Lower bounds for the probability that an Ito process stay in a tube and applications to finance

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1 Framework

We consider the following objects :

A deterministic differentiable curve : $x_t, 0 \leq t \leq T$,

A stochastic process : $X_t, 0 \leq t \leq T, \quad X_t \in R^n$,

A time depending radius $R_t > 0, 0 \leq t \leq T$.

We define the stopping time

$$\tau_R = \inf\{t : |X_t - x_t| \geq R_t\}$$

and we assume that

$$X_t = x + \sum_{i=1}^{\infty} \int_0^t \sigma_j(s,\omega,X_s)dW_s^j + \int_0^t b(s,\omega,X_s)ds \quad for \quad t \leq \tau_R.$$

We give lower bounds for

$$P(\tau_R \geq T) = P(|X_t - x_t| \leq R_t, \forall t \leq T).$$

Hypothesis.

Adapted : $t \to \sigma_j(t,\omega,X_t), t \to b(t,\omega,X_t)$ are adapted to the filtration of W.

Locally Bounded :

$$\left|b(t\wedge\tau_R,\omega,X_{t\wedge\tau_R})\right| + \sum_{j=1}^{\infty}\left|\sigma_j(t\wedge\tau_R,\omega,X_{t\wedge\tau_R})\right| \leq c_t.$$

Locally Lipshitz continuous : for every $t < s < \tau_R$

$$\sum_{j=1}^{\infty} E_t(\left|\sigma_j(s,\omega,X_s) - \sigma_j(t,\omega,X_t)\right|^2 1_{\{\tau_R<s\}}) \leq L_t^2(s-t).$$

Locally Elliptic :

$$\lambda_t \leq \sigma\sigma^*(t,\omega,X_t) \leq \gamma_t$$

Growth condition : For $h > 0, \mu \geq 1$ we define $L(\mu, h)$ to be the class of non negative function such that

$$f(t) \leq \mu f(s) \quad if \quad |s-t| \leq h.$$

Remark : Given f we may take

$$\mu = \text{sup}\{\frac{f(t)}{f(s)} : |s-t| \leq h\}.$$

If $f(s) = 0$ for some s then $\mu = \infty$!

Hypothesis : We assume that c_t, L_t, λ_t and γ_t belong to some $L(\mu, h)$.

Rate function :

$$F_x(t) = \frac{1}{h} + c_t^2 L_t^2 (\frac{1}{\lambda_t} + \frac{1}{R_t}) + \frac{|\partial_t x_t|^2}{\lambda_t}.$$

MAINE RESULT

$$P(\tau_R \geq T) \geq \exp(-Q_n(1 + \int_0^T F_x(t)dt))$$

with $\quad Q_n = 8^{4n+4}\pi^n e^2 \mu^{2n^2+19n+25}$.

Related topics :

Osanger-Machlup function (see Ikeda-Watanabe) : for an elliptic diffusion process one may compute :

$$\lim_{R\to 0} R^2 \ln P(\sup_{t\leq T} \left| X_t^x - x_t^y \right| \leq R) = c_T(x,y).$$

Lower bounds for the density of the law :

$$p_T(x,y) = P(X_t^x \in dy) \geq q_T(x,y).$$

Elliptic framework. X_t verifies on **the whole space** the equation

$$X_t = x + \sum_{i=1}^{\infty} \int_0^t \sigma_j(s, \omega, X_s) dW_s^j + \int_0^t b(s, \omega, X_s) ds$$

and the coefficients verifies **in the tube** the hypothesis **:**

$$\left| b(t \wedge \tau_R, \omega, X_{t\wedge\tau_R}) \right| + \sum_{j=1}^{\infty} \left| \sigma_j(t \wedge \tau_R, \omega, X_{t\wedge\tau_R}) \right| \leq c.$$

Locally Lipshitz continuous : for every $t < s < \tau_R$

$$\sum_{j=1}^{\infty} E_t(\left|\sigma_j(s, \omega, X_s) - \sigma_j(t, \omega, X_t)\right|^2 1_{\{\tau_R < s\}}) \leq L^2(s-t).$$

Locally Elliptic :

$$\lambda \leq \sigma\sigma^*(t, \omega, X_t) \leq \gamma.$$

Warning : the bounds does no more depend on the time !

