The Method of Intrinsic Scaling

José Miguel Urbano
CMUC, University of Coimbra, Portugal
jmurb@mat.uc.pt

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Lecture Notes in Mathematics

José Miguel Urbano

The Method of Intrinsic Scaling

A Systematic Approach to Regularity for Degenerate and Singular PDEs

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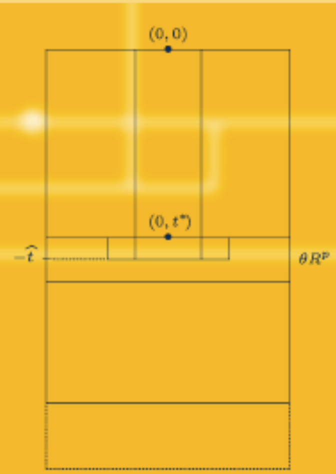


Springer

The parabolic p-Laplace equation

$$u_t - \mathrm{div}\, |\nabla u|^{p-2} \nabla u = 0$$

- Degenerate if $p>2$
- Singular if $1<p<2$
- Results are local but extend up to the boundary
- Theory allows for lower-order terms

Hilbert's 19th problem

- *Are solutions of regular problems in the Calculus of Variations always necessarily analytic?*

- Minimize the functional

$$\mathcal{I}[w] = \int_{\Omega} f\left[\nabla w(x)\right]\, dx$$

- The problem is regular if the Lagrangian is regular and convex

Euler-Lagrange equation

- A minimizer solves the corresponding Euler-Lagrange equation

$$\left(f_{\zeta_i}(\nabla u)\right)_{x_i} = 0$$

- and its partial derivatives solve the elliptic PDE

$$\left(a_{ij}(x)\, u_{x_j}\right)_{x_i} = 0$$

with coefficients

$$a_{ij}(x) := f_{\zeta_i \zeta_j}\left(\nabla w^*(x)\right).$$

Schauder estimates

(bootstrapping...)

$$w^* \in C^{1,\alpha} \Rightarrow a_{ij} := f_{\zeta_i\zeta_j}(\nabla w^*) \in C^{0,\alpha} \Rightarrow v_k \in C^{1,\alpha}$$

$$\Downarrow$$

$$v_k \in C^{2,\alpha} \Leftarrow a_{ij} := f_{\zeta_i\zeta_j}(\nabla w^*) \in C^{1,\alpha} \Leftarrow w^* \in C^{2,\alpha}$$

$$\Downarrow$$

$$w^* \in C^{3,\alpha} \Rightarrow \qquad \cdots$$

A beautiful problem

- Direct methods give existence in H^1 (in the spirit of Hilbert's 20th problem)
- Around 1950, the problem was to go from

$$w^* \in H^1 \Rightarrow a_{ij} := f_{\zeta_i \zeta_j}(\nabla w^*) \in L^\infty$$

to

$$v_k \in C^{0,\alpha} \Rightarrow w^* \in C^{1,\alpha}$$

De Giorgi - Nash - Moser

- No use is made of the regularity of the coefficients

- Nonlinear approach

- *[...] it was an unusual way of doing analysis, a field that often requires the use of rather fine estimates, that the normal mathematician grasps more easily through the formulas than through the geometry.*

The quasilinear elliptic case

$$\operatorname{div}\, \mathbf{a}(x,u,\nabla u) = 0$$

- Structure assumptions (p>1)

$$\begin{cases} \mathbf{a}(x,u,\nabla u)\cdot\nabla u \geq C_0\,|\nabla u|^p - C \\ |\mathbf{a}(x,u,\nabla u)| \leq C\left(|\nabla u|^{p-1}+1\right), \end{cases}$$

