TOWARDS A NON-LINEAR CALDERÓN-ZYGMUND THEORY

GIUSEPPE MINGIONE

University of Parma

For those interested

- **These slides will be posted on my web page**

Overture: The standard CZ theory

- **Consider the model case**

$$\triangle u = f \qquad \text{in } \mathbb{R}^n$$

Then

$$f \in L^q \quad \text{implies} \quad D^2u \in L^q \qquad 1 < q < \infty$$

with natural failure in the borderline cases $q = 1, \infty$

- **As a consequence (Sobolev embedding)**

$$Du \in L^{\frac{nq}{n-q}} \qquad q < n$$

The singular integral approach

- **Representation via Green's function**

$$u(x) \approx \int G(x,y) f(y)\, dy$$

with

$$G(x,y) = \begin{cases} |x-y|^{2-n} & \text{if} \quad n > 2 \\ -\log|x-y| & \text{if} \quad n = 2 \end{cases}$$

- **Differentiation**

$$D^2 u(x) = \int K(x,y) f(y)\, dy$$

and $K(x,y)$ is a singular integral kernel, and the conclusion follows

The fractional integral approach

- **Again differentiating**

$$|Du(x)| \lesssim I_1(|f|)(x)$$

- **where I_1 is a fractional integral**

$$I_\beta(g)(x) := \int \frac{g(y)}{|x-y|^{n-\beta}}\, dy \qquad \beta \in [0,n)$$

- **and then**

$$I_\beta \colon L^q \to L^{\frac{nq}{n-\beta q}} \qquad \beta q < n$$

- **This is in fact equivalent to the original proof of Sobolev embedding theorem for the case $q > 1$, which uses that**

$$|u(x)| \lesssim I_1(|Du|)(x)$$

The fractional integral approach

- **Important remark: the theory of fractional integral operators substantially differs from that of singular ones.**
- **In fact, while the latter is based on cancelation properties of the kernel, the former only considers the size of the kernel.**
- **As a consequence all the estimates related to the operator I_β degenerate when $\beta \to 0$.**

Another linear case

- **Higher order right hand side**

$$\triangle u = \text{div } Du = \text{div } F$$

Then

$$F \in L^q \Longrightarrow Du \in L^q \qquad q > 1$$

just “simplify" the divergence operator!!

Interpolation approach

- **Define the operator**

$$T\colon F \mapsto T(F) := \text{ gradient of the solution to } \triangle u = \text{div } F$$

- **Then**

$$T\colon L^2 \to L^2$$

by testing with the solution, and

$$T\colon L^\infty \to \text{BMO}$$

by regularity estimates (hard part).

- **Campanato-Stampacchia interpolation**

$$T\colon L^q \to L^q \qquad 1 < q < \infty$$

The case of continuous coefficients

- **Constant coefficients**

$$\text{div } (ADu) = \text{div } F$$

Then just repeat the previous approach

- **Variable coefficients**

$$\text{div } (A(x)Du) = \text{div } F$$

where $A(x)$ is an elliptic and bounded matrix with continuous entries

- **Freezing**

$$\text{div } (A(x_0)Du) = \text{div } ((A(x_0) - A(x))Du) + \text{div } F$$

and assume, eventually considering small sub-domains, that

$$\text{osc } A(\cdot) \leq \varepsilon$$

The case of continuous coefficients

- **Then we have**

$$\|Du\|_{L^q} \leq c\,[\text{osc } A(\cdot)]\,\|Du\|_{L^q} + c\|F\|_{L^q}$$

and therefore

$$\|Du\|_{L^q} \leq c\varepsilon\|Du\|_{L^q} + c\|F\|_{L^q}$$

and conclude with

$$\|Du\|_{L^q} \leq c\|F\|_{L^q}$$

This argument maybe made rigorous via (localization and approximation) or fixed point arguments. Observe that the strict ellipticity provides uniqueness.

- **Approach by Stampacchia & Murthy, Trudinger**

BMO/VMO

- **Define**

$$(v)_{B_s} := \frac{1}{|B_s|}\int_{B_s} v\,dx$$

and

$$\omega(R) := \sup_{s\le R}\frac{1}{|B_s|}\int_{B_s} |v-(v)_{B_s}|\,dx$$

- **A map v belongs to BMO iff**

$$\omega(R) < \infty$$

- **A map v belongs to VMO iff**

$$\lim_{R\to 0}\omega(R) = 0$$

Mild discontinuities

- **Chiarenza & Frasca & Longo (TAMS 93)**

$$A(x)D^2u = f$$

and

$$f \in L^q \implies D^2u \in L^q \qquad 1 < q < \infty$$

provided

$$A(\cdot) \in \text{VMO}$$

- **Main tools: commutator estimate**

$$\|I[\varphi, g]\|_{L^q} \leq \|\varphi\|_{\text{BMO}} \|g\|_{L^q}$$

$$I[\varphi, g] = K(\varphi g) - \varphi K(g)$$

Part 1: First non-linear cases

- **Iwaniec's result (Studia Math. 83)**

$$\text{div}\,(|Du|^{p-2}Du) = \text{div}\,(|F|^{p-2}F)$$

Then it holds that

$$F \in L^q \Longrightarrow Du \in L^q \qquad p \leq q < \infty$$

- **Di Benedetto & Manfredi (Amer. J. Math. 93) prove this for the p-Laplacean system**
- **Kinnunen & Zhou (Comm. PDE 99) consider the case**

