Randomized Approximation of Functions

Stefan Heinrich
Department of Computer Science
University of Kaiserslautern
D-67653 Kaiserslautern, Germany
e-mail: heinrich@informatik.uni-kl.de
homepage:
http://www.uni-kl.de/AG-Heinrich

$1 \leq p \leq \infty$, $r \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$,
Q a bounded Lipschitz domain

Sobolev space

$$W_p^r(Q) = \{f \in L_p(Q) : \ D^\alpha f \in L_p(Q), \ |\alpha| \leq r\}$$

$\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, $|\alpha| := \sum_{j=1}^d \alpha_j \leq r$,
$D^\alpha f$ generalized partial derivative

norm

$$\|f\|_{W_p^s(Q)} = \left(\sum_{|\alpha| \leq r} \|D^\alpha f\|_{L_q(Q)}^p \right)^{1/p}$$

if $p < \infty$, and

$$\|f\|_{W_\infty^r(Q)} = \max_{|\alpha| \leq r} \|D^\alpha f\|_{L_\infty(Q)}.$$

Note:

$$r = 0 \quad W_p^0(Q) = L_p(Q)$$

Approximation problem

$1 \leq p, q \leq \infty,\ r, s \in \mathbb{N}_0,$

$$\frac{r-s}{d} > \max\left(\frac{1}{p} - \frac{1}{q}, 0\right)$$

Approximate

$$J : W_p^r(Q) \to W_q^s(Q)$$

deterministic algorithms $\mathcal{A}_n^{\text{det}}$

$$A : W_p^r(Q) \to W_q^s(Q),$$

$$A(f) = \sum_{i=1}^{n} f(x_i)\psi_i \quad x_i \in Q,\ \psi_i \in W_q^s(Q)$$

error:

$$e(J, A, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q))$$
$$= \sup_{f \in \mathcal{B}_{W_p^r(Q)}} \|Jf - A(f)\|_{W_q^s(Q)}$$

deterministic n-th minimal error:
(linear sampling numbers)

$$e_n^{\mathrm{det}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q))$$
$$= \inf_{A \in \mathcal{A}_n^{\mathrm{det}}} e(J, A, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q))$$

randomized algorithms $\mathcal{A}_n^{\text{ran}}$:

$(\Omega, \Sigma, \mathbb{P})$ probability space,

$$(A_\omega)_{\omega \in \Omega}, \quad A_\omega : W_p^r(Q) \to W_q^s(Q)$$

$$A_\omega(f) = \sum_{i=1}^n f(x_{i,\omega}) \psi_{i,\omega}$$

$x_{i,\omega} \in Q$, $\psi_{i,\omega} \in W_q^s(Q)$ $(\omega \in \Omega)$,

error:

$$e(J, (A_\omega), \mathcal{B}_{W_p^r(Q)}, W_q^s(Q))$$
$$= \sup_{f \in \mathcal{B}_{W_p^r(Q)}} \mathbb{E} \, \|Jf - A_\omega(f)\|_{W_q^s(Q)}$$

randomized n-th minimal error:

(randomized linear sampling numbers)

$$e_n^{\text{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q))$$
$$= \inf_{(A)_\omega \in \mathcal{A}_n^{\text{ran}}} e(J, (A)_\omega, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q))$$

$$e_n^{\mathrm{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \leq e_n^{\mathrm{det}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q))$$

$e_n^{\mathrm{det}}: \quad \Omega = \{\omega_0\}$

embedding condition

$W_p^r(Q)$ is embedded into $C(Q)$ iff

$$\left.\begin{array}{llll} & p = 1 & \text{and} & r/d \geq 1 \\ \text{or} & & & \\ & 1 < p \leq \infty & \text{and} & r/d > 1/p \end{array}\right\} \quad (1)$$

The case $s \geq 0$, embedding condition

Theorem 1. *(many authors)*
Let $r, s \in \mathbb{N}_0$, $1 \leq p, q \leq \infty$, *with*

$$\frac{r-s}{d} > \max\left(\frac{1}{p} - \frac{1}{q}, 0\right),$$

let Q *be a bounded Lipschitz domain, and assume that (1) holds, i.e.,* $W_p^r(Q)$ *is embedded into* $C(Q)$*. Then in the deterministic setting*

$$e_n^{\det}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \;\asymp\; n^{-\frac{r-s}{d}+\left(\frac{1}{p}-\frac{1}{q}\right)_+}$$

and in the randomized setting

$$e_n^{\mathrm{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \;\asymp\; n^{-\frac{r-s}{d}+\left(\frac{1}{p}-\frac{1}{q}\right)_+}$$

The case $s \geq 0$, no embedding condition

deterministic setting:

$$e_n^{\text{det}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \text{ does not make sense,}$$

$$e_n^{\text{det}}(J, \mathcal{B}_{W_p^r(Q)} \bigcap C(Q), W_q^s(Q)) \asymp ??$$

randomized setting:

$$e_n^{\text{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \asymp ??$$

Theorem 2. *(H., 2007/08)*
Let $r, s \in \mathbb{N}_0$, $1 \le p, q \le \infty$, with

$$\frac{r-s}{d} > \max\left(\frac{1}{p} - \frac{1}{q}, 0\right),$$

let Q be a bounded Lipschitz domain, and assume that (1) does not hold, i.e., $W_p^r(Q)$ is not embedded into $C(Q)$. Then

$$\begin{aligned} e_n^{\text{det}}(J, \mathcal{B}_{W_p^r(Q)} \cap C(Q), W_q^s(Q)) &\asymp 1 \\ e_n^{\text{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) &\asymp n^{-\frac{r-s}{d} + \left(\frac{1}{p} - \frac{1}{q}\right)_+} \end{aligned}$$