- Prototype

$$\operatorname{div}\, |\nabla u|^{p-2}\nabla u = 0$$

From elliptic to parabolic

- Linear

$$u_t - \left(a_{ij}(x,t)\, u_{x_j}\right)_{x_i} = 0$$

- Quasilinear

$$u_t - \mathrm{div}\ \mathbf{a}(x,t,u,\nabla u) = 0$$

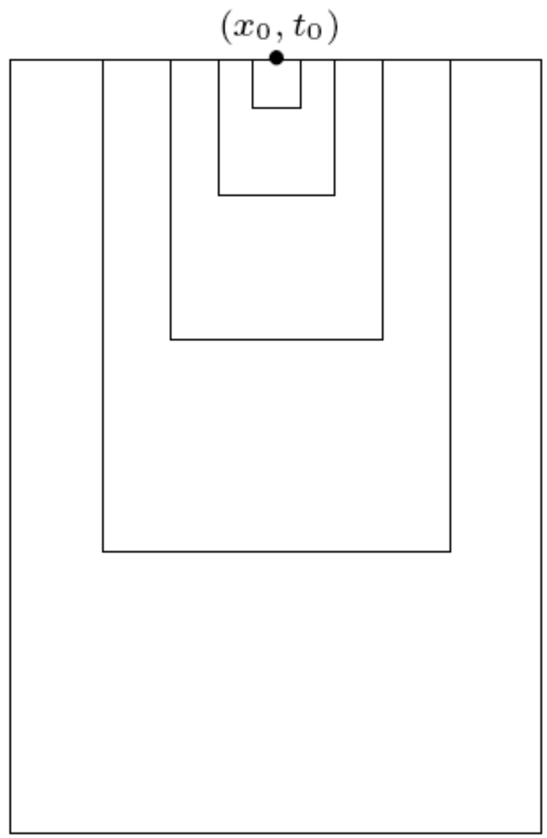
only for p=2

- Prototype

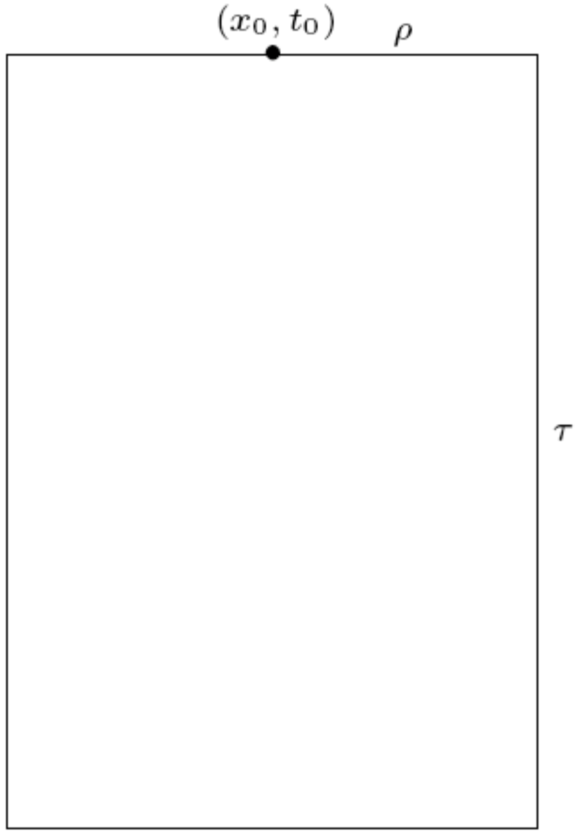
$$u_t - \mathrm{div}\ |\nabla u|^{p-2}\nabla u = 0$$

Measuring the oscillation

- iterative method
- measures the oscillation in a sequence of nested and shrinking cylinders
- based on (*homogeneous*) integral estimates on level sets - the building blocks of the theory
- nonlinear approach



The cylinders



- (x_0, t_0) is the vertex
- ρ is the radius
- τ is the height

notation:

$$(x_0, t_0) + Q(\tau, \rho) := K_\rho(x_0) \times (t_0 - \tau, t_0)$$

Energy estimates

$$\sup_{-\tau<t<0} \int_{K_\rho \times \{t\}} (u-k)_\pm^2 \, \zeta^p \, dx + \int_{-\tau}^0 \int_{K_\rho} |\nabla (u-k)_\pm \zeta|^p \, dx \, dt$$

$$\leq \int_{K_\rho \times \{-\tau\}} (u-k)_\pm^2 \, \zeta^p \, dx + C \int_{-\tau}^0 \int_{K_\rho} (u-k)_\pm^p \, |\nabla \zeta|^p \, dx \, dt$$

$$+p \int_{-\tau}^0 \int_{K_\rho} (u-k)_\pm^2 \, \zeta^{p-1} \, \zeta_t \, dx \, dt.$$

