WAVE FOCUSING IN ONE DIMENSION

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(joint with J. Rose of Iowa State U)

Plasma-wave eq (PWE)

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- time-domain approach
- assume V has no bound states
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- scattering sol behaving like a plane wave as $t \to -\infty$

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  $g_l(x,t) = 0$ for $x > t$, causality

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- Example 1: wave probing with Dirac-delta wavefront

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Aktosun and Rose, *J. Math. Phys.* **43** (2002) 247–266

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- the tail starts reappearing
- the tail at any moment is determined by V

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- $g_{\mathrm{l}}(x, t; x_0) = 0$ for $x > t + x_0$, causality
- $g_{\mathrm{l}}(x, \cdot; x_0) \in L^2(\mathbf{R})$ if V has no bound states

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  how to construct $V(x_0)$ from $g_{\mathrm{l}}(x, t; x_0)$

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SE: $\quad \dfrac{d^2\psi}{dx^2} + k^2\psi = V(x)\, \psi, \quad x \in \mathbf{R}.$

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- $L(k)$, $R(k)$ left and right reflection coeff
- $T(k)$ transmission coeff, with simple poles at $i\kappa_j$

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Marchenko method: $\quad V(x) = 2\dfrac{dK(x,x^-)}{dx}$

Probing (nonfocusing) sols to PWE

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- if $V \equiv 0$ for $x < 0$, then

$$u_{\mathrm{l}}(x,t) = \delta(x-t) + I(-x-t), \quad x < 0$$

$$\hat{I}(t) := \hat{L}(-t) + \sum_{j=1}^{N} c_{\mathrm{r}j}^2\, e^{\kappa_j t} \text{ impulse response}$$

- inverse problem when $V \equiv 0$ for $x < 0$

find $V(x)$ for $x \in (0, x_1)$ from $\hat{I}(t)$ for $t \in (0, 2x_1)$

Marchenko method: $\quad V(x) = 2\dfrac{dK(x,x^-)}{dx}$

$$K(x,y) + I(x+y) + \int_0^x dz\, K(x,z)\, I(z+y) = 0, \quad 0 < y < x$$

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$$\frac{1}{4\pi}\int_{-\infty}^{\infty} dk\left[\psi_{\mathrm{l}}(k,x)\,\overline{\psi_{\mathrm{l}}(k,x_0)}+\psi_{\mathrm{r}}(k,x)\,\overline{\psi_{\mathrm{r}}(k,x_0)}\right] + \sum_{j=1}^{N}\varphi_j(x)\,\varphi_j(x_0)=\delta(x-x_0)$$

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$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,T(k)\,f_{\mathrm{l}}(k,x)\,f_{\mathrm{r}}(k,x_0)+\sum_{j=1}^{N}\varphi_j(x)\,\varphi_j(x_0)=\delta(x-x_0)$$

Focusing sols to PWE

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- relation to Marchenko method

Recovery of $V(x_0)$ by focusing to x_0

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If V is continuous at x_0, then for any $t \in \mathbf{R}$

$$V(x_0) = V(x_0 + t) + 2\, g_l(x_0^- + t, t; x_0)^2 + 4\, \frac{\partial g_l(x_0^- + t, t; x_0)}{\partial x}$$

Some generalizations

- radial SE: $u_{xx} - u_{tt} = V(x)\,u, \quad x > 0, \quad t \in \mathbf{R}$

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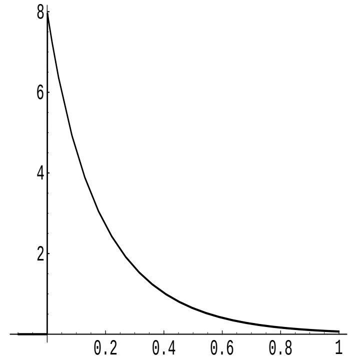
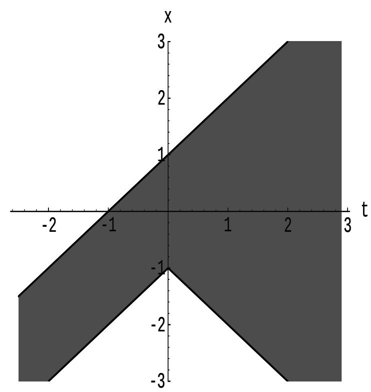
- variable-speed wave eq: $u_{xx} - \dfrac{1}{c(x)^2}\,u_{tt} = V(x)\,u, \quad x, t \in \mathbf{R}$

