Approximation of Functionals of SDEs and Application to a Recent Multilevel MC Method

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SDE

Consider the SDE

$$\begin{cases} X_0 = x_0, \\ dX_t = \sigma(t, X_t)\, dW_t + b(t, X_t)\, dt, \end{cases}$$

where $x_0 \in \mathbb{R}$, $\sigma, b : [0, T] \times \mathbb{R} \to \mathbb{R}$ and W is a standard one-dimensional Brownian motion.

Assumptions

Assume that $\sigma, b \in C([0,T] \times \mathbb{R})$ and for $f \in \{\sigma, b\}$ there exists a constant C_T such that

1) $|f(t,x) - f(t,y)| \leq C_T|x-y|$
2) $|f(t,x) - f(s,x)| \leq C_T(1+|x|)|t-s|$
3) X_T has a bounded density.

Assumption 3) may be replaced by the uniform ellipticity condition

3′) $\sigma, b \in C_b^\infty([0,T] \times \mathbb{R})$ and
$\sigma(t,x) \geq \beta > 0$ for all $(t,x) \in [0,T] \times \mathbb{R}$.

Theorem 1 (Caballero, Fernández, Nualart 1998)

Assume that σ and b are C^2 in x, the second derivatives have polynomial growth, the functions $|\sigma(0,x)|$, $|\sigma_x(t,x)|$, $|b(0,x)|$ and $|b_x(t,x)|$ are bounded, and

$$\mathbb{E}\left(\left|\int_0^t \sigma(s,X_s)^2\,ds\right|^{-p_0/2}\right) < \infty$$

for some $p_0 > 2$ and for all $t \in (0,T]$. Then for all $t \in (0,T]$ there exists a continuous density f_{X_t} of X_t such that for all $p > 1$

$$f_{X_t}(x) \le C_p \left\| \left(\int_0^t \sigma(s,X_s)^2\,ds \right)^{-1/2} \right\|_p$$

for some constant $C_p > 0$.

Euler scheme

Let π_n be the equidistant partition $0 = t_0 < t_1 < \cdots < t_n = T$ of the interval $[0, T]$ with mesh size $|\pi_n| = T/n$.
Define $X^{n,E}$ to be the Euler approximation relative to π_n , i.e.

$$X_0^{n,E} = x_0, \text{ and}$$
$$X_{t_{i+1}}^{n,E} = X_{t_i}^{n,E} + b(t_i, X_{t_i}^{n,E})(t_{i+1} - t_i) + \sigma(t_i, X_{t_i}^{n,E})(W_{t_{i+1}} - W_{t_i}).$$

Then $X_T^{n,E}$ is the equidistant Euler approximation of X_T evaluated at the endpoint $T = t_n$.

Theorem 2 (Classical)

If $1 \le p < \infty$, *then* $\left\| X_T - X_T^{n,E} \right\|_p \le C_p \sqrt{|\pi_n|}$.
Here
$$C_p \le e^{M(x_0, T, C_T)p^2}.$$

Question

Include a function in the error term.
What is the rate of $\mathbb{E}|g(X_T) - g(X_T^{n,E})|^2$ if $g(x) = \chi_{[K,\infty)}(x)$?

If g is Lipschitz, then $\mathbb{E}|g(X_T) - g(X_T^{n,E})|^2 \leq L^2 \mathbb{E}|X_T - X_T^{n,E}|^2$.

$\rightsquigarrow$ non-Lipschitz functions

Indicator functions

Theorem 3

Let $0 < p < \infty$ *and suppose that* X *is a random variable.*

- *If* X *has a bounded density* f_X*, then for all* $K \in \mathbb{R}$ *and for all random variables* $\hat{X}$ *we have*
  $\mathbb{E}|\chi_{[K,\infty)}(X) - \chi_{[K,\infty)}(\hat{X})| \leq 3(\sup f_X)^{\frac{p}{p+1}} \left\| X - \hat{X} \right\|_p^{\frac{p}{p+1}},$
  where $\frac{p}{p+1}$ *is the optimal exponent.*
- *If there exist* $p_0 > 0$ *and* $B_X > 0$ *such that for all* $p_0 \leq p < \infty$*, all* $K \in \mathbb{R}$*, and all random variables* $\hat{X}$ *we have*
  $$\mathbb{E}|\chi_{[K,\infty)}(X) - \chi_{[K,\infty)}(\hat{X})| \leq B_X \left\| X - \hat{X} \right\|_p^{\frac{p}{p+1}},$$
  then X *has a bounded density.*

Functions of bounded variation

Let BV be the class of functions of bounded variation on the real line and denote the variation of $g \in BV$ by $V(g)$.
For any $g \in BV$ (up to continuity and normalization) there exist a unique σ-finite signed measure μ such that

$$g(x) = \mu((-\infty, x)).$$

If $\mu = \mu_1 - \mu_2$ is the Jordan decomposition, then $|\mu| = \mu_1 + \mu_2$.

