Solvability Complexity Index (=SCI) and Towers of Algorithms

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Aalto SCI

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  on the Solvability Complexity Index,
- J. Ben-Artzi, A. Hansen, O. Nevanlinna , M. Seidel

Definition of a Tower

PROBLEM

Ω: primary set, e.g $\mathcal{B}(\ell^2(\mathbb{N}))$
Λ: evaluation set, e.g. $f_{ij} : A \mapsto < Ae_i, e_j >$ for $A \in \mathcal{B}(\ell^2(\mathbb{N}))$
$\mathcal{M}$: metric space
Ξ: problem function $\Omega \to \mathcal{M}$, such as $\sigma(A)$ for $A \in \mathcal{B}(\ell^2(\mathbb{N}))$

TOWER

$\Xi(A) = \lim_{n_k \to \infty} \Gamma_{n_k}(A)$
$\Gamma_{n_k}(A) := \lim_{n_{k-1} \to \infty} \Gamma_{n_k, n_{k-1}}(A)$
.....
.....
.....
$\Gamma_{n_k,..,n_2}(A) := \lim_{n_1 \to \infty} \Gamma_{n_k,..,n_2,n_1}(A)$

Matrices first

$A \in B(\mathbb{C}^n)$ solve for $\pi_A(z) = 0$

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- $n \leq 3$: generally convergent rational iteration exists (McMullen 1987)
- $n \leq 5$: a tower of generally convergent rational iterations (Doyle, McMullen 1989)
- $n > 5$: no such towers (Doyle, McMullen 1989)

Matrices continues

radicals, $z \mapsto |z|$ available, then convergent iterations exist for solving roots of polynomials

input finite: the complex coefficients of the polynomial

Computabilities...

"Turing view": problem computable if a computing device exists which solves the problem

Computation in the limit and higher hierarchies

BSS (Blum, Shub, Smale) $\mathbb{R}$-machine model

IBC (infromation based complexity) uses BSS, "tractability"

constructivism, computability on $\mathbb{Z}$ and within computable numbers

Any compact can be spectrum

Represent compact $K \subset \mathbb{C}$ from outside:

$$K = \bigcap K_n$$

where

$$\cdots \subset K_{n+1} \subset K_n \subset \cdots$$

and testing $z \notin K_n$ "easy"

Any compact can be spectrum, so look at Julia sets

We first look at the Julia set $\mathcal{J}$ for the quadratic polynomial $z^2 + 4$.

Consider the question

$$z \in \mathcal{J} \quad ?$$

Then the corresponding question for the spectrum $\sigma(A)$ is

$$\lambda \in \sigma(A) \quad ?$$

The natural formulation of these questions is, can you decide whether the answer is yes or no?

2.1 Julia set $\mathcal{J}$ for $z^2 + 4$

Let

$$p(z) = z^2 + 4$$

Iterate

$$z_{n+1} = p(z_n)$$

If $z_n \to \infty$ then $z_0 \notin \mathcal{J}$.
Note that if $|z_k| > 1 + \sqrt{5}$ for some k, then $|z_{k+1}| > 2|z_k|$ and then $z_n \to \infty$.
For this $p(z)$ the Julia set is homeomorphic to a Cantor set.
Observe that $\mathbb{C} \setminus \mathcal{J}$ is open.

S. Smale and coworkers: $\mathcal{J}$ is not decidable
("semidecidable")

Computation in the limit...

Output as follows:

if $|z_k| \leq 1 + \sqrt{5}$, then $Out(k) = 1$
if $|z_k| > 1 + \sqrt{5}$, then $Out(k) = 0$.

So depending on the initial value we obtain sequences of the form

$$1, 1, \ldots, 1, 0, 0, 0 \ldots$$

and

$$1, 1, 1, \ldots$$

In either case the limit exists; and then you (would) know

Similar question for the spectrum in abstract Banach algebra

Consider the subalgebra generated by just one element a (say, in Banach algebra $\mathcal{A}$). Then the spectrum within the subalgebra is $\mathit{fill}(\sigma(a))$.
If we are allowed to produce polynomials of a and compute their norms but inverting is not allowed, then:

The question

$$\lambda \notin \mathit{fill}(\sigma(a))$$

is semidecidable as follows:

If answer positive: finite termination with sure answer, while

if negative, you will never know (the one you look after does not exist)

What exists is easier to find!