Recovering the homogeneity

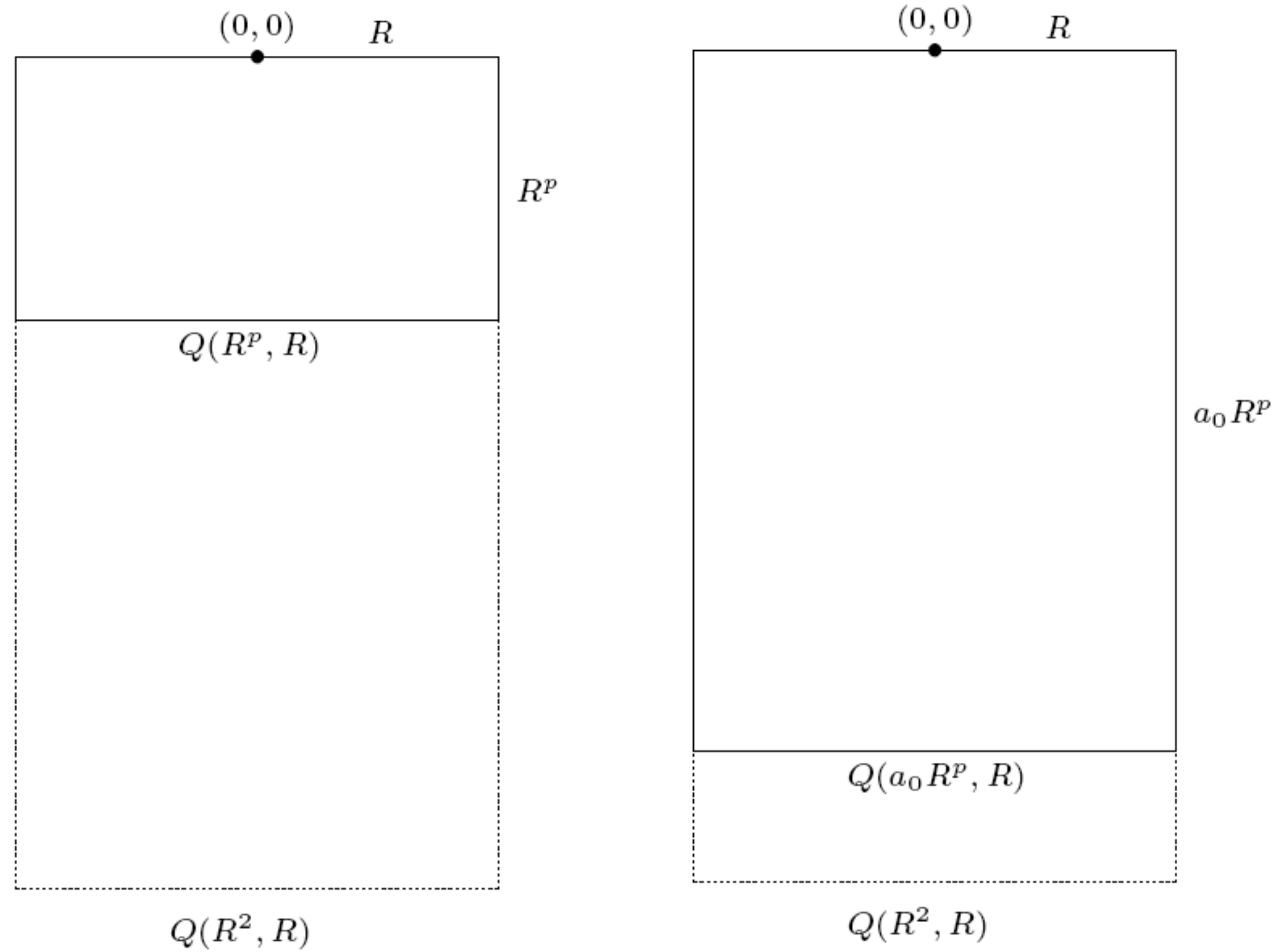
$$(u^{p-1})_t - \operatorname{div} |\nabla u|^{p-2} \nabla u = 0$$

is homogeneous; how does it compare with the p-Laplace equation?

$$\left(\frac{u}{c}\right)^{2-p} (u^{p-1})_t - \operatorname{div} |\nabla u|^{p-2} \nabla u = 0$$

scaling factor

Intrinsic scaling - DiBenedetto



The scaling factor

$$Q(a_0 R^p, R) = K_R(0) \times (-a_0 R^p, 0)$$

$$a_0 = \left(\frac{\omega}{2\lambda}\right)^{2-p}$$

scaling factor

$$\left(\frac{u}{c}\right)^{2-p} (u^{p-1})_t - \operatorname{div} |\nabla u|^{p-2} \nabla u = 0$$

Local weak solutions

- A local weak solution is a measurable function

$$u \in C_{\mathrm{loc}}\left(0,T;L^2_{\mathrm{loc}}(\Omega)\right) \cap L^p_{\mathrm{loc}}\left(0,T;W^{1,p}_{\mathrm{loc}}(\Omega)\right)$$

such that, for every compact K and every subinterval $[t_1,t_2]$,

$$\int_K u\varphi\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \left\{-u\varphi_t + |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\right\}\,dx\,dt = 0,$$

for all $\varphi \in H^1_{\mathrm{loc}}\left(0,T;L^2(K)\right) \cap L^p_{\mathrm{loc}}\left(0,T;W^{1,p}_0(K)\right)$.