$$\text{div}\,(\langle c(x)Du, Du\rangle^{\frac{p-2}{2}}Du) = \text{div}\,(|F|^{p-2}F)$$

where $c(x)$ is an elliptic matrix with VMO entries

First non-linear cases

- **The local estimate**

$$\left(\fint_{B_R} |Du|^q \, dz\right)^{\frac{1}{q}} \leq c \left(\fint_{B_{2R}} |Du|^p \, dz\right)^{\frac{1}{p}} + c \left(\fint_{B_{2R}} |F|^q \, dz\right)^{\frac{1}{q}}$$

General elliptic problems

- **In the same way the non-linear result of Iwaniec extends to all elliptic equations in divergence form of the type**

$$\text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

where $a(\cdot)$ is p-monotone in the sense of the previous slides

- **and to all systems with special structure**

$$\text{div } (g(|Du|)Du) = \text{div } (|F|^{p-2}F)$$

- **Moreover, VMO-coefficients can be considered too, along the methods of Kinnunen & Zhou**

$$\text{div } [c(x)a(Du)] = \text{div } (|F|^{p-2}F)$$

General systems - the elliptic case

- **The previous result cannot hold**

$$\text{div}\ a(Du) = \text{div}\ (|F|^{p-2}F)$$

with $a(\cdot)$ being a general p-monotone in the sense of the previous slide. The failure of the result, which happens already in the case $p = 2$, can be seen as follows.

- **Consider the homogeneous case**

$$\text{div}\ a(Du) = 0$$

The validity of the result would imply $Du \in L^q$ for every $q < \infty$, and, ultimately, that

$$u \in L^\infty$$

- **But Sverák & Yan (Proc. Natl. Acad. Sci. USA 02) recently proved the existence of unbounded solutions, even when $a(\cdot)$ is non-degenerate and smooth**

The up-to-a-certain-extent CZ theory

- **Kristensen & Min. (ARMA 06) - General elliptic systems**

$$\text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

for $a(Du)$ being a p-monotone vector field and

$$p \leq q < p + \frac{2p}{n-2} + \delta \qquad n > 2$$

Then it holds that $F \in L^q \Longrightarrow Du \in L^q$

The Dirichlet problem

- **Applications to singular sets estimates**
- **Consider the Dirichlet problem**

$$\begin{cases} -\text{div}\ a(Du) = 0 & \text{in } \Omega \\ u = v & \text{on } \partial\Omega \end{cases}$$

Then it holds that

$$\int_\Omega |Du|^q\, dx \leq c \int_\Omega (|Dv|^q + 1)\, dx$$

Discussion about the sharpness - $p = 2$

- **For general elliptic systems**

$$\text{div } a(Du) = \text{div } (F)$$

we have

$$2 \leq q \leq 2 + \frac{4}{n-2} = \frac{2n}{n-2} \qquad n > 2$$

Then it holds that $F \in L^q \Longrightarrow Du \in L^q$

- **Take** $F \equiv 0$ then the example of Sverák & Yan gives the existence of an unbounded solution, while $Du \in L^q$ for some $q > 2n/(n-2)$ would imply that u is bounded

$$\frac{2n}{n-2} + \delta > n \Longleftrightarrow n \leq 4$$

- **The result is sharp at least when** $n = 2, 3, 4$

Non-uniformly elliptic problems

- **Acerbi & Min. (Crelle J. 05)**

$$\text{div}\,(|Du|^{p(x)-2}Du) = \text{div}\,(|F|^{p(x)-2}F)$$

Then it holds that

$$|F|^{p(x)} \in L^q \Longrightarrow |Du|^{p(x)} \in L^q \qquad 1 \leq q < \infty$$