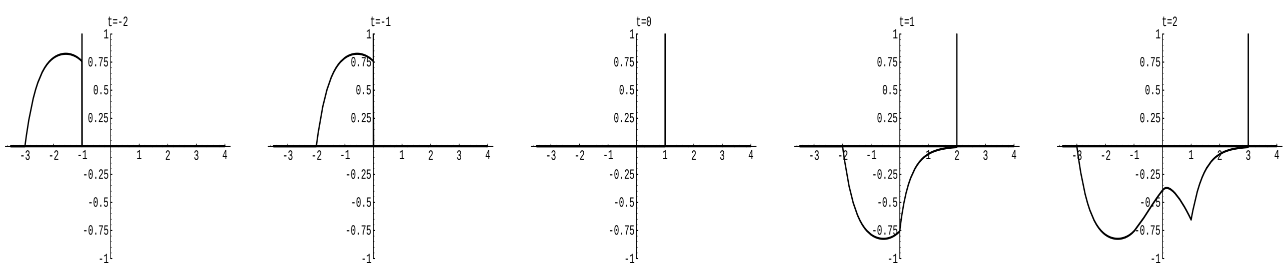
Example 2: focusing within nonhomogeneity



Potential V, vanishing when $x < 0$

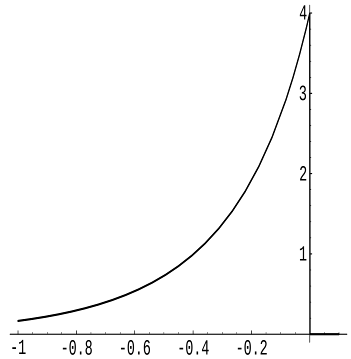


Support of u_{l}, with $x_0 = 1$

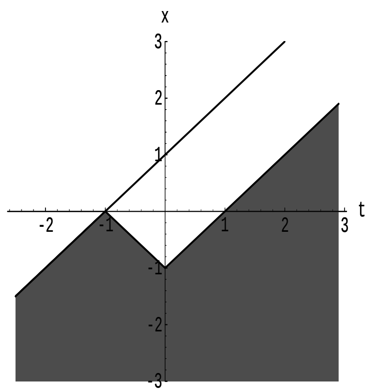


u_{l} with $x_0 = 1$ at $t = -2, -1, 0, 1, 2.$

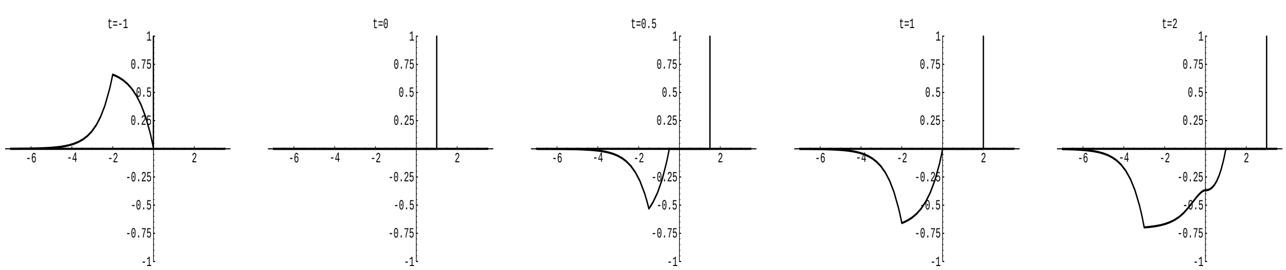
Example 3: focusing behind nonhomogeneity