E.g., $1 < p = q < \infty$, $s = 0$, $\frac{r}{d} = \frac{1}{p}$

speedup: $n^{-(1-\delta)}$ for arbitrary small $\delta > 0$ possible

The case $s < 0$

$$W_q^{-s}(Q) := \widetilde{W}_{q^*}^s(Q)^* \qquad \left(\frac{1}{q} + \frac{1}{q^*} = 1\right)$$

Motivation

$m \in \mathbb{N}$, bilinear form a on $W_2^m(Q)$,

$$a(u,v) = \sum_{\alpha|,|\beta| \leq m} \int_Q a_{\alpha\beta}(x) D^\alpha u(x) D^\beta v(x) dx,$$

$a_{\alpha\beta} \in C(Q)$

assume a is $\widetilde{W}_2^m(Q)$-elliptic

$$\begin{aligned} |a(u,v)| &\leq c_1 \|u\|_{W_2^m(Q)} \|v\|_{W_2^m(Q)} \\ a(u,u) &\geq c_2 \|u\|^2_{W_2^m(Q)} \end{aligned}$$

$(u, v \in \widetilde{W}_2^m(Q))$.

associated differential operator:

$$\mathscr{L} u = \sum_{\alpha|,|\beta| \leq m} D^\beta (a_{\alpha\beta}\, D^\alpha u)$$

weak elliptic problem associated with a:

Given $f \in W_2^{-m}(Q)$, find $u \in \widetilde{W}_2^m(Q)$ such that for all $v \in \widetilde{W}_2^m(Q)$

$$a(u, v) = f(v).$$

The problem has a unique solution $S_0 f \in \widetilde{W}_2^m(Q)$,

$$S_0 : W_2^{-m}(Q) \to \widetilde{W}_2^m(Q)$$

is an isomorphism

$r \in \mathbb{N}_0$, $1 \leq p \leq \infty$

$$\frac{r+m}{d} > \frac{1}{p} - \frac{1}{2},$$

Solve the weak problem for $f \in W_p^r(Q)$: solution operator

$$S = S_0 J : W_p^r(Q) \xrightarrow{J} W_2^{-m}(Q) \xrightarrow{S_0} \widetilde{W}_2^m(Q)$$

Corollary 1.

$$\begin{aligned} & e_n^{\mathrm{ran}}(S, \mathcal{B}_{W_p^r(Q)}, \widetilde{W}_2^m(Q)) \\ \asymp \ & e_n^{\mathrm{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_2^{-m}(Q)), \end{aligned}$$

and analogously for e_n^{det}*.*

Deterministic case:

Theorem 3. *Let* $r \in \mathbb{N}_0$, $s \in \mathbb{N}$, $1 \le p, q \le \infty$,

$$\frac{r+s}{d} > \frac{1}{p} - \frac{1}{q}.$$

1. (Novak, Triebel, 2006, Vybiral, 2007) Assume that $W_p^r(Q)$ *is embedded into* $C(Q)$. *Then*

$$e_n^{\mathrm{det}}(J : W_p^r(Q) \to W_q^{-s}(Q)) \asymp n^{-\gamma_1}$$

where

$$\gamma_1 = \min\left(\frac{r+s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+, \frac{r}{d}\right).$$

2. (H., 2008) If $W_p^r(Q)$ *is not embedded into* $C(Q)$, *then*

$$e_n^{\mathrm{det}}(J : W_p^r(Q) \to W_q^{-s}(Q)) \asymp 1.$$

Randomized case:

Theorem 4. *(H., 2008) Let* $r \in \mathbb{N}_0$, $s \in \mathbb{N}$, $1 \le p, q \le \infty$ *and*

$$\frac{r+s}{d} > \frac{1}{p} - \frac{1}{q}.$$

Then

$$e_n^{\mathrm{ran}}(J : W_p^r(Q) \to W_q^{-s}(Q)) \asymp_{log} \; n^{-\gamma_2}$$

where

$$\gamma_2 = \min\left(\frac{r+s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+, \frac{r}{d} + 1 - \frac{1}{\bar{p}}\right)$$

and $\bar{p} = \min(p, 2)$.

solves Problem 25 of Novak, Woźniakowski (Tractability of Multivariate Problems, Volume 1)

$W_p^r(Q)$ is embedded into $C(Q)$

$J: W_p^r(Q) \to W_q^{-s}(Q)$	e_n^{det}	e_n^{ran}
$1 < q \le p \le \infty,$	$n^{-\frac{r}{d}}$	$n^{-\frac{r}{d}-\min(s/d, 1-1/\bar{p})}$
$1 \le p < q \le \infty,$ $s/d > 1/p - 1/q$	$n^{-\frac{r}{d}}$	$n^{-\frac{r}{d}-\min(s/d-(1/p-1/q), 1-1/\bar{p})}$
$1 \le p < q \le \infty,$ $s/d < 1/p - 1/q$	$n^{-(r+s)/d+(1/p-1/q)}$	$n^{-(r+s)/d+(1/p-1/q)}$

$W_p^r(Q)$ is not embedded into $C(Q)$

$J: W_p^r(Q) \to W_q^{-s}(Q)$	e_n^{det}	e_n^{ran}
$r/d < 1/p$ or $r/d = 1/p$ and $1 < p \le \infty$	1	$n^{-\min\left(\frac{r+s}{d}-\left(\frac{1}{p}-\frac{1}{q}\right)_+, \frac{r}{d}+1-\frac{1}{\bar{p}}\right)}$