An equivalent definition

- A local weak solution is a measurable function

$$u \in C_{\mathrm{loc}}\left(0,T;L^2_{\mathrm{loc}}(\Omega)\right) \cap L^p_{\mathrm{loc}}\left(0,T;W^{1,p}_{\mathrm{loc}}(\Omega)\right)$$

such that, for every compact K and every 0<t<T-h,

$$\int_{K\times\{t\}} \left\{(u_h)_t\,\varphi + \left(|\nabla u|^{p-2}\nabla u\right)_h \cdot \nabla\varphi\right\} dx = 0,$$

for all $\varphi \in W^{1,p}_0(K)$.

Energy estimates

- $(x_0,t_0) = (0,0)$
- smooth cutoff function in $Q(\tau,\rho) \subset \Omega_T$ such that

$$|\nabla\zeta| < \infty \qquad \text{and} \qquad \zeta(x,t) = 0\ , \quad x \notin K_\rho.$$

$$\sup_{-\tau<t<0} \int_{K_\rho\times\{t\}} (u-k)_\pm^2\, \zeta^p\, dx + \int_{-\tau}^{0}\int_{K_\rho} |\nabla(u-k)_\pm\zeta|^p\ dx\, dt$$

$$\leq \int_{K_\rho\times\{-\tau\}} (u-k)_\pm^2\, \zeta^p\, dx + C\int_{-\tau}^{0}\int_{K_\rho} (u-k)_\pm^p\, |\nabla\zeta|^p\, dx\, dt$$

$$+p\int_{-\tau}^{0}\int_{K_\rho} (u-k)_\pm^2\, \zeta^{p-1}\, \zeta_t\, dx\, dt.$$

