Boundary Harnack inequalities for operators of p-Laplace type

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The p-Laplace operator

$G \subset \mathbf{R}^n$ bounded domain, $1 < p < \infty$.

u is p-harmonic in G provided $u \in W^{1,p}(G)$ and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0, \quad \forall \theta \in W_0^{1,p}(G).$$

If u is smooth and $\nabla u \neq 0$ in G,

$$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0 \text{ in } G.$$

Generic set-up: $w \in \partial G$, $0 < r < r_0$, u, v are positive p-harmonic functions in $G \cap B(w, 4r)$, u, v are continuous in $\bar{G} \cap B(w, 4r)$ and $u = 0 = v$ on $\partial G \cap B(w, 4r)$.

p-Harmonic functions in Lipschitz domains: our results

- Boundary Harnack inequality for positive p-harmonic functions vanishing on a portion of the boundary of a Lipschitz domain.
- $C^{0,\alpha}$-estimates for quotients of positive p-harmonic functions vanishing on a portion of the boundary of a Lipschitz domain.
- Resolution of the p-Martin boundary problem in convex, C^1-domains and in flat Lipschitz domains.
- Regularity of ∇u:

$$\log|\nabla u| \in \text{BMO (Lipschitz domains)},$$
$$\log|\nabla u| \in \text{VMO } (C^1\text{-domains}).$$

- Free boundary regularity: $\log|\nabla u| \in$ VMO implies $n \in$ VMO.
- Free boundary regularity: $C^{1,\gamma}$-regularity of Lipschitz free boundaries in general two-phase problems for the p-Laplace operator.

Theorem.

Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, suppose that u and v are positive p-harmonic functions in $\Omega \cap B(w, 2r)$. Assume also that u and v are continuous in $\bar{\Omega} \cap B(w, 2r)$ and $u = 0 = v$ on $\partial\Omega \cap B(w, 2r)$. Then there exist $c, 1 \leq c < \infty$, and α, $\alpha \in (0, 1)$, both depending only on p, n, and M, such that

$$\left|\log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)}\right| \leq c\left(\frac{|y_1 - y_2|}{r}\right)^{\alpha}$$

whenever $y_1, y_2 \in \Omega \cap B(w, r/c)$.

Techniques - small/large Lipschitz constant

1. Category 1: domains which are ‘flat’ in the sense that their boundaries are well-approximated by hyperplanes.
2. Category 2: Lipschitz domains and domains which are well approximated by Lipschitz graph domains.

1. Domains in category 1 are called Reifenberg flat domains with small constant or just Reifenberg flat domains and include domains with small Lipschitz constant, C^1-domains and certain quasi-balls.
2. Domains in category 2 include Lipschitz domains with large Lipschitz constant and certain Ahlfors regular NTA-domains, which can be well approximated by Lipschitz graph domains in the Hausdorff distance sense.

- The purpose of this talk is to present a paper in which we highlight the techniques labeled as category 1 and how we use these techniques to prove new results for operators of p-Laplace type with variable coefficients (joint work with J. Lewis and N. Lundström).
- In future papers we intend to highlight the techniques labeled as category 2 and to use these techniques to prove new results for operators of p-Laplace type with variable coefficients in Lipschitz domains (joint work with B. Avelin and J. Lewis).

Definition 1.1.

Let $p, \beta, \alpha \in (1, \infty)$ and $\gamma \in (0, 1)$. Let $A = (A_1, ..., A_n) : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$. We say that the function A belongs to the class $M_p(\alpha, \beta, \gamma)$ if the following conditions are satisfied whenever x, $y, \xi \in \mathbf{R}^n$ and $\eta \in \mathbf{R}^n \setminus \{0\}$:

$$
\begin{aligned}
&(i) && \alpha^{-1} |\eta|^{p-2} |\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial A_i}{\partial \eta_j}(x, \eta) \xi_i \xi_j, \\
&(ii) && \left| \frac{\partial A_i}{\partial \eta_j}(x, \eta) \right| \leq \alpha |\eta|^{p-2}, 1 \leq i, j \leq n, \\
&(iii) && |A(x, \eta) - A(y, \eta)| \leq \beta |x - y|^{\gamma} |\eta|^{p-1}, \\
&(iv) && A(x, \eta) = |\eta|^{p-1} A(x, \eta / |\eta|).
\end{aligned}
$$

$M_p(\alpha) := M_p(\alpha, 0, \gamma)$

Operators of p-Laplace type

Definition 1.2.