- **Assumptions on the exponent function**

$$1 < \gamma_1 \leq p(x) \leq \gamma_2 < \infty$$

$$|p(x) - p(y)| \leq \omega(|x-y|)$$

$$\lim_{R\to 0} \omega(R) \log\left(\frac{1}{R}\right) = 0$$

- **Further developments by Habermann (Math. Z. 08)**

The parabolic case

- **Kinnunen & Lewis (Duke Math. J. 00)**

$$u_t - \text{div}\,(|Du|^{p-2}Du) = \text{div}\,(|F|^{p-2}F)$$

for

$$p > \frac{2n}{n+2} \qquad \text{(optimal)}$$

Then it holds that

$$F \in L^q \Longrightarrow Du \in L^q \qquad \text{for} \qquad p \leq q < p + \delta(n, N, p)$$

- **This is actually available for more general systems, being a result in the spirit of Gehring's lemma**
- **The case of higher order systems is treated by Bögelein (Ann. Acad. Sci. Fenn., 08)**

The parabolic case

- **Acerbi & Min. (Duke Math. J. 07)**

$$u_t - \text{div}\,(|Du|^{p-2}Du) = \text{div}\,(|F|^{p-2}F)$$

for

$$p > \frac{2n}{n+2} \qquad \text{(optimal)}$$

Then it holds that

$$F \in L^q \Longrightarrow Du \in L^q \qquad \text{for} \qquad p \leq q < \infty$$

- **The elliptic approach via maximal operators only works in the case** $p = 2$
- **The result also works for systems, that is when** $u(x,t) \in \mathbb{R}^N$**,** $N \geq 1$
- **First Harmonic Analysis free approach to non-linear Calderón-Zygmund estimates**

The parabolic case

- **The result is new already in the case of equations i.e. $N = 1$, the difficulty being in the lack of homogenous scaling of parabolic problems with $p \neq 2$, and not being caused by the degeneracy of the problem, but rather by the polynomial growth.**
- **The result extends to all parabolic equations of the type**

$$u_t - \text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

with $a(\cdot)$ being a monotone operator with p-growth. More precisely we assume

$$\begin{cases} \nu(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_2 - z_1|^2 \leq \langle a(z_2) - a(z_1), z_2 - z_1 \rangle \\ \\ |a(z)| \leq L(s^2 + |z|^2)^{\frac{p-1}{2}} \ , \end{cases}$$

The parabolic case

- **The result also holds for systems with a special structure (sometimes called Uhlenbeck structure). This means**

$$u_t - \text{div}\, a(Du) = \text{div}\, \left(|F|^{p-2}F\right)$$

with $a(\cdot)$ being p-monotone in the sense of the previous slide, and satisfying the structure assumption

$$a(Du) = g(|Du|)Du$$

- **The p-Laplacean system is an instance of such a structure**

The parabolic UTCE CZ theory

- **Duzaar & Min. & Steffen (08) - General parabolic systems**

$$u_t - \text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

for $a(Du)$ being a p-monotone vector field and

$$p \leq q < p + \frac{4}{n} + \delta$$

Then it holds that $F \in L^q \Longrightarrow Du \in L^q \qquad p \leq q < \infty$

Discontinuous coefficients

- **Duzaar & Min. & Steffen (08)**

$$u_t - \text{div}\, [c(x)a(t, Du)] = \text{div}\, (|F|^{p-2}F)$$

equations or systems (in this last case with limitations on q), or

$$u_t - \text{div}\, (c(x)b(t)|Du|^{p-2}Du) = \text{div}\, (|F|^{p-2}F)$$

we have

$$F \in L^q \Longrightarrow Du \in L^q$$

- **Discontinuous coefficients**

$$c(x) \in \text{VMO} \qquad b(t) \;\text{ is just measurable}$$

- **This extends recent linear results of Krylov and his students, valid for linear parabolic equations**

Elliptic vs parabolic local estimates

- **Elliptic estimate**

$$\left(\fint_{B_R} |Du|^q\,dz\right)^{\frac{1}{q}} \leq c\left(\fint_{B_{2R}} |Du|^p\,dz\right)^{\frac{1}{p}} + c\left(\fint_{B_{2R}} |F|^q\,dz\right)^{\frac{1}{q}}$$

- **Parabolic estimate -** $p \geq 2$

$$\left(\fint_{Q_R} |Du|^q\,dz\right)^{\frac{1}{q}} \leq c\left[\left(\fint_{Q_{2R}} |Du|^p\,dz\right)^{\frac{1}{p}} + \left(\fint_{Q_{2R}} |F|^q\,dz\right)^{\frac{1}{q}} + 1\right]^{\frac{p}{2}}$$

- **Parabolic cylinders** $Q_R \equiv B_R \times (t_0 - R^2, t_0 + R^2)$
- **The exponent** $p/2$ **is the scaling deficit of the system**

Interpolation nature of local estimates

- **Parabolic local estimate -** $p \geq 2$

$$\left(\fint_{Q_R} |Du|^q \, dz\right)^{\frac{1}{q}} \leq c(n,N,p)\left[\left(\fint_{Q_{2R}} |Du|^p \, dz\right)^{\frac{1}{p}} + c(q)\left(\fint_{Q_{2R}} |F|^q \, dz\right)^{\frac{1}{q}} + 1\right]^{\frac{p}{2}}$$