The intrinsic geometry

- starting cylinder

$$Q(R^2, R) \subset \Omega_T$$

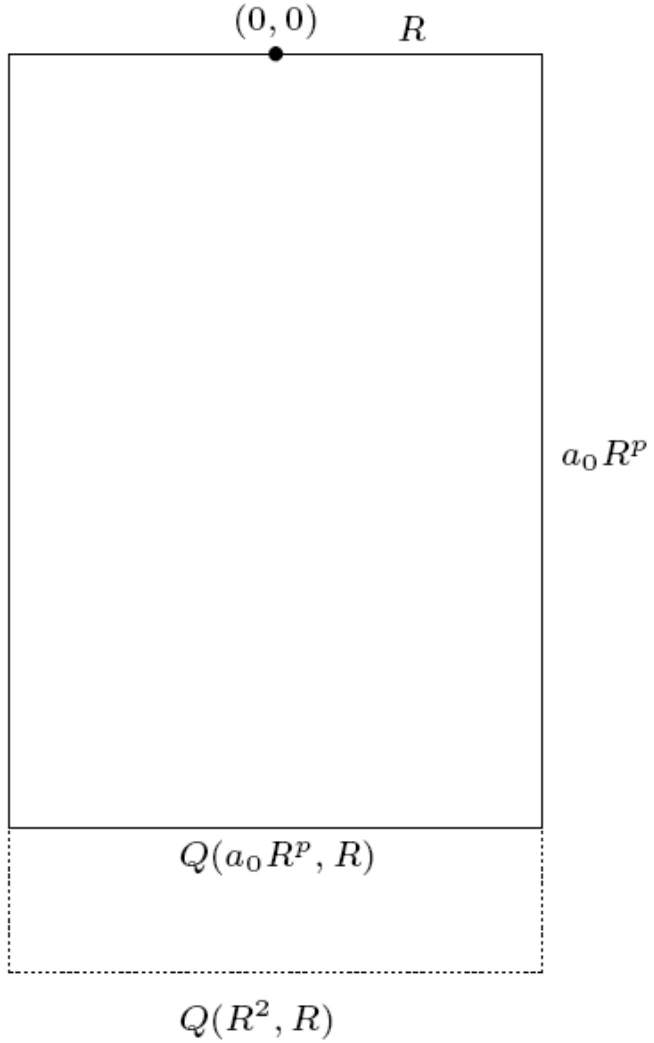
- measure the oscillation there

- construct the rescaled cylinder

$$Q(a_0 R^p, R)$$

- the scaling factor is

$$a_0 = \left(\frac{\omega}{2\lambda}\right)^{2-p}$$



Subdividing the cylinder

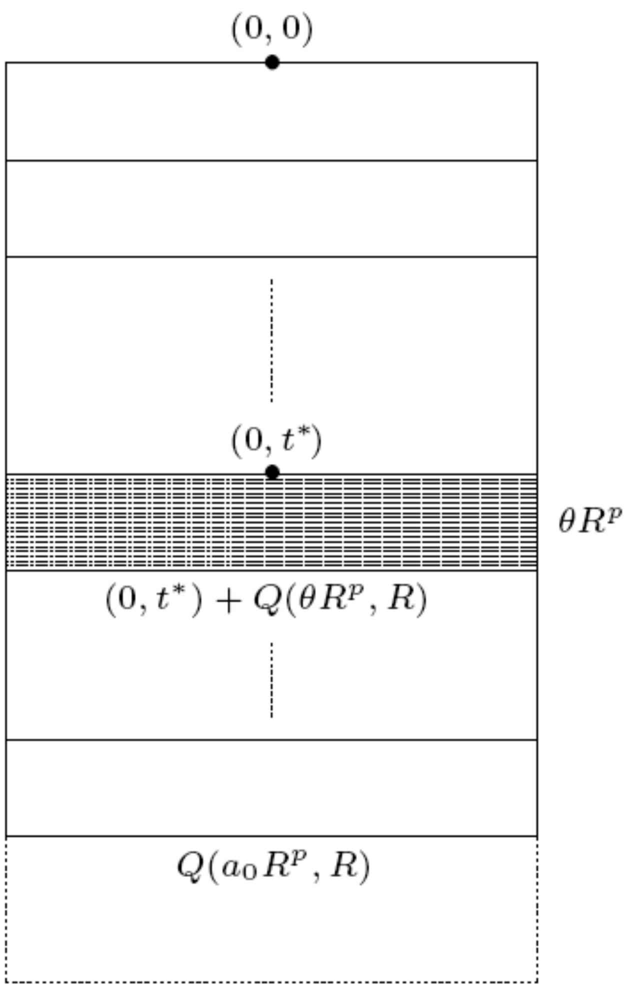
- subcylinders

$$(0, t^*) + Q(\theta R^p, R)$$

- with

$$\theta = \left(\frac{\omega}{2}\right)^{2-p}$$

- division in an integer number of congruent subcylinders



The first alternative

For a constant $\nu_0 \in (0,1)$, depending only on the data, there is a cylinder

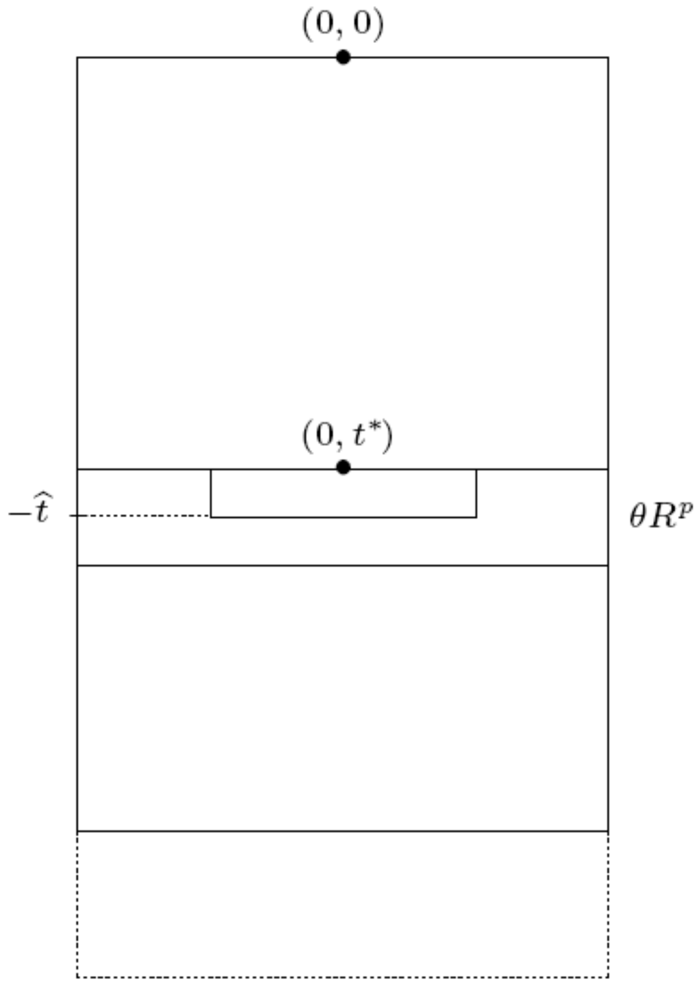
$$(0, t^*) + Q(\theta R^p, R)$$

such that

$$\frac{\left|\left\{(x,t) \in (0,t^*) + Q(\theta R^p, R) \ : \ u(x,t) < \mu^- + \frac{\omega}{2}\right\}\right|}{|Q(\theta R^p, R)|} \leq \nu_0$$