Let $p \in (1, \infty)$ and let $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Given a bounded domain G we say that u is A-harmonic in G provided $u \in W^{1,p}(G)$ and

$$\int \langle A(y, \nabla u(y)), \nabla \theta(y) \rangle \, dy = 0 \tag{1.3}$$

whenever $\theta \in W_0^{1,p}(G)$. As a short notation for (1.3) we write $\nabla \cdot (A(y, \nabla u)) = 0$ in G.

An important class of equations:

$$\nabla \cdot \left[\langle A(y) \nabla u, \nabla u \rangle^{p/2-1} A(y) \nabla u \right] = 0 \text{ in } G \tag{1.4}$$

where $A = A(y) = \{a_{i,j}(y)\}$.

Definition 1.5.

A bounded domain Ω is called non-tangentially accessible (NTA) if there exist $M \geq 2$ and r_0 such that the following are fulfilled whenever $w \in \partial\Omega, 0 < r < r_0$:

(i) *interior corkscrew condition,*

(ii) *exterior the corkscrew condition,*

(iii) *Harnack chain type condition.*

M will denote the NTA-constant.

Geometry- Reifenberg flat domains

Definition 1.6.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain, $w \in \partial\Omega$, and $0 < r < r_0$. Then $\partial\Omega$ is said to be uniformly (δ, r_0)-approximable by hyperplanes, provided there exists, whenever $w \in \partial\Omega$ and $0 < r < r_0$, a hyperplane Λ containing w such that

$$h(\partial\Omega \cap B(w,r), \Lambda \cap B(w,r)) \leq \delta r.$$

$\mathcal{F}(\delta, r_0)$: the class of all domains Ω which satisfy the definition.

Definition 1.7.

Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain with constants M and r_0. Then Ω and $\partial\Omega$ are said to be (δ, r_0)-Reifenberg flat provided $\Omega \in \mathcal{F}(\delta, r_0)$.

Main Results - Boundary Harnack inequalities

Theorem 1.

Let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0)-Reifenberg flat domain.
Let p, $1 < p < \infty$, be given and assume that $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Let $w \in \partial\Omega, 0 < r < r_0$, and suppose that u, v are positive A-harmonic functions in $\Omega \cap B(w, 4r)$, continuous in $\bar{\Omega} \cap B(w, 4r)$, and $u = 0 = v$ on $\partial\Omega \cap B(w, 4r)$. Then there exist $\tilde{\delta}, \sigma > 0,$, $\sigma \in (0, 1)$, and $c \geq 1$, all depending only on $p, n, \alpha, \beta, \gamma$, such that if $0 < \delta < \tilde{\delta}$, then

$$\left|\log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)}\right| \leq c \left(\frac{|y_1 - y_2|}{r}\right)^{\sigma}$$

whenever $y_1, y_2 \in \Omega \cap B(w, r/c)$.

Main Results - Martin Boundary problem

Theorem 2.

Let $\Omega \subset \mathbf{R}^n$, δ, r_0, p, α, β, γ, and A be as in the statement of Theorem 1. Then there exists $\delta^* = \delta^*(p, n, \alpha, \beta, \gamma) > 0$, such that the following is true. Let $w \in \partial\Omega$ and suppose that $\hat{u}, \hat{v}$ are positive A-harmonic functions in Ω with $\hat{u} = 0 = \hat{v}$ continuously on $\partial\Omega \setminus \{w\}$. If $0 < \delta < \delta^*$, then $\hat{u}(y) = \lambda\hat{v}(y)$ for all $y \in \Omega$ and for some constant λ.

Main Results - The case $p = 2$

Remark. *We note that Theorems 1 and 2 are well known, in the case of the operators in (1.4), for* $p = 2$ *in NTA-domains under less restrictive assumptions on* A :

- *L. Caffarelli, E. Fabes, S. Mortola, S. Salsa.* Boundary behavior of nonnegative solutions of elliptic operators in divergence form, *Indiana J. Math.* **30** *(4) (1981) 621-640.*
- *D. Jerison and C. Kenig,* Boundary behavior of harmonic functions in non-tangentially accessible domains, *Advances in Math.* **46** *(1982), 80-147.*

Steps in the proof of Theorem 1 - outline

Step 0. - Prove Theorem 1, for $A \in M_p(\alpha)$, in the upper half plane.
Step A. - Uniform non-degeneracy of $|\nabla u|$ - the 'fundamental inequality'.
Step B. - Extension of $|\nabla u|^{p-2}$ to an A_2-weight.
Step C. - Deformation of A-harmonic functions - an associated linear pde.
Step D. - Boundary Harnack inequalities for degenerate elliptic equations.

