. If $S^{\varepsilon,l} = 0$ define $\bar{X}_t^{3,\varepsilon}(x) = x$, otherwise $\bar{X}_t^{3,\varepsilon,l}(x) = x + h(x) l(Z^{\varepsilon,l})^{-1} Z^{\varepsilon,l}$.

Lemma

1.Assume that $\int_{|y|>\varepsilon} |y|^2 (l(y)^{-1} - 1) + |y|^{p+3} |l(y)^{-1} - 1|^{p+2} \nu(dy) \leq Ct$ *and* $\left| C_{\varepsilon,l}^{-1} P\left[S^{\varepsilon,l} = 1\right] - t \right| \leq Ct^2$ *then*

$$\left| E\left[f(\bar{X}_t^{3,\varepsilon,l}) \right] - f(x) - tL_{d+1}^3 f(x) \right| \leq Ct^2 \left\| f \right\|_{C_p^2} (1 + |x|^{p+2}).$$

That is, condition $\mathcal{R}(2, t^2)$ is satisfied.
2. Assume that $\sup_{\varepsilon \in (0,1]} \max_{j=1,\ldots,p} \int_{|y|>\varepsilon} l(y)^{1-j} |y|^j \nu(dy) < \infty$.*then assumption* $(\mathcal{M})$ *is satisfied with*

$$E\left[\left| \bar{X}_{d+1}^{3,\varepsilon}(x) \right|^p \right] \leq (1 + Kt)|x|^p + K't$$

Example: Tempered stable

Let a Lévy measure ν defined on $\mathbf{R}_0$ be given by

$$\nu(dy) = \frac{1}{|y|^{1+\alpha}}\left(c_+ e^{-\lambda_+|y|}1_{y>0} + c_- e^{-\lambda_-|y|}1_{y<0}\right)dy$$

- Gamma: $\lambda_+, c_+ > 0$, $c_- = 0$, $\alpha = 0$.
- Variance gamma: $\lambda_+, \lambda_-, c_+, c_- > 0$, $\alpha = 0$.
- Tempered stable: $\lambda_+, \lambda_-, c_+, c_- > 0$, $0 < \alpha < 2$.

Then, we have that for $\alpha \in [0, 1)$

$$\int_{|y|\leq\varepsilon} |y|^k \nu(dy) \sim \varepsilon^{k-\alpha}, \quad k \geq 1.$$

Therefore $\sup_{\varepsilon\in(0,1]} \int_{|y|\leq\varepsilon} |y|\nu(dy) < \infty$, then the conditions of the approximation Lemma (*) are satisfied if $r \geq \alpha$, $r + \alpha \leq 4$ and $\varepsilon = t^{\frac{1}{3-\alpha}}$. approximation Lemma (**) is satisfied for example in the following case. Let $P\left[S^\varepsilon = 1\right] = e^{-C_\varepsilon a(\varepsilon,t)}$ where $a(\varepsilon, t) = -\varepsilon^\alpha \log\left(\left(t^2 + t\right)\varepsilon^{-\alpha}\right)$ as $\varepsilon = t^{\frac{1}{3-\alpha}}$ then we have that

$$a(\varepsilon(t), t) = -t^{\frac{\alpha}{3-\alpha}} \log\left((t+1)t^{\frac{3-2\alpha}{3-\alpha}}\right).$$

Other topics:

- Design other approximation schemes

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- Irregular coefficients. A CIR type example
  Consider the operators

$$Lf(x) = k(a-x)f'(x) + \sigma x f''(x)$$

$$L_\varepsilon f(x) = k(a-x)f'(x) + \sigma \phi_\varepsilon(x) f''(x)$$

  where $\phi_\varepsilon(x) = x$ for $x > \varepsilon$, $|\phi_\varepsilon(x)| \leq \varepsilon$ for $x \in (0, \varepsilon)$ and $\phi_\varepsilon \in C_b^\infty(\mathbf{R}_+)$ and $2ka \geq \sigma^2$. Then we have that

$$|Lf(x) - L_\varepsilon f(x)| \leq 2\sigma\varepsilon \left| f''(x) \right|.$$

  Take $\varepsilon = T/n$.

Other topics:

- Design other approximation schemes
- At most kn in n intervals
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  Take $\varepsilon = T/n$.
- Irregular functions f: Consider the right stochastic representation and concatenate.But there is a technical problem with jump type processes !

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