Potential V, vanishing when $x > 0$

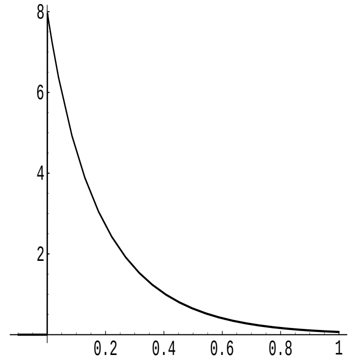


Support of $u_{\rm l}$, with $x_0 = 1$

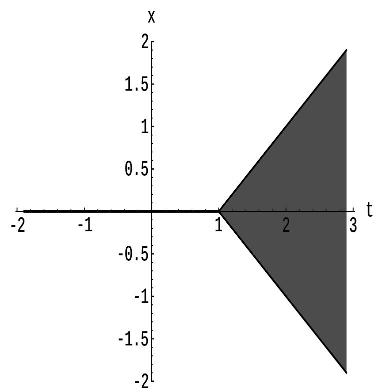


$u_{\rm l}$ with $x_0 = 1$ at $t = -1, 0, 0.5, 1, 2$.

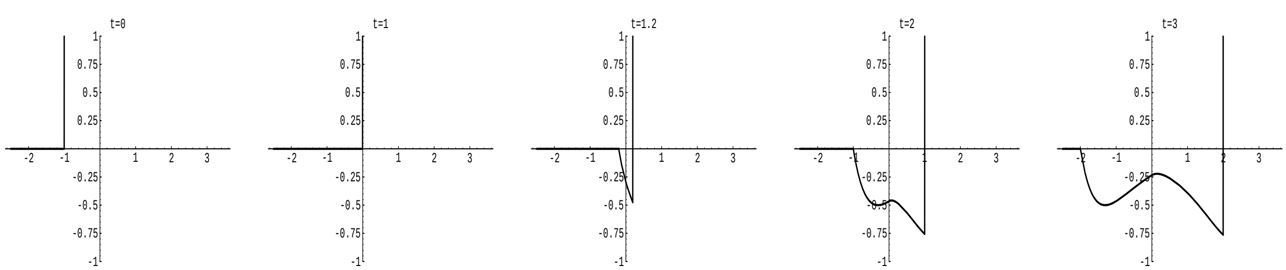
Example 4: focusing in front of nonhomogeneity