Then

$$u(x,t) > \mu^- + \frac{\omega}{4}\,, \quad \text{a.e. in} \ \ (0,t^*) + Q\left(\theta\left(\frac{R}{2}\right)^p, \frac{R}{2}\right).$$


$(0,0)$
$(0,t^*)$
$-\widehat{t}$
θR^p

Proof - getting started

- Sequence of *radii*

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \qquad n = 0, 1, \ldots$$

- Sequence of nested and shrinking cylinders

$$Q(\theta R_n^p, R_n)$$

- Sequence of cutoff functions $0 \le \zeta_n \le 1$ such that

$$\zeta_n = 1 \quad \text{in} \quad Q\left(\theta R_{n+1}^p, R_{n+1}\right); \qquad \zeta_n = 0 \quad \text{on} \quad \partial_p Q\left(\theta R_n^p, R_n\right);$$

$$|\nabla \zeta_n| \le \frac{2^{n+1}}{R}; \qquad 0 \le (\zeta_n)_t \le \frac{2^{p(n+1)}}{\theta R^p}.$$

Proof - using the estimates

- Sequence of *levels*

$$k_n = \mu^- + \frac{\omega}{4} + \frac{\omega}{2^{n+2}}\,, \qquad n = 0, 1, \ldots$$

- Energy inequalities over these cylinders for

$$(u - k_n)_-$$

and $\zeta = \zeta_n$.

Proof - the functional framework revealed

- Change the time variable: $\bar{t} = t/\theta$

$$\overline{u}(\cdot, \bar{t}) := u(\cdot, t) \ , \qquad \overline{\zeta_n}(\cdot, \bar{t}) := \zeta_n(\cdot, t)$$

- The right functional framework:

$$V_0^p(\Omega_T) = L^\infty\left(0, T; L^p(\Omega)\right) \cap L^p\left(0, T; W_0^{1,p}(\Omega)\right)$$

$$\|u\|^p_{V^p(\Omega_T)} = \operatorname*{ess\ sup}_{0 \le t \le T} \|u(\cdot, t)\|^p_{p,\Omega} + \|\nabla u\|^p_{p,\Omega_T}$$

- A crucial embedding:

$$\|v\|^p_{p,\Omega_T} \le \gamma \mid |v| > 0 \mid^{\frac{p}{d+p}} \|v\|^p_{V^p(\Omega_T)}$$

Proof - a recursive relation

- Define

$$A_n = \int_{-R_n^p}^{0} \int_{K_{R_n}} \chi_{\{(\overline{u}-k_n)_->0\}} \, dx \, d\bar{t}$$

and obtain

$$\begin{aligned}
\frac{1}{2^{p(n+2)}} \left(\frac{\omega}{2}\right)^p A_{n+1} &= |k_n - k_{n+1}|^p \; A_{n+1} \\
&\leq \|(\overline{u} - k_n)_-\|^p_{p,Q(R^p_{n+1},R_{n+1})} \\
&\leq \left\|(\overline{u} - k_n)_- \overline{\zeta_n}\right\|^p_{p,Q(R^p_n,R_n)} \\
&\leq C \left\|(\overline{u} - k_n)_- \; \overline{\zeta_n}\right\|^p_{V^p(Q(R^p_n,R_n))} A_n^{\frac{p}{d+p}} \\
&\leq C \frac{2^{pn}}{R^p} \left(\frac{\omega}{2}\right)^p A_n^{1+\frac{p}{d+p}}.
\end{aligned}$$

Proof - fast geometric convergence

- Divide through by $\left|Q(R_{n+1}^p, R_{n+1})\right|$ to obtain

$$X_{n+1} \leq C\, 4^{pn}\, X_n^{1+\frac{p}{d+p}}$$

where $X_n = \dfrac{A_n}{|Q(R_n^p, R_n)|}$

- If $X_0 \leq C^{-\frac{d+p}{p}} 4^{-\frac{(d+p)^2}{p}} =: \nu_0$ then $X_n \longrightarrow 0.$