- **Taking** $F = 0$ **and letting** $q \to \infty$ **yields**

$$\sup_{Q_R} |Du| \leq c\left[\left(\fint_{Q_{2R}} |Du|^p \, dz\right)^{\frac{1}{p}} + 1\right]^{\frac{p}{2}}$$

- **This is the original sup estimate of Di Benedetto & Friedman (Crelles J. 84)**

The ghost of the maximal operator

- **In the estimate**

$$\left(\fint_{Q_R} |Du|^q \, dz\right)^{\frac{1}{q}}$$

$$\leq c(n,N,p)\left[\left(\fint_{Q_{2R}} |Du|^p \, dz\right)^{\frac{1}{p}} + c(q)\left(\fint_{Q_{2R}} |F|^q \, dz\right)^{\frac{1}{q}} + 1\right]^{\frac{p}{2}}$$

- **the dependence of $c(q)$ is maximal function like**

$$c(q) \approx \frac{q}{q-p}$$

- **This apparent instability for $q \searrow p$ is recovered via Gehring's lemma**

The local estimate in the singular case

- **The singular case**

$$\frac{2n}{n+2} < p < 2$$

- **The local estimate is**

$$\left(\fint_{Q_R} |Du|^q \, dz\right)^{\frac{1}{q}}$$
$$\leq c(n,N,p)\left[\left(\fint_{Q_{2R}} |Du|^p \, dz\right)^{\frac{1}{p}} + c(q)\left(\fint_{Q_{2R}} |F|^q \, dz\right)^{\frac{1}{q}} + 1\right]^{\frac{2p}{p(n+2)-2n}}$$

- **Observe that**

$$\frac{2p}{p(n+2)-2n} \nearrow \infty \qquad \text{when} \qquad p \searrow \frac{2n}{n+2}$$

Obstacle problems

- **Results of Bögelein & Duzaar & Min. (08)**
- **The elliptic case**

$$\text{Min} \int |Dv|^p \, dx$$

in the class

$$K = \{v \in W^{1,p} \, : \, v(x) \geq \Phi(x)\}$$

- **Then**

$$\Phi \in W^{1,q} \Longrightarrow u \in W^{1,q} \qquad q \geq p$$

- **Local estimate**

$$\left(\fint_{B_R} |Du|^q \, dz \right)^{\frac{1}{q}} \leq c \left(\fint_{B_{2R}} |Du|^p \, dz \right)^{\frac{1}{p}} + c \left(\fint_{B_{2R}} |D\Phi|^q \, dz \right)^{\frac{1}{q}}$$

Parabolic obstacle problems

- **Parabolic variational inequalities**

$$\int_{\Omega_T} v_t(v-u) + \langle a(Du), D(v-u)\rangle \, dx\, dt + (1/2)\|v(\cdot,-T)\|^2_{L^2(\Omega)} \geq 0,$$

in the class

$$K = \{v \in L^p(-T,0;W_0^{1,p}(\Omega)) \cap L^2(\Omega_T) : v(x,t) \geq \Phi(x)\}$$

- **Then, again**

$$D\Phi \in L^q \Longrightarrow Du \in L^q \qquad q \geq p$$

- **The case when Φ depends on time is allowable**

Elliptic and Parabolic coverings

- **Cover the level sets with balls**

$$E(\lambda) := \{|Du| \geq \lambda\}$$

with usual Euclidean balls

$$B(x_0, R)$$

and study their decay via the well-known formula

$$\int |Du|^q = \int^{\infty} \lambda^{q-1}|E(\lambda)|\, d\lambda$$

- **Take a vitali covering of** $E(\lambda)$, that is $\{B_i\}$

$$\frac{1}{|B_i|}\int_{B_i} |Du|^p\, dx \approx \lambda^p$$

Elliptic and Parabolic coverings

- **Use the comparison maps** v_i

$$\triangle_p v_i = 0 \qquad v_i = u \text{ on } \partial B_i$$

  and their local regularity properties to infer those of the solution u
- **The parabolic case - DiBenedetto's intrinsic geometry**
- **Cover the level set** $E(\lambda)$ with intrinsic cubes

$$Q_i \equiv B(x_i, R) \times (t_i - \lambda^{2-p} R^2, t_i + \lambda^{2-p} R^2)$$

  for $p \geq 2$, and

$$Q_i \equiv B(x_i, \lambda^{\frac{p-2}{2}} R) \times (t_i - R^2, t_i + R^2)$$

  for $p < 2$, such that

$$\frac{1}{|Q_i|} \int_{Q_i} |Du|^p \, dx \, dt \approx \lambda^p$$

Elliptic and Parabolic coverings

- **These are intrinsic cubes**, as described by Urbano in his lectures
- **They cannot generate a family with an naturally associated maximal operator**
- **Use the comparison maps** v_i