Step A - the 'fundamental inequality'

Let $w \in \partial\Omega, 0 < r < r_0$, and suppose that u is a positive A-harmonic functions in $\Omega \cap B(w, 4r)$, continuous in $\bar{\Omega} \cap B(w, 4r)$, and $u = 0$ on $\partial\Omega \cap B(w, 4r)$.

There exist $\delta_1 = \delta_1(p, n, \alpha, \beta, \gamma)$, $\hat{c}_1 = \hat{c}_1(p, n, \alpha, \beta, \gamma)$ and $\bar{\lambda} = \bar{\lambda}(p, n, \alpha, \beta, \gamma)$, such that if $0 < \delta < \delta_1$, then

$$\bar{\lambda}^{-1} \frac{u(y)}{d(y, \partial\Omega)} \leq |\nabla u(y)| \leq \bar{\lambda} \frac{u(y)}{d(y, \partial\Omega)}$$

whenever $y \in \Omega \cap B(w, r/\hat{c}_1)$.

Step B - Extension of $|\nabla u|^{p-2}$ to an A_2-weight

$\lambda(y, \tau)$ is said to belong to the class $A_2(B(w, 2\hat{r}))$ if there exists a constant γ such that

$$r^{-2n} \int_{B(\tilde{w},\tilde{r})} \lambda(y,\tau)\, dy \cdot \int_{B(\tilde{w},\tilde{r})} \lambda(y,\tau)^{-1} dy \leq \gamma$$

whenever $\tilde{w} \in B(w, 2\hat{r})$ and $0 < \tilde{r} \leq 2\hat{r}$.

There exist $\delta_2 = \delta_2(p, n, \alpha, \beta, \gamma)$, $\hat{c}_2 = \hat{c}_2(p, n, \alpha, \beta, \gamma)$ such that if $0 < \delta < \delta_2$, then $|\nabla u|^{p-2}$ extends to an $A_2(B(w, r/(\hat{c}_1 \hat{c}_2))$-weight with constant depending only on $p, n, \alpha, \beta, \gamma$.

Step C - Deformation of A-harmonic functions

To simplify let $r^* = r/c$ and assume,

$$0 \le u \le v/2 \text{ and } v \le c \text{ in } \bar{\Omega} \cap \bar{B}(w, 4r^*).$$

Let $\tilde{u}(\cdot, \tau)$ be the A-harmonic function in $\Omega \cap B(w, 4r^*)$ with continuous boundary values

$$\tilde{u}(y, \tau) = \tau v(y) + (1 - \tau) u(y)$$

whenever $y \in \partial(\Omega \cap B(w, 4r^*))$ and $\tau \in [0, 1]$.

From our assumption, we have

$$0 < \frac{\tilde{u}(\cdot, t) - \tilde{u}(\cdot, \tau)}{t - \tau} = v - u \le c$$

on $\partial(\Omega \cap B(w, 4r^*))$ and in $\Omega \cap B(w, 4r^*)$.

Step C - an associated linear pde

Step A: $|\nabla u(\cdot,\tau)|$ satisfies the fundamental inequality in $\Omega \cap B(w, 16r')$, $r' = r^*/\hat{c}$, $\tau \in [0,1]$.

Differentiating the equation, $\nabla \cdot (A(y, \nabla \tilde{u}(y,\tau)) = 0$ with respect to τ we find that $\zeta = \tilde{u}_\tau(y,\tau)$ satisfies,

$$
\begin{aligned}
\tilde{L}\zeta &= \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i} \left(\tilde{b}_{ij}(y,\tau) \zeta_{y_j}(y) \right) = 0, \\
\tilde{b}_{ij}(y,\tau) &= \frac{\partial A_i}{\partial \eta_j}(y, \nabla \tilde{u}(y,\tau)), \\
\alpha^{-1} \tilde{\lambda}(y,\tau)|\xi|^2 &\leq \sum_{i,j} \tilde{b}_{ij}(y,\tau)\xi_i\xi_j \leq \alpha \tilde{\lambda}(y,\tau)|\xi|^2
\end{aligned}
$$

for $y \in \Omega \cap B(w, r')$, $\tilde{\lambda}(y,\tau) = |\nabla \tilde{u}(y,\tau)|^{p-2}$, $\xi \in \mathbf{R}^n$.