Result for BV

Theorem 4 (Rate for equidistant Euler scheme)

Let $1 \le p < \infty$ *and* $g \in BV$. *Then there exists* $n_0 \in \mathbb{N}$ *such that for* $n \ge n_0$ *we have*

$$\mathbb{E}|g(X_T) - g(X_T^{n,E})|^p \le 3^p(\sup f_{X_T} \vee \sqrt{\sup f_{X_T}})V(g)^p n^{-\frac{1}{2}+\frac{M}{(\log n)^{1/3}}},$$

where M *depends on* x_0, T *and the Lipschitz coefficients.*

Idea of the proof

Start by considering the case $g = \chi_{[K,\infty)}$.
For $1 \le q < \infty$,

$$\begin{aligned}
\mathbb{E}|\chi_{[K,\infty)}(X_T) - \chi_{[K,\infty)}(X_T^{n,E})| &\le 3(\sup f_{X_T})^{\frac{q}{q+1}} \left\| X_T - X_T^{n,E} \right\|_q^{\frac{q}{q+1}} \\
&\le 3(\sup f_{X_T})^{\frac{q}{q+1}} C_q^{\frac{q}{q+1}} n^{-\frac{1}{2}\frac{q}{q+1}}.
\end{aligned}$$

Find optimal q by computing

$$\inf_{1 \le q < \infty} C_q^{\frac{q}{q+1}} n^{-\frac{1}{2}\frac{q}{q+1}}.$$

Then there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have

$$\mathbb{E}|\chi_{[K,\infty)}(X_T) - \chi_{[K,\infty)}(X_T^{n,E})| \le 3(\sup f_{X_T} \vee \sqrt{\sup f_{X_T}})n^{-\frac{1}{2}+\frac{M}{(\log n)^{1/3}}}.$$

Idea of the proof

Then

$$g(x) = \mu((-\infty, x)) = \int_{\mathbb{R}} \chi_{(-\infty,x)}(z)\, d\mu(z) = \int_{\mathbb{R}} \chi_{(z,\infty)}(x)\, d\mu(z).$$

and

$$\begin{aligned}
\mathbb{E}|g(X_T) - g(X_T^{n,E})| &= \mathbb{E}\left|\int_{\mathbb{R}} \chi_{(z,\infty)}(X_T) - \chi_{(z,\infty)}(X_T^{n,E})\, d\mu(z)\right| \\
&\leq \int_{\mathbb{R}} \mathbb{E}\left|\chi_{[z,\infty)}(X_T) - \chi_{[z,\infty)}(X_T^{n,E})\right| d|\mu|(z) \\
&\leq 3(\sup f_{X_T} \vee \sqrt{\sup f_{X_T}})V(g)n^{-\frac{1}{2}+\frac{M}{(\log n)^{1/3}}}.
\end{aligned}$$

Lower bound

Let S be the geometric Brownian motion.

Theorem 5

There exist $K_0 > 0$ such that

$$\liminf_{n\to\infty} \sqrt{n} \sup_{K \geq K_0} \mathbb{E}|\chi_{[K,\infty)}(S_1) - \chi_{[K,\infty)}(S_1^{n,E})| > 0.$$

Therefore, since $V(\chi_{[K,\infty)}) = 1$, it is impossible to find $\gamma > \frac{1}{2}$ such that

$$\mathbb{E}|g(S_1) - g(S_1^{n,E})| \leq cV(g)\left(\frac{1}{n}\right)^{\gamma}.$$

Whether $\gamma = \frac{1}{2}$ is possible remains open.

Multilevel Monte Carlo Method (Michael B. Giles, 2006)

Define payoff $P := g(X_T)$.

Goal: to approximate the expected payoff $\mathbb{E}P = \mathbb{E}g(X_T)$.

1° Define time steps $h_l := \frac{T}{M^l}$ for $l = 0, 1, \ldots, L$

2° Approximate X_T by a numerical discretization $\hat{X}_l$ using time step h_l (e.g. Euler)

3° Approximate P by $\hat{P}_l = g(\hat{X}_l)$

4° Write

$$\mathbb{E}\hat{P}_L = \mathbb{E}\hat{P}_0 + \sum_{l=1}^{L} \mathbb{E}[\hat{P}_l - \hat{P}_{l-1}]$$

5° Let $\hat{Y}_0$ be an estimator of $\mathbb{E}\hat{P}_0$ using N_0 samples

6° Let $\hat{Y}_l$, $l \geq 1$, be an estimator of $\mathbb{E}[\hat{P}_l - \hat{P}_{l-1}]$ using N_l paths

7° Consider the combined estimator

$$\hat{Y} = \sum_{l=0}^{L} \hat{Y}_l$$

Multilevel Monte Carlo Method

Example:

$$\hat{Y} = \sum_{l=0}^{L} \hat{Y}_l,$$

where

$$\hat{Y}_l = \frac{1}{N_l} \sum_{i=1}^{N_l} \left(\hat{P}_l^{(i)} - \hat{P}_{l-1}^{(i)} \right).$$