$$(v_i)_t - \text{div}\,(|Dv_i|^{p-2}Dv_i) = 0 \qquad v_i = u \text{ on } \partial_{par}Q_i$$

this is like

$$(v_i)_t - \text{div}\,(\lambda^{p-2}Dv_i) = 0$$

since, in some sense, we still have

$$\frac{1}{|Q_i|}\int_{Q_i} |Dv_i|^p \, dx \, dt \approx \lambda^p$$

Elliptic and Parabolic coverings

- **Make the change of variables, for** $p > 2$

$$w(y,s) := v(x_i + R^2 y, t_i + \lambda^{2-p} R s) \qquad (y,s) \in B_1 \times (-1,1)$$

  and then

$$w_t - \triangle w = 0$$

- **On intrinsic cylinders solutions behave as caloric functions**

The ghost of the good-λ inequality principle

- **Exit time on the composite quantity**

$$\frac{1}{|Q_i|}\int_{Q_i} |Du|^p \, dx \, dt + \frac{M^p}{|Q_i|}\int_{Q_i} |F|^p \, dx \, dt \approx \lambda^p$$

therefore, while on Q_i we approximately have

$$|Du| \approx \lambda$$

we have

$$|F| \approx \frac{\lambda}{M}$$

- **Large M-inequality principle**

Open problems

- **The full range**

$$p - 1 < q < \infty$$

Compare with the linear case $p = 2$.

- **This is the case below the duality exponent**, when

$$\text{div}\,(|F|^{p-2}F) \notin W^{-1,p'}$$

- **Iwaniec & Sbordone (Crelle J., 94), Lewis (Comm. PDE, 93)**

$$p - \varepsilon \le q < \infty \qquad \varepsilon \equiv \varepsilon(n,p)$$

- **Parabolic case: Kinnunen & Lewis (Ark. Math, 02), Bögelein (Ph. D. Thesis, 07)**

A few problems

- **Differentiability of Du for**

$$\text{div}\,(|Du|^{p-2}Du) = f$$

- **Estimates below the duality exponents**
- **Estimates in non-rearrangement invariant spaces**
- **Estimates in two-scale spaces**
- **Estimates in borderline cases**

Results from two papers:

- The Calderón-Zygmund theory for elliptic problems with measure data, in Ann. SNS Pisa 07
- Gradient estimates below the duality exponent, Preprint 2008

Part 2: Differentiability of the gradient

Measure data problems - a review

- **Model case**

$$\begin{cases} -\text{div } a(x, Du) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and $|\mu|(\Omega) < \infty$.

- **Assumptions**

$$\begin{cases} \nu(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_2 - z_1|^2 \leq \langle a(x, z_2) - a(x, z_1), z_2 - z_1 \rangle \\ \\ |a(x, z)| \leq L(s^2 + |z|^2)^{\frac{p-1}{2}} \ , \end{cases}$$

with

$$2 \leq p \leq n \qquad \text{standing assumption}$$

- **Linear case** Classical work by Littman & Stampacchia & Weinberger (Ann. SNS Pisa, 1963)

Notion of solution

- **Distributional**

$$\int_\Omega a(x, Du)D\varphi\, dx = \int_\Omega \varphi\, d\mu, \quad \text{for every } \varphi \in C_c^\infty(\Omega)\ .$$

- **Minimal integrability** that is $u \in W^{1,p-1}$
- **Approximations**

$$\begin{cases} -\text{div}\ a(x, Du_k) = f_k \in L^\infty & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega \end{cases}$$

$$f_k \to \mu$$

$$u_k \to u \qquad \text{strongly in } W^{1,p-1}$$

Basic model examples

- **The p-Laplacean operator**

$$-\text{div}(|Du|^{p-2}Du) = f$$

- **Its measurable variations**

$$-\text{div}[c(x)(s^2 + |Du|^2)^{\frac{p-2}{2}}Du] = f$$

- **Its non-autonomous variations**

$$-\text{div}[c(x,u)(s^2 + |Du|^2)^{\frac{p-2}{2}}Du] = f$$

The fundamental solution

- **Consider the problem**

$$\begin{cases} -\text{div}(|Du|^{p-2}Du) = \delta & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- **And its unique solution (in the approximation class), called the fundamental solution**

$$G_p(x) \approx \begin{cases} |x|^{\frac{p-n}{p-1}} & \text{if} \quad p \neq n \\ \\ -\log|x| & \text{if} \quad n = p \end{cases}$$

- $G_p(x)$ **can be used to asses the optimality of several regularity estimates**