Step C - an associated linear pde

A key observation is that $\zeta = \tilde{u}(\cdot, \tau)$ is also a weak solution to $\tilde{L}$ in $\Omega \cap B(w, r')$. Indeed, using the homogeneity in Definition 1.1 (iv) we see that

$$\begin{aligned}\sum_j \tilde{b}_{ij}(y,\tau)\tilde{u}_{y_j}(y,\tau) &= \sum_j \frac{\partial A_i}{\partial \eta_j}(y, \nabla \tilde{u}(y,\tau))\tilde{u}_{y_j}(y,\tau) \\ &= (p-1)A_i(y, \nabla \tilde{u}(y,\tau)).\end{aligned}$$

Hence $\zeta = \tilde{u}(\cdot, \tau)$ is also a weak solution to $\tilde{L}$.

Step D - BHI for degenerate elliptic equations

$\zeta = u_\tau(\cdot,\tau)$ and $\zeta = u(\cdot,\tau)$ satisfy $\tilde{L}\zeta = 0$ in $\Omega \cap B(w,r')$:

$$\begin{aligned} \tilde{L}\zeta &= \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(\tilde{b}_{ij}(y,\tau)\zeta_{y_j}(y)\right) = 0, \\ \tilde{b}_{ij}(y,\tau) &= \frac{\partial A_i}{\partial \eta_j}(y,\nabla\tilde{u}(y,\tau)). \end{aligned}$$

$u_\tau(y,\tau) = u(y,\tau) = 0$ whenever $y \in \partial\Omega \cap B(w,r')$.

$$\alpha^{-1}\tilde{\lambda}(y,\tau)|\xi|^2 \le \sum_{i,j} \tilde{b}_{ij}(y,\tau)\xi_i\xi_j \le \alpha\tilde{\lambda}(y,\tau)|\xi|^2.$$

Step B: $\tilde{\lambda}(\cdot,\tau)$, $\tau \in [0,1]$, can be extended to A_2-weights in $B(w,4r'')$, $r'' = r'/(4\hat{c}_2)$.

Step D - BHI for degenerate elliptic equations

The fundamental theorem of calculus (heuristic deduction, can be made rigorous):

$$\log\left(\frac{v(y)}{u(y)}\right) = \log\left(\frac{u(y,1)}{u(y,0)}\right) = \int_0^1 \frac{u_\tau(y,\tau)}{u(y,\tau)} d\tau, \;\; y \in \Omega \cap B(w,r').$$

Claim. *Let $\tau \in (0,1]$ be fixed, $\tilde{L}$ as defined above. Let v_1 and v_2 be non-negative solutions to the operator $\tilde{L}$ in $\Omega \cap B(w,r')$, vanishing continuously on $\partial\Omega \cap B(w,r')$. Then there exist $c = c(p,n,\alpha,\beta,\gamma)$, $1 \leq c < \infty$, and $\sigma = \sigma(p,n,\alpha,\beta,\gamma)$, $\sigma \in (0,1)$, such that if $r''' = r''/c$, then*

$$\left|\log\frac{v_1(y_1)}{v_2(y_1)} - \log\frac{v_1(y_2)}{v_2(y_2)}\right| \leq c\left(\frac{|y_1 - y_2|}{r''}\right)^\sigma,$$

whenever $y_1, y_2 \in \Omega \cap B(w,r''')$.

Step D - BHI for degenerate elliptic equations

The Claim follows by using results from

- E. Fabes, C. Kenig, and R. Serapioni, *The local regularity of solutions to degenerate elliptic equations,* Comm. Partial Differential Equations, **7** (1982), no. 1, 77 - 116.
- E. Fabes, D. Jerison, and C. Kenig, *Boundary behavior of solutions to degenerate elliptic equations.* Conference on harmonic analysis in honor of Antonio Zygmund, Vol I, II Chicago, Ill, 1981, 577-589, Wadsworth Math. Ser, Wadsworth Belmont CA, 1983.
- E. Fabes, D. Jerison, and C. Kenig, *The Wiener test for degenerate elliptic equations*, Ann. Inst. Fourier (Grenoble) **32** (1982), no. 3, 151-182.

Work in progress!