Potential V, vanishing when $x < 0$

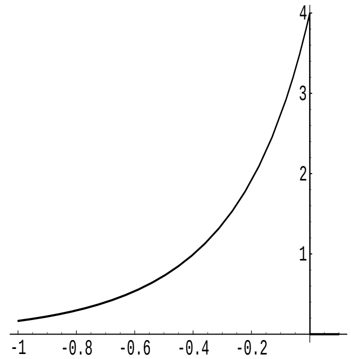


Support of u_{l}, with $x_0 = -1$

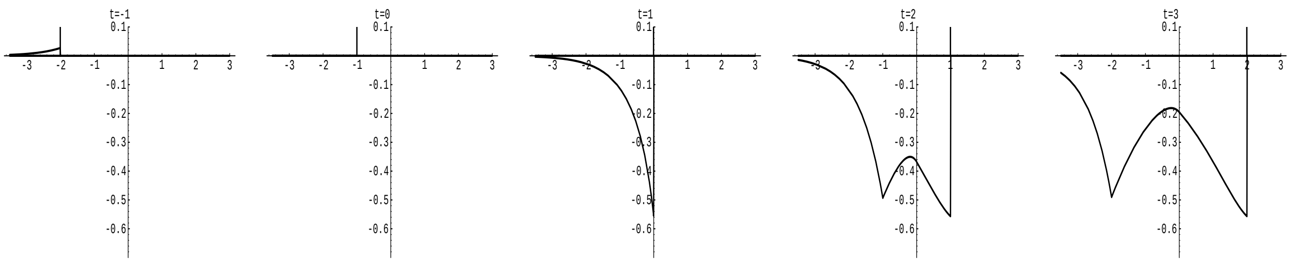


u_{l} with $x_0 = -1$ at $t = 0, 1, 1.2, 2, 3.$

Example 5: focusing within nonhomogeneity

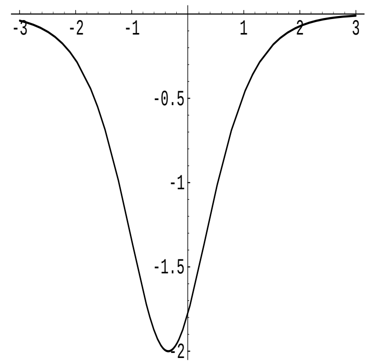


Potential V, vanishing when $x > 0$

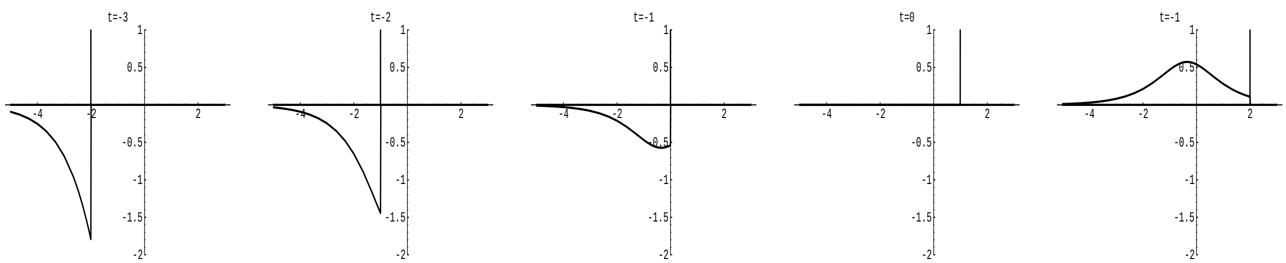


u_{l} with $x_0 = -1$ at $t = -1, 0, 1, 2, 3$.

Example 6: focusing with bound states

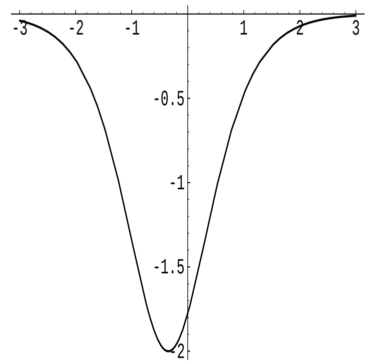


Potential V, nonvanishing, negative everywhere



u_1 with with $x_0 = 1$ at $t = -3, -2, -1, 0, 1$.

Example 7: focusing with bound states



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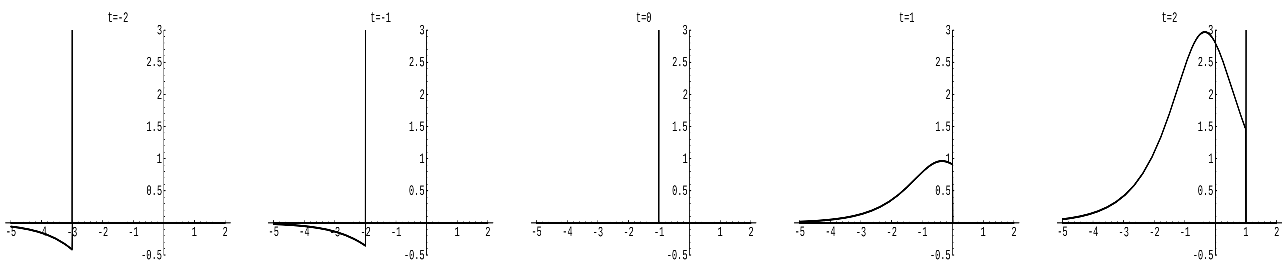

u_1 with $x_0 = -1$ at $t = -2, -1, 0, 1, 2$.

Example 8: focusing with Dirac delta distribution

Example 9: focusing to edge of potential

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  (focusing sol to $x_0 = 0$) = (probing physical sol)