The role of logarithmic estimates

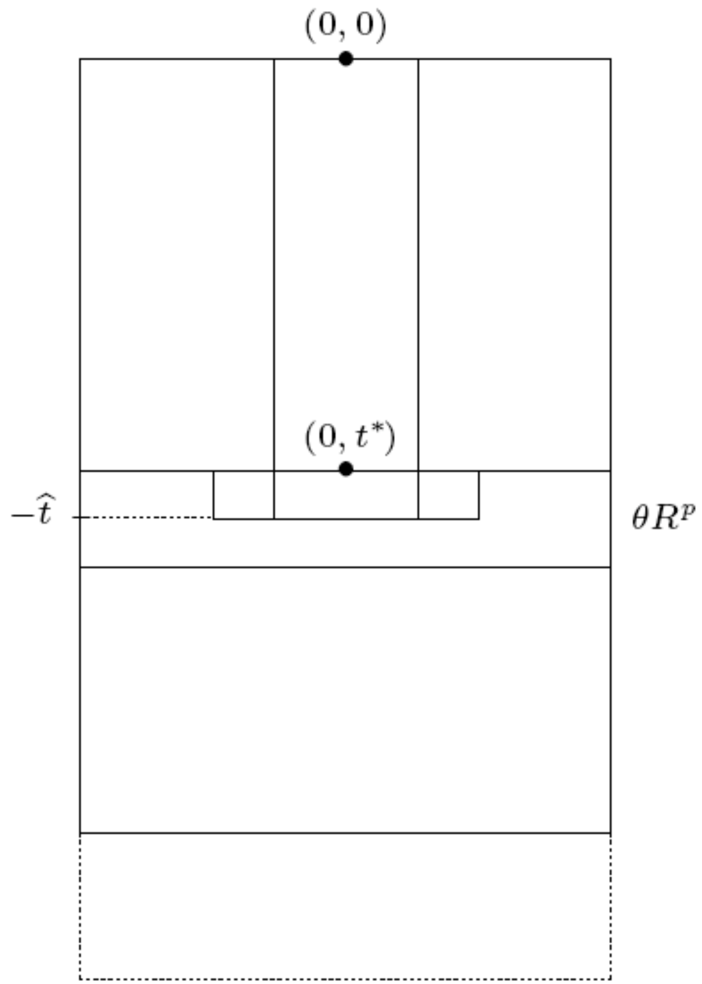
- get the conclusion for a *full* cylinder

$$Q(\tau, \rho)$$

- look at

$$-\widehat{t} := t^* - \theta \left(\frac{R}{2}\right)^p$$

as an *initial time*



Reduction of the oscillation

- There exists a constant $\sigma \in (0,1)$, depending only the data, such that

$$\operatorname*{ess\,osc}_{Q\left(\theta(\frac{R}{8})^p, \frac{R}{8}\right)} u \le \sigma\,\omega$$

The recursive argument

☑ There exists a positive constant C, depending only the data, such that, defining the sequences

$$R_n = C^{-n} R \qquad \text{and} \qquad \omega_n = \sigma^n \omega,$$

and constructing the family of cylinders

$$Q_n = Q(a_n R_n^p, R_n)\ , \qquad \text{with} \qquad a_n = \left(\frac{\omega_n}{2^\lambda}\right)^{2-p},$$

we have

$$Q_{n+1} \subset Q_n \qquad \text{and} \qquad \operatorname*{ess\ osc}_{Q_n} u \le \omega_n$$

The Hölder continuity

- There exist constants $\gamma>1$ and $\alpha \in (0,1)$, that can be determined a priori only in terms of the data, such that

$$\operatorname*{ess\,osc}_{Q(a_0\rho^p,\rho)} u \leq \gamma\,\omega\left(\frac{\rho}{R}\right)^{\alpha}$$

for all $0 < \rho \leq R$.

Generalizations

$$u_t - \operatorname{div} \mathbf{a}(x,t,u,\nabla u) = b(x,t,u,\nabla u) \qquad \text{in} \quad \mathcal{D}'(\Omega_T),$$

$$\begin{array}{ll} (\mathrm{A}_1) \qquad & \mathbf{a}(x,t,u,\nabla u)\cdot\nabla u \geq C_0|\nabla u|^p - \varphi_0(x,t); \\ (\mathrm{A}_2) \qquad & |\mathbf{a}(x,t,u,\nabla u)| \leq C_1|\nabla u|^{p-1} + \varphi_1(x,t); \\ (\mathrm{A}_3) \qquad & |b(x,t,u,\nabla u)| \leq C_2|\nabla u|^p + \varphi_2(x,t), \end{array}$$