Conclude: Think positive, construct the resolvent

$$\mathbb{C} \setminus \mathit{fill}(\sigma(A)) \to B(X)$$

$$\lambda \mapsto (\lambda - A)^{-1}$$

instead!

Get a multicentric holomorphic calculus - but not during this talk...

Computation in the limit

Example

Let A be diagonal operator in $\ell_2(\mathbb{N})$ such that $a_{ii} \in \{0, 1\}$.
Input information: read one diagonal element in time, in a fixed enumeration.
Then

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- $\sigma_{ess}(A) \neq \emptyset$: this can also be build in
- $1 \in \sigma(A)$: this cannot be be computed except at the limit
- $1 \in \sigma_{ess}(A)$ this needs "two limits", i.e. a "tower"

How to get the answers

$1 \in \sigma(A)$

- define function for each n

$$\Gamma_n(A) = 1, \text{ if } \sum_{i=1}^{n} a_{ii} > 0,$$
$$0, \text{ otherwise}$$

and set

$$\Gamma(A) = \lim_{n\to\infty} \Gamma_n(A).$$

Then, answer is "yes", when $\Gamma(A) = 1$

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- Using quantifiers: $\exists n \; (\sum_{i=1}^{n} a_{ii} > 0)$

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$$\Gamma_{m,n}(A) = 1, \text{ if } \sum_{i=1}^{n} a_{ii} > m,$$
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- With two quantifiers: $\forall m \, \exists n \, (\sum_{i=1}^{n} a_{ii} > m)$

Another example

We define $A \in B(\ell_2(\mathbb{N}))$ using diagonal blocks:

$$A = \bigoplus_{j=1}^{\infty} A_{k(j)}$$

where A_k are $k \times k$-matrices with number 1's in the corners, like

$$A_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and $k(j) \geq 2$ is some sequence. Thus, A is algebraic,
$\sigma(A) = \sigma_{ess}(A) = \{0, 2\}$.

Constructivism, computability

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- then one can "tailor" a computing machine which computes the spectrum in a finite number of operations

Constructivism, computability 2

- The operator

$$B = \bigoplus_{j=1}^{\infty} \beta_j A_{k(j)}$$

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- Then,

- the spectrum is computable.

Constructivism, computability 3

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- In this theory effectively described bounded self-adjoint operators have computable spectra
- but
- there exists an effectively determined bounded non-selfadjoint operator which has a noncomputable real as an eigenvalue.

Computability; towers

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- algorithm given for a class of operators $A = (a_{ij}) \in B(\ell_2(\mathbb{N}))$
- can be adaptive but only based on what it has already computed
- input enters by reading one element a_{ij} at a time

Example

Then for each such fixed algorithm one can "tailor" a sequence $\{k(j)\}$ such that the algorithm keeps the number 1 as a candidate for the spectrum for the operator

$$A = \bigoplus_{j=1}^{\infty} A_{k(j)}$$

Example continues

In fact, the algorithm would be made to see a finite matrix consisting of diagonal blocks $A_{k(j)}$ and a block having just one nonzero element

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{pmatrix}$$

Thus,

- just one limit would give wrong answer

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Thus,

- just one limit would give wrong answer
- but limits on two levels work

Idea of a tower for the example

Let $A = A^* \in B(\ell_2(\mathbb{N}))$ and denote by $\gamma_{m,n}(t)$ the smallest singular value of the $n \times m$- matrix $A_{nm}(t)$ representing

$$P_n(A - tI)$$

when restricted to the range of P_m: $P_m \ell_2(\mathbb{N})$.

Example continues

Applied to

$$A = \bigoplus_{j=1}^{\infty} A_{k(j)}$$

the matrices $A_{nm}(t)$ shall consist of a finite number of square blocks and possibly one rectangle which for fixed m and all large enough n is of the form

$$\begin{pmatrix} 1-t & 0 & 0 & \cdot \\ 0 & -t & 0 & \cdot \\ \cdot & \cdot & -t & \cdot \\ \cdot & & & \\ \boxed{1} & & & \\ 0 & & & \\ \cdot & & & \end{pmatrix}$$

Proto for the tower at the Example

Since $\boxed{1}$ appears, the rectangle has full rank at $t = 1$.

- For example

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$$\forall m \, \exists n_m \, \{n > n_m \implies \Gamma_{m,n}(A) = \{0, 2\}\}$$

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- $\Gamma(A) = \lim_{m\to\infty} \Gamma_m(A) = \{0, 2\} = \sigma(A)$.