It is easy to show that

$$Var(\hat{Y}_l) = \frac{Var(\hat{P}_l - \hat{P}_{l-1})}{N_l}$$

and

$$C(\hat{Y}_l) \leq \frac{cN_l}{h_l}.$$

Multilevel Monte Carlo Method

Theorem 6 (Giles 2006)

Suppose that there exist independent estimators $\hat{Y}_l$ based on N_l Monte Carlo samples, and positive constants $\alpha \geq \frac{1}{2}$, β, c_1, c_2, c_3, such that

i) $|\mathbb{E}[\hat{P}_l - P]| \leq c_1 h_l^\alpha$,

ii) $\mathbb{E}\hat{Y}_l = \begin{cases} \mathbb{E}\hat{P}_0, & l = 0, \\ \mathbb{E}[\hat{P}_l - \hat{P}_{l-1}], & l > 0, \end{cases}$

iii) $Var(\hat{Y}_l) \leq c_2 N_l^{-1} h_l^\beta$,

iv) $C(\hat{Y}_l) \leq c_3 N_l h_l^{-1}$.

Multilevel Monte Carlo Method

Then there exists $c_4 > 0$ such that for any $\varepsilon < 1/e$ there are values L and N_l for which the multilevel estimator

$$\hat{Y} = \sum_{l=0}^{L} \hat{Y}_l$$

has

$$MSE = \mathbb{E}(\hat{Y} - \mathbb{E}P)^2 < \varepsilon^2$$

and

$$C(\hat{Y}) \leq \begin{cases} c_4 \varepsilon^{-2}, & \beta > 1, \\ c_4 \varepsilon^{-2} (\log \varepsilon)^2, & \beta = 1, \\ c_4 \varepsilon^{-2-\frac{(1-\beta)}{\alpha}}, & 0 < \beta < 1. \end{cases}$$

Application

Consider the Euler scheme and the estimator

$$\hat{Y}_l = \frac{1}{N_l} \sum_{i=1}^{N_l} \left(\hat{P}_l^{(i)} - \hat{P}_{l-1}^{(i)} \right).$$

We have to determine the parameters α and β in the conditions

i) $|\mathbb{E}[\hat{P}_l - P]| \leq c_1 h_l^{\alpha}$, and

iii) $Var(\hat{Y}_l) \leq c_2 N_l^{-1} h_l^{\beta}$,

Parameter α

We have $\alpha = 1$ in

i) $|\mathbb{E}[\hat{P}_l - P]| = |\mathbb{E}g(\hat{X}_l) - \mathbb{E}g(X_T)| \leq c_1 h_l^\alpha$

by weak convergence results:

- Talay, Tubaro (1990): $g \in C_{pol}^\infty$, under the assumption $\sigma, b \in C_b^\infty$
- Bally, Talay (1996): g measurable and bounded, under uniform hypoellipticity
- Guyon (2006): extension to measurable functions with exponential growth, under uniform ellipticity

Parameter β

For $\hat{Y}_l$ the condition is

iii) $Var(\hat{Y}_l) = N_l^{-1} Var(\hat{P}_l - \hat{P}_{l-1}) \leq c_2 N_l^{-1} h_l^{\beta}$,

so we have to determine β in $Var(\hat{P}_l - \hat{P}_{l-1}) \leq c_2 h_l^{\beta}$.

$$\begin{aligned} Var(\hat{P}_l - \hat{P}_{l-1}) &\leq \left(\sqrt{Var(\hat{P}_l - P)} + \sqrt{Var(\hat{P}_{l-1} - P)} \right)^2 \\ &\leq \left(\sqrt{\mathbb{E}(\hat{P}_l - P)^2} + \sqrt{\mathbb{E}(\hat{P}_{l-1} - P)^2} \right)^2 \end{aligned}$$

If g is Lipschitz, then $\beta = 1$.
If $g \in BV$, then Theorem 4 implies that $\beta = \frac{1}{2} - \frac{A}{((l \log M) \vee B)^{1/3}}$.

Theorem 3 applied to the Giles' method

Corollary 7

Given $g \in BV$ and the assumptions of Theorem 6 with $\alpha = 1$ and $\beta = \frac{1}{2} - \frac{A}{((l \log M) \vee B)^{1/3}}$, there exists $c_4 > 0$ such that for any $\varepsilon < 1/e$ there are values L and N_l for which the multilevel estimator

$$\hat{Y} = \sum_{l=0}^{L} \hat{Y}_l$$

has

$$MSE < \varepsilon^{2-\delta(\varepsilon)},$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and

$$C(\hat{Y}) \leq c_4 \varepsilon^{-2-\frac{(1-1/2)}{1}} = c_4 \varepsilon^{-2.5}.$$

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