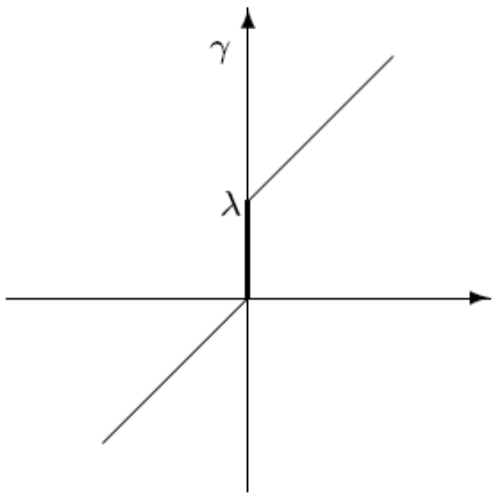
$$\varphi_0, \quad \varphi_1^{\frac{p}{p-1}}, \quad \varphi_2 \ \in \ L^{q,r}(\Omega_T)$$

$$\frac{1}{r} + \frac{d}{pq} \in (0,1).$$

Phase transitions

$$\gamma(u)_t - \mathrm{div}\,|\nabla u|^{p-2}\nabla u = 0$$

- Phase transition at constant temperature
- Nonlinear diffusion
- Degenerate if p>2
- Singular if 1<p<2
- Singular in time - maximal monotone graph



Regularize

- Regularization of the maximal monotone graph

$$\gamma_\epsilon(s) = s + \lambda H_\epsilon(s)$$

- Smooth approximation of the Heaviside function

$$H_\epsilon(s) = 0 \quad \text{if} \quad s \leq 0 \; ; \qquad H_\epsilon(s) = 1 \quad \text{if} \quad s \geq \epsilon$$

- Lipschitz, together with its inverse:

$$1 \leq \gamma_\epsilon'(s) \leq 1 + \lambda L_\epsilon, \quad s \in \mathbb{R}, \qquad L_\epsilon \equiv \mathcal{O}(\tfrac{1}{\epsilon})$$

Approximate solutions are Hölder

- They satisfy

$$u_t - \operatorname{div} |\nabla \beta_\epsilon(u)|^{p-2} \nabla \beta_\epsilon(u) = 0$$

with $u = \gamma_\epsilon(\theta_\epsilon)$.

- Structure assumptions:

$$\left| |\nabla \beta_\epsilon(u)|^{p-2} \nabla \beta_\epsilon(u) \right| \leq |\nabla u|^{p-1}$$

$$|\nabla \beta_\epsilon(u)|^{p-2} \nabla \beta_\epsilon(u) \cdot \nabla u \geq \left(\frac{1}{1 + \lambda L_\epsilon} \right)^{p-1} |\nabla u|^p$$

Idea of the proof

- Show the sequence of approximate solutions is
  - uniformly bounded
  - equicontinuous
- Obtain estimates that are independent of the approximating parameter

A new power in the energy estimates

$$\sup_{-\tau<t<0}\int_{K_\rho\times\{t\}}(\theta_\epsilon - k)_-^2\zeta^p\,dx + \int_{-\tau}^{0}\int_{K_\rho}|\nabla(\theta_\epsilon - k)_-\zeta|^p\,dx\,dt$$

$$\leq C\int_{K_\rho\times\{-\tau\}}(\theta_\epsilon - k)_-^{1}\zeta^p\,dx + C\int_{-\tau}^{0}\int_{K_\rho}(\theta_\epsilon - k)_-^p|\nabla\zeta|^p\,dx\,dt$$

$$+C\int_{-\tau}^{0}\int_{K_\rho}(\theta_\epsilon - k)_-^{1}\zeta^{p-1}\zeta_t\,dx\,dt$$

Three powers?

- The constants will depend on the oscillation - this makes the analysis compatible.

- Modulus of continuity is defined implicitly.

- The Hölder character is lost in the limit...

The intrinsic geometry

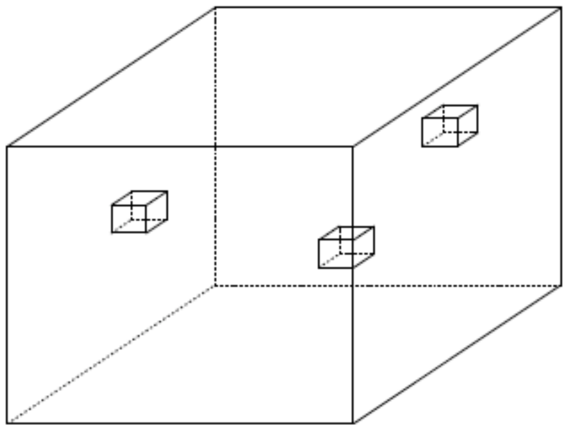
- starting cylinder
- measure the oscillation there
- construct the rescaled cylinder

$$Q\left(a_0 R^p, c_0 R\right)$$

- the scaling factors are

$$a_0 = \left(\frac{\omega}{A}\right)^{(1-p)(2-p)}$$

$$c_0 = \left(\frac{\omega}{B}\right)^{p-2}$$



$Q(a_0 R^p, c_0 R)$