Regularity estimates, existence

- **Talenti (Ann. SNS Pisa 76, Annali di Matematica 79)**
- **Lindqvist (Crelles J. 86)**
- **Heinonen & Kilpeläinen & Martio (Oxford book and contributions from the eighties)**
- **Boccardo & Gallöuet (J. Funct. Anal. 89)**
- **Kilpeläinen & Malý (Ann. SNS Pisa. 92, Acta Math. 94)**
- **Greco & Iwaniec & Sbordone (manus. math. 97)**
- **Dolzmann & Hungerbühler & Müller (Ann. IHP. 97, Clelles 00)**
- **Kilpeläinen-Shanmugalingam-Zhong, Ark. Math. 08**
- **A large number of authors contributed to such estimates, in several different, and interesting ways**
- **There exists a solution such that**

$$|Du|^{p-1} \in \mathcal{M}^{\frac{n}{n-1}} \qquad p \leq n$$

- **The regularity is optimal, as revealed by** $G_p(\cdot)$

Limiting spaces

- **Marcinkiewicz spaces (weak L^γ)**

$$f \in \mathcal{M}^\gamma \Longleftrightarrow \sup_{\lambda>0} \lambda^\gamma |\{|f| > \lambda\}| =: \|f\|^\gamma_{\mathcal{M}^\gamma} < \infty$$

- **Prominent example: potentials**

$$\frac{1}{|x|^{n/\gamma}} \in \mathcal{M}^\gamma \qquad\qquad \frac{1}{|x|^{n/\gamma}} \not\in L^\gamma$$

- **Inclusions**

$$L^\gamma \subset \mathcal{M}^\gamma \subset L^{\gamma-\varepsilon} \qquad \varepsilon > 0$$

- $L \log L$**-spaces**

$$f \in L \log L \Longleftrightarrow \int |f| \log(e + |f|)\, dx < \infty$$

Limiting spaces

- **Definition comes from**

$$
\begin{aligned}
|\{|f| > \lambda\}| &= \int_{\{|f|>\lambda\}} \\
&\leq \int_{\{|f|>\lambda\}} \frac{|f|^\gamma}{\lambda^\gamma} \\
&\leq \frac{1}{\lambda^\gamma} \int |f|^\gamma \\
&= \frac{1}{\lambda^\gamma} \|f\|_{L^\gamma}^\gamma
\end{aligned}
$$

- **therefore**

$$
\begin{aligned}
\|f\|_{\mathcal{M}^\gamma}^\gamma &= \sup_{\lambda>0} \lambda^\gamma |\{|f| > \lambda\}| \\
&\leq \|f\|_{L^\gamma}^\gamma
\end{aligned}
$$

Function data

– Boccardo & Gallöuet (Comm. PDE 93)

- **Optimal integrability of Du**

$$f \in L^{\gamma} \qquad p < n$$

$$|Du|^{p-1} \in L^{\frac{n\gamma}{n-\gamma}}$$

- **In the case below the duality exponent**

$$1 < \gamma < \frac{np}{np - n + p} = (p^*)'$$

- **In fact**

$$L^{\gamma} \subset W^{-1,p'} \qquad \text{when} \qquad \gamma \geq (p^*)'$$

- **Finally**

$$f \in L \log L \Longrightarrow |Du|^{p-1} \in L^{\frac{n}{n-1}}$$

Function data – Kilpeläinen & Li (Diff. Int. Equ. 00)

- **Optimal integrability of Du**

$$f \in \mathcal{M}^{\gamma} \qquad p < n$$

$$|Du|^{p-1} \in \mathcal{M}^{\frac{n\gamma}{n-\gamma}}$$

- **In the case below the duality exponent**

$$1 < \gamma < \frac{np}{np - n + p} = (p^*)'$$

- **The borderline case $\gamma = (p^*)'$ is left by the authors as a tempting open problem, on which I will come back later; progress on this has been made by Zhong (Ph. D. Thesis)**
- **This case is interesting since we would have $Du \in \mathcal{M}^p$, almost at the natural integrability**

Differentiability of Du for measure data

- **Differentiability**

$$\begin{cases} \nu(s^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_2-z_1|^2 \leq \langle a(x,z_2)-a(x,z_1), z_2-z_1\rangle \\ |a(x,z_2)-a(x,z_1)| \leq L(s^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_2-z_1| \\ |a(x,0)| \leq Ls^{p-1} \end{cases}$$

- **Differentiable coefficients**

$$|a(x,z)-a(x_0,z)| \leq L|x-x_0|(s^2+|z|^2)^{\frac{p-1}{2}}$$

Fractional Sobolev spaces

- **We have that $v \in W^{s,\gamma}$ with**

$$0 < s < 1 \qquad \gamma \geq 1$$

iff

$$\int\int \frac{|v(x) - v(y)|^\gamma}{|x-y|^{n+s\gamma}}\, dx\, dy < \infty$$

- **Intuitively**

$$\int\int \frac{|v(x) - v(y)|^\gamma}{|x-y|^{n+s\gamma}}\, dx\, dy \approx \int |D^s v(x)|^\gamma\, dx$$

