The Nonequispaced FFT and its Applications
A Mini Course at Helsinki University of Technology

http://math.tkk.fi/numericsyear/NFFT

Solution 1   **Basics and Matlab**

Exercise 1:

The function $g(x) = \sum_{r \in \mathbb{Z}} f(x+r)$ is one-periodic and integrable on $[-\frac{1}{2}, \frac{1}{2}]$. Moreover, the right hand side converges absolutely. The Fourier coefficients of g obey

$$
\begin{aligned}
c_k(g) &= \int_{-1/2}^{1/2} g(x) \, e^{-2\pi i k x} dx \\
&= \int_{-1/2}^{1/2} \sum_{r \in \mathbb{Z}} f(x+r) \, e^{-2\pi i k x} e^{-2\pi i k r} dx \\
&= \sum_{r \in \mathbb{Z}} \int_{r-1/2}^{r+1/2} f(x) \, e^{-2\pi i k x} dx \\
&= \int_{\mathbb{R}} f(x) e^{2\pi i k x} dx.
\end{aligned}
$$

Due to the decay of $\hat{f}$, these Fourier coefficients are summable and thus the Fourier series of g converges absolutely.

Exercise 2:

1. For $p = 1$, $k \in \mathbb{Z}$, and $f \in C^1(\mathbb{T})$ holds

$$
\begin{aligned}
c_k(f^{(1)}) &= \int\limits_{-1/2}^{1/2} f^{(1)}(x) \, e^{-2\pi i k x} dx \\
&= 0 - \left[-2\pi i k \int\limits_{-1/2}^{1/2} f(x) \, e^{-2\pi i k x} dx \right] \\
&= (2\pi i k)^1 \, c_k(f).
\end{aligned}
$$

2. For real-valued $f \in L^2(\mathbb{T})$ and $k \in \mathbb{Z}$ the Fourier-coefficients satisfy

$$\overline{c_{-k}(f)} = \overline{\int_{-1/2}^{1/2} f(x) e^{2\pi i k x} dx}$$

$$= \int_{-1/2}^{1/2} \overline{f(x)} e^{-2\pi i k x} dx$$

$$= c_k(f).$$

3. For even $f \in L^2(\mathbb{T})$, i.e. $f(x) = f(-x)$, and $k \in \mathbb{Z}$ the Fourier-coefficients satisfy

$$c_{-k}(f) = \int_{-1/2}^{1/2} f(x) e^{2\pi i k x} dx$$

$$= -\int_{1/2}^{-1/2} f(-x) e^{-2\pi i k x} dx$$

$$= c_k(f).$$

4. For $j, k = -N/2, \ldots, N/2 - 1$ holds, see slides, page 15 and 19-20,

$$\left(\tilde{\mathbf{F}}_N^H \tilde{\mathbf{F}}_N \right)_{j,k} = \frac{1}{N} \sum_{l=-N/2}^{N/2-1} e^{2\pi i l (j-k)/N}$$

$$= \delta_{k,j}.$$

Exercise 3:
Convolution theorems.

1.

$$c_k(f *_p g) = c_k \left(\int_{\mathbb{T}} f(t) g(x-t) dt \right)$$

$$= \int_{\mathbb{T}} \int_{\mathbb{T}} f(t) g(x-t) e^{-2\pi i k x} dt dx$$

$$= \int_{\mathbb{T}} f(t) e^{-2\pi i k t} \left[\int_{\mathbb{T}} g(x-t) e^{-2\pi i k (x-t)} dx \right] dt$$

$$= c_k(f) c_k(g).$$

$$(\mathbf{c}\,(f) *_d \mathbf{c}\,(g))_k = \sum_{l \in \mathbb{Z}} c_l\,(f)\,c_{(k-l)}\,(g)$$

$$= \sum_{l \in \mathbb{Z}} \int_{\mathbb{T}} f\,(t)\,\mathrm{d}t \int_{\mathbb{T}} g\,(x)\,\mathrm{e}^{-2\pi\mathrm{i}(k-l)x}\mathrm{d}x$$

$$= \int_{\mathbb{T}} f\,(t) \left[\sum_{l \in \mathbb{Z}} \left(\int_{\mathbb{T}} g\,(x)\,\mathrm{e}^{-2\pi\mathrm{i}(k-l)x}\mathrm{d}x \right) \mathrm{e}^{2\pi\mathrm{i}(k-l)t} \right] \mathrm{e}^{-2\pi\mathrm{i}kt}\mathrm{d}t$$

$$= \int_{\mathbb{T}} f\,(t) \left[\sum_{l \in \mathbb{Z}} c_{k-l}\,(g)\,\mathrm{e}^{2\pi\mathrm{i}(k-l)t} \right] \mathrm{e}^{-2\pi\mathrm{i}kt}\mathrm{d}t$$

$$= \int_{\mathbb{T}} f\,(t)\,g\,(t)\,\mathrm{e}^{-2\pi\mathrm{i}kt}\mathrm{d}t$$

$$= c_k\,(fg)\,.$$

$$\hat{f}_k\hat{g}_k = \sum_{j=-n/2}^{n/2-1} f_j\mathrm{e}^{-2\pi\mathrm{i}kj/n} \sum_{l=-n/2}^{n/2-1} g_l\mathrm{e}^{-2\pi\mathrm{i}kl/n}$$

$$= \sum_{j=-n/2}^{n/2-1} f_j\mathrm{e}^{-2\pi\mathrm{i}kj/n} \sum_{l=-n/2}^{n/2-1} g_{l-j}\mathrm{e}^{-2\pi\mathrm{i}k(l-j)/n}$$

$$= \sum_{l=-n/2}^{n/2-1} \left[\sum_{j=-n/2}^{n/2-1} f_j g_{l-j} \right] \mathrm{e}^{-2\pi\mathrm{i}kl/n}$$

$$= (\mathbf{F}_n\,(\mathbf{f} *_c \mathbf{g}))_k\,.$$

Exercise 4:

3.
```
function f=ndft(x,f_hat)

M=length(x);
freq=(-N/2):(N/2-1);
f=zeros(M,1);
for k=1:N
  f=f+f_hat(k).*exp(-2*pi*i*x*freq(k));
end;
```

Exercise 5:
Owing to Exercise 3.3 and 2.4, we have

$$\mathbf{G} = \tilde{\mathbf{F}}_n^H \mathrm{diag}(\tilde{\mathbf{F}}_n \mathbf{g}) \tilde{\mathbf{F}}_n$$

Fast matrix vector multiplication with a Toeplitz matrix can be realised by

```
function b=fast_toeplitz(c,r,x)

c1=[c;flipud(r(2:end))];
b=(size(c,1)+size(r,1)-1)*fft(ifft(c1).*ifft([x;zeros(size(c))]));
b=b(1:size(c,1));
```

Exercise 6:
The Taylor expansion based nfft can be realised by

```
function b=taylor_nfft(x,f_hat,sigma,m)

M=length(x);
n=sigma*2^ceil(log2(N));
sigma=n/N;
f=zeros(size(x));
freq=-2*pi*i*(-(N/2):(N/2-1))';
ix=round(n*(x+0.5));
dx=x-(ix/n-0.5);
ix=mod(ix,n)+1;
for l=0:m
  g_hat=[zeros((n-N)/2,1);f_hat.*(freq.^l);zeros((n-N)/2,1)];
  g=fftshift(fft(fftshift(g_hat)));
  f=f+g(ix).*(dx.^l)/prod(1:l);
end;
```

Exercise 7:
Images
   1a   2a   3a
   4a   5a   6a

Spectra
   3b   2b   6b
   5b   4b   1b