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- Limits in the Hausdorff distance between compact sets in $\mathbb{C}$

$$\mathrm{dist}_H(K, M) = \max\{\sup_{z\in K} \inf_{w\in M} |z-w|, \sup_{w\in M} \inf_{z\in K} |z-w|\}$$

Definition of Tower

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$\mathcal{M}$: metric space
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TOWER

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Definition of SCI

k = height of tower

SCI = $\min k$ of towers solving the problem for arbitrary $A \in \Omega$

SCI $= 3$ for bounded operators, $\Xi = \sigma(A)$

- a tower of height 3 works for all $A \in \mathcal{B}(\ell_2(\mathbb{N}))$

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- a tower of height 3 works for all $A \in \mathcal{B}(\ell_2(\mathbb{N}))$
- we have a construction which shows that three limits are needed in general

SCI=2, subsets of $\mathcal{B}(\ell_2(\mathbb{N}))$, for $\sigma(A)$

- Self-adjoint operators $A^* = A$, and further

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- A is similar to normal: $A = TNT^{-1}$ where N is normal with a known constant C such that $\|T\|\|T^{-1}\| \leq C$ (but the decomposition is not known), so that

$$\|(\lambda - A)^{-1}\| \leq \frac{C}{\mathrm{dist}(\lambda, \sigma(\mathrm{A}))}.$$

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- there is a known function g such that for $\lambda \notin \sigma(A)$

$$\|(\lambda - A)^{-1}\| \leq 1/g(\mathrm{dist}(\lambda, \sigma(\mathrm{A}))).$$

Dispersion known, again lowers the index

Dispersion: there is a known function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\max\{\|(I - P_{f(n)})AP_n\|, \|P_n A(I - P_{f(n)})\|\} \to 0, \text{ as } n \to \infty$$

For example, if bandwidth $= d$ one has $f(n) = n + d$.

If f is known for A, then $SCI = 2$

and if both resolvent control g and dispersion function f are known, then SCI=1.

SCI=1 for $\sigma(A)$ with $A \in \mathcal{B}(\ell_2(\mathbb{N}))$ compact

So, this is the situation in which computing eigenvalues of finite sections $A_n = (a_{ij})_{i,j \leq n}$ and studing their limit behavior is ok.

Computing the essential spectrum $\sigma_{ess}(A)$

Again $A \in \mathcal{B}(\ell^2(\mathbb{N}))$

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Computing the essential spectrum $\sigma_{ess}(A)$

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- If we only know that A is bounded , then SCI=3.
- If additionally both f and g are known, then SCI=2
- if we know that A is compact, then SCI=0, since $\sigma_{ess}(A) = \{0\}$.

Schrödinger as an example

Let

$$H = -\Delta + V \text{ where } V : \mathbb{R}^d \to \mathbb{C}.$$

- If V is bounded and in a certain total variation space. The evaluation functions are pointwise evaluations $x \mapsto V(x)$. Then SCI ≤ 2.

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- If V is bounded and in a certain total variation space. The evaluation functions are pointwise evaluations $x \mapsto V(x)$. Then SCI ≤ 2.
- If V is continuous, $|V(x)| \to \infty$ as $\|x\| \to \infty$ and its values are in a sector with opening less than π and including the positive real axis, then the resolvent of H is compact and SCI$=1$.

References

Arithmetic hierarchy

J. Knight. The Kleene-Mostowski hierarchy and the Davis-Mostowski hierarchy. In Andrzej Mostowski and Foundational Studies. IOS Press, 2008.

J. R. Shoenfield. On degrees of unsolvability. Ann. of Math. (2), 69:644653, 1959.

Baire functions

Baire, R. (1905), Leons sur les fonctions discontinues, Professees au College de France, Gauthier-Villars

References

Curtis McMullen

S. Smale. The work of Curtis T McMullen. In Proceedings of the International Congress of Mathematicians I, Berlin, Doc. Math. J. DMV, pages 127132. 1998.

BSS-model and Julia sets

L. Blum, F. Cucker, M. Shub, S. Smale, and R. M. Karp. Complexity and real computation. Springer, New York, Berlin, Heidelberg, 1998.

Computability in Analysis

Marian B.Pour-el, J.Ian Richards, Computability in Analysis and Physics, Springer 1989
[Second Main Theorem, p 128 and Theorem 5 (Noncomputable eigenvalues, p 132)]