We take $y \in R^n$ and we consider the **straight line**

$$x_t = x + \frac{t}{T}(y - x),$$

Then $|\partial_t x_t| = \frac{1}{T}|y - x|$ so that

$$F_x(t) = \frac{1}{h} + c^2 L^2 (\frac{1}{\lambda} + \frac{1}{R}) + \frac{|y - x|^2}{\lambda T^2}$$

and

$$\int_0^T F_x(t)dt = T(\frac{1}{h} + c^2 L^2 (\frac{1}{\lambda} + \frac{1}{R})) + \frac{|y - x|^2}{\lambda T}.$$

RESULT :

$$P(\sup_{t \le T} |X_t - x_t| \le R) \ge \exp(-C_T(T + \frac{|y - x|^2}{\lambda T})).$$

In particular we have the **ball estimate**

$$P(|X_T - y| \le R) \ge \exp(-C_T(T + \frac{|y - x|^2}{\lambda T})).$$

Log-normal framework. X_t verifies on **the whole space** the equation

$$X_t = x + \sum_{i=1}^{\infty} \int_0^t \sigma_j(s,\omega,X_s)dW_s^j + \int_0^t b(s,\omega,X_s)ds$$

and the coefficients verifies **in the tube** the hypothesis **:**

$$i) \quad |b(t,\omega,X_t)| + \sum_{j=1}^{\infty} \left|\sigma_j(,\omega,X_t)\right| \leq c\,|X_t|$$

$$ii) \quad \sum_{j=1}^{\infty} E_t(\left|\sigma_j(s,\omega,X_s) - \sigma_j(t,\omega,X_t)\right|^2) \leq L^2(s-t)E_t(\sup_{t\leq u\leq s} |X_u|^2)$$

$$iii) \quad \lambda \min_{i=1,n} \left|X_t^i\right| \leq \sigma\sigma^*(t,\omega,X_t) \leq \gamma\,|X_t|$$

Exponential path :

$$x_t^i = x^i \exp(\frac{t}{T}(\ln y^i - \ln x^i).$$

Lemma : Our basic hypothesis holds in the tube of radius $R_t = R\,|x_t|$ with

$$c_t = c(1+R)\,|x_t|\,, \quad L_t = L(1+R)\,|x_t|\,, \quad \gamma_t = \gamma(1+R)\,|x_t| \quad \lambda_t = \lambda(1-R)\min_t x_t^i.$$

Example 1. : Log-normal diffusion :

$$\frac{dX_t^i}{X_t^i} = \sum_{j=1} \sigma_j^i(t,\omega)dW_t^j + b^i(t,\omega)dt, \quad i = 1, n.$$

Example 2. : Baskets : $\quad Y_t = \sum_{j=1}^n q_i X_t^i.$

MAINE ESTIMATE :

$$P(|X_t - x_t| \leq R\,|x_t|\,, \forall t \leq T) \geq \exp(-C_T(\theta(x)+\theta(y))^2(1+\frac{d^2(x,y)}{T\lambda}))$$

with

$$d(x,y) = \max_{i=1,n}\left|\ln y^i - \ln x^i\right|, \qquad \theta(x) = \max_{i,j}\frac{x^i}{x^j}.$$

Corollary : The "**appropriate distance**" is $d(x,y)$ and we obtain :

$$P(\sup_{t\leq T} d(X_t, x_t) \leq R) \geq \exp(-C_T(\theta(x)+\theta(y))^2(1+\frac{d^2(x,y)}{T\lambda})).$$

Expectation Estimates.

Class of functions : Let $y \in R^n$, $\varepsilon > 0$ and $\mu \geq 1$. We define $L_y^d(\mu, \varepsilon)$ to be the class of positive functions such that

$$f(y) \leq \mu f(x) \quad \forall x \in B_\varepsilon^d(y).$$

Elliptic framework $(d(x,y) = |x-y|)$: For every $f \in L_{y_0}^d(\mu, \varepsilon)$ one has

$$E(f(X_T^x) 1_{B_\varepsilon(y_0)}(X_T^x)) \geq \frac{1}{\mu^2} \int_{B_\varepsilon(y_0)} f(z) \exp(-\frac{C}{|x-y_0|^2 \varepsilon^2}(1 + \frac{|x-z|^2}{\lambda T}))dz.$$

Remarks : 1. Small balls (ε very small) $\rightarrow zero.$

2. In the neighborhood of x ($|x - y_0| \rightarrow 0$) $\rightarrow zero.$

3. Far from x, with a "big" ball : Gaussian estimate.

Distribution function : Let $y_0 = (y_0^1, ..., y_0^n)$ with $y_0^i > x^i, i = 1, ..., n$. Then

$$P(X_T^{x,i} > y_0^i, i = 1, ..., n) \geq \int_{\{z^i > y_0^i, i=1,n\}} f(z) \exp(-\frac{C}{|x-y_0|^2}(1 + \frac{|x-z|^2}{\lambda T}))dz.$$

Log-normal framework $(d(x,y) = \max_{i=1,n} \left| \ln y^i - \ln x^i \right|)$: For every $f \in L^d_{y_0}(\mu, \varepsilon)$ one has

$$E(f(X_T^x) 1_{B_\varepsilon^d(y_0)}(X_T^x))$$
$$\geq \frac{1}{\mu^2} \int_{B_\varepsilon^d(y_0)} f(z) \exp(-\frac{C}{d^2(x,y_0)\varepsilon^2}(\theta(x)+\theta(z))^2(1+\frac{d^2(x,z)}{\lambda T}))dz.$$