- **Fractional Sobolev embedding**

$$W^{s,\gamma} \hookrightarrow L^{\frac{n\gamma}{n-s\gamma}} \qquad s\gamma < n$$

Further fractional spaces

- **Nikolskii -** $v \in N^{s,\gamma}$ **with**

$$0 \leq s \leq 1 \qquad \gamma \geq 1$$

iff

$$\|v(\cdot + h) - v(\cdot)\|_{L^\gamma} \leq L|h|^s$$

- $v \in \tilde{N}^{s,\gamma}$

$$|v(x) - v(y)| \leq [g(x) + g(y)]\,|x - y|^s$$

for some $g \in L^\gamma$

- **Inclusions**

$$W^{s,\gamma} \subset \tilde{N}^{s,\gamma} \subset N^{s,\gamma}$$

- **and**

$$N^{s+\varepsilon_1,\gamma} \subset \tilde{N}^{s+\varepsilon_2,\gamma} \subset W^{s,\gamma} \qquad \varepsilon_1 > \varepsilon_2 > 0$$

Differentiability of Du for measure data - $p = 2$

- **You cannot have** $Du \in W^{1,1}$, already in the linear case (otherwise $Du \in L^{\frac{n}{n-1}}$)
- **Full integrability** $Du \in \mathcal{M}^{\frac{n}{n-1}}$ vs total lack of differentiability of Du, but “almost nothing" is actually lost, in fact we have

Differentiability of Du for measure data - $p = 2$

- **Theorem 1**

$$Du \in W^{1-\varepsilon,1} \qquad \text{for every } \varepsilon > 0$$

Differentiability of Du for measure data - $p = 2$

- **Alternatively**

$$Du \in \tilde{N}^{1-\varepsilon,1} \qquad \text{for every } \varepsilon > 0$$

or

$$Du \in N^{1-\varepsilon,1} \qquad \text{for every } \varepsilon > 0$$

Caccioppoli's estimate

- **Local estimate**

$$\int_{B_{R/2}}\int_{B_{R/2}} \frac{|Du(x)-Du(y)|}{|x-y|^{n+1-\varepsilon}}\,dx\,dy \leq \frac{c}{R^{1-\varepsilon}}\int_{B_R}(|Du|+s)\,dx + cR^{\varepsilon}|\mu|(B_R)$$

A first open problem

- **Metaprinciple: “behind every $\forall\, \varepsilon > 0$ there is a limiting case"; in this situation, a limiting space**
- **Question (Koskela, Koch): find a full differentiability scale space X such that $Du \in X$**
- **Question, consider the approximating solutions**

$$\begin{cases} -\text{div}\; a(x, Du_k) = f_k \in L^\infty & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega \end{cases}$$

- **Prove the uniform bound (in k)**

$$\|D^2 u_k\|_{\mathcal{M}^1} \leq c$$

- **Recall that**

$$|D^2 G_2(x)| \lesssim \frac{1}{|x|^n} \in \mathcal{M}^1$$

- **Warning: these are not distributional derivatives**

A first open problem

- **Alternatively, we may try an estimate of the type**

$$\sup_{\lambda \geq 0} \lambda |\{x \in \Omega : |Du(x+h) - Du(x)| > \lambda\}| \leq c|h|$$

- **Observe** how the problem with the weak distributional derivatives not well-defined in $\mathcal{M}^1$ now turns into a problem of weak convergence in L^1 for approximate difference quotients

A first open problem

- **More in general we define** $v \in \mathcal{M}N^{s,\gamma}$ for

$$0 \leq s \leq 1 \qquad \gamma \geq 1$$

if

$$\sup_{\lambda \geq 0} \lambda^{\gamma} |\{x \in \Omega \, : \, |v(x+h) - v(x)| > \lambda\}| \leq c|h|^{s\gamma}$$

- **And, accordingly**

$$\begin{aligned} &\|v\|_{\mathcal{M}N^{s,\gamma}} \\ &\quad := \|v\|_{L^{\gamma}} \\ &\quad + \sup_{|h|>0, \lambda \geq 0} \left(\lambda |h|^{-s} |\{x \in \Omega \, : \, |v(x+h) - v(x)| > \lambda\}|^{\frac{1}{\lambda}} \right) \end{aligned}$$

- **These, we believe, are the right spaces to study the exact differentiability properties of solutions to measure data problems**