Distribution function : Let $y_0 = (y_0^1, ..., y_0^n)$ with $y_0^i > x^i, i = 1, ..., n$. Then

$$P(X_T^{x,i} > y_0^i, i = 1, ..., n)$$
$$\geq \int_{\{z^i > y_0^i, i=1,n\}} \exp(-\frac{C}{d^2(x,y_0)}(\theta(x)+\theta(z))^2(1+\frac{d^2(x,z)}{\lambda T}))dz.$$

Lower bounds for Option Prices. The stocks verify

$$\frac{dX_t^i}{X_t^i} = \sum_{j=1} \sigma_j^i(t,\omega) dW_t^j + b^i(t,\omega)dt.$$

Hypothesis : σ_j, b are adapted, bounded and Lipschiz continuos processes.

Call on a basket : We assume the **ellipticity condition**

$$\sum_{i=1}^{n} \left|\sigma_j^i(t,\omega)\right|^2 \geq \lambda > 0 \quad \forall i = 1, n$$

Let $Y_t = \sum_{i=1}^n q_i X_t^{x,i}$. Then

$$E((Y_T - K)_+) \geq \int_0^\infty (y-K)_+ p_T^K(x,y)dy$$

with

$$p_T^K(x,y) = \exp(-\frac{C_T}{1 \wedge (y-K)^2}(1 + \left|\ln y - \ln \sum_{i=1}^n q_i x^i\right|^2).$$

Proof : We use the previous results for Y_t (which is **not a Markov process**, but is an Ito process).

Asian Options. Ellipticity assumption : $\sigma\sigma^*(t,\omega) \geq \lambda > 0$.

$$E((\int_0^T \sum_{i=1}^n q_i X_t^{x,i} - K)_+) \geq \int_0^\infty (y-K)_+ p_T^K(x,y)dy$$

with

$$p_T^K(x,y) = \exp(-\frac{C_T}{1 \wedge (y-K)_+} \times \frac{1+\max_{i=1,n}\left|\ln y - \ln Tq_i x^i\right|^2}{1 \wedge \min_{i=1,n}\left|\ln y - \ln Tq_i x^i\right|}.$$

Idea of proof :

Step 1 Given $z \in R_+$ we look for $\overline{z} \in R^n$ such that

$$\int_0^T x_t^i(x,\overline{z})dt = \int_0^T x^i e^{\frac{1}{T}(\ln \overline{z}^i - \ln x^i)} dt = z..$$

Step 2.

$$P(d(\int_0^T \sum_{i=1}^n q_i X_t^{x,i} dt, z) \leq R) = P(d(\int_0^T \sum_{i=1}^n q_i X_t^{x,i} dt, \int_0^T x_t^i(x,\overline{z})dt) \leq R)$$
$$\leq P(\sup_{t\leq T} d(X_t^x, x_t(x,\overline{z})) \leq \frac{RT}{K}).$$

Step 3. We use the previous results for X_t^x and we get lower bounds for the ball and then for the expectation.

Sketch of the proof of the Main Result

Step 1. Decomposition For $t, h > 0$

$$X_{t+h} = X_t + G_{t,h} + R_{t,h}$$

with

$$G_{t,h} = \sum_{j=1}^{\infty} \sigma_j(t, \omega, X_t)(W^j_{t+h} - W^j_t) \qquad \mathit{Gaussian}$$

$$R_{t,h} = \sum_{j=1}^{\infty} \int_t^{t+h} (\sigma_j(s, \omega, X_s) - \sigma_j(t, \omega, X_t)) dW^j + \int_t^{t+h} b(s, \omega, X_s) ds \quad \sim \quad h.$$

Step 2. Short time behavior There exists α depending on the parameters $c_t, L_t, \lambda_t, \gamma_t$ such that, if

$$i) \quad h \leq \alpha,$$
$$ii) \quad h^{\frac{1}{n+1}}\sqrt{h} \leq \delta \leq \sqrt{h}$$

then, on the set $\{|X_t - z| \leq h\}$ we have

$$\frac{1}{\delta^n} P_t(|X_{t+h} - z| \leq \delta) \geq \frac{c}{h^{n/2}}.$$

Step 3. Chain argument We choose $h \leq \alpha$ and $h^{\frac{1}{n+1}}\sqrt{h} \leq \delta \leq \sqrt{h}$, we denote $t_i = ih$ and $N = 1/h$ and we write

$$P(\left|X_{t_i} - x_{t_i}\right| \leq \delta, i = 1, ..., N) \geq \prod_{i=1}^{N} P(\left|X_{t_{i+1}} - x_{t_{i+1}}\right| \leq \delta \mid \left|X_{t_i} - x_{t_i}\right| \leq \delta)$$
$$\geq a^N = \exp(-N \ln \frac{1}{a}).$$