Caccioppoli's estimate - scaling vs optimality

- **Local estimate**

$$\int_{B_{R/2}}\int_{B_{R/2}} \frac{|Du(x) - Du(y)|}{|x-y|^{n+1-\varepsilon}}\,dx\,dy$$

$$\leq \frac{c}{R^{1-\varepsilon}}\int_{B_R}(|Du|+s)\,dx + cR^{\varepsilon}|\mu|(B_R)$$

- **For the case div $a(Du) = f \in L^2$, no ε appears**

$$\int_{B_{R/2}} |D^2u|^2\,dx\,dy$$

$$\leq \frac{c}{R^2}\int_{B_R}(|Du|+s)^2\,dx + c\int_{B_R}|f|^2\,dx$$

the Caccioppoli's estimate we are looking for

- **The following cannot hold**

$$\int_{B_{R/2}} |D^2u| dx\, dy \leq \frac{c}{R} \int_{B_R} (|Du| + s)\, dx + c|\mu|(B_R)$$

- **Maybe**

$$\sup_{\lambda,h} \lambda |h|^{-1} |\{x \in B_{R/2} \,:\, |Du(x+h) - Du(x)| > \lambda\}| \leq \frac{c}{R} \int_{B_R} (|Du| + s)\, dx + c|\mu|(B_R)$$

Weak Schauder theory

- **Consider the equation**

$$\text{div } a(x, Du) = 0$$

with

$$|a(x,z) - a(x_0,z)| \leq L|x - x_0|^\alpha (1 + |z|) \qquad \alpha \in (0,1]$$

that is

$$\frac{a(\cdot, z)}{(1 + |z|)} \in C^{0,\alpha} \equiv N^{\alpha,\infty}$$

Then

$$Du \in C^\alpha \equiv N^{\alpha,\infty}$$

Weak Schauder theory

- **In the vectorial case this is not true, but nevertheless**

$$Du \in N^{\alpha,2}$$

Min. (ARMA 03)

- **The rate of integrability decreases due to singularities, but the rate of differentiability is preserved**
- **Now consider the equation**

$$\text{div}\ a(x, Du) = \mu$$

with

$$|a(x,z) - a(x_0,z)| \leq L|x-x_0|^\alpha(1+|z|) \qquad \alpha \in (0,1]$$

Then

$$Du \in N^{\alpha-\varepsilon,1} \qquad \forall\, \varepsilon > 0$$

- **The differentiability and the integrability scales separate**

A related problem

- **Prove/disprove that**

$$Du \in \mathcal{M}N^{\alpha,1}$$

that is

$$\sup_{\lambda \geq 0} \lambda |\{x \in \Omega \,:\, |Du(x+h) - Du(x)| > \lambda\}| \leq c|h|^{\alpha}$$

Differentiability of Du for $p \neq 2$

- **For $W^{1,p}$-solutions to** div$(|Du|^{p-2}Du) = 0$ the existence of second derivatives is an open problem while

$$Du \in W^{\frac{2}{p},p}$$

- **Non-linear uniformization phenomenon**

$$V_s(Du) := (s^2 + |Du|^2)^{\frac{p-2}{4}} Du \qquad V(Du) := |Du|^{\frac{p-2}{2}} Du$$

then

$$V_s(Du) \in W^{1,2}$$

Uhlenbeck (Acta Math. 77)

- **Loss of integrability - gain in differentiability**

$$|V(Du)| \approx |Du|^{\frac{p}{2}}$$

Differentiability of Du for $p \neq 2$

- **What should we expect?** We have

$$\frac{1}{|x|^{\beta}} \in W^{s,\gamma}(B) \Longleftrightarrow \beta < \frac{n}{\gamma} - s$$

We apply this fact to the fundamental solution

$$|Du| \approx \frac{1}{|x|^{\frac{n-1}{p-1}}}$$

with the natural choice $\gamma = p - 1$, to optimize the differentiability parameter. This yields

$$s < \frac{1}{p-1}$$

and we expect

$$Du \in W^{s,p-1} \qquad \forall \;\; s < \frac{1}{p-1}$$

Indeed....

- **Theorem 2**

$$Du \in W^{\frac{1-\varepsilon}{p-1},p-1} \qquad \text{for every } \varepsilon > 0$$

- **Recall that we are assuming** $p \geq 2$ so that

$$\frac{1}{p-1} \leq 1$$

Integrability recovered

- **Corollary**

$$|Du|^{p-1} \in L^{\frac{n}{n-1}-\varepsilon} \qquad \text{since} \qquad W^{\frac{1-\varepsilon}{p-1},p-1} \hookrightarrow L^{\frac{n(p-1)}{n-1+\varepsilon}}$$

  This gives back the first result of Boccardo & Gallöuet (J. Funct. Anal. 89)

- **Theorem 2 is optimal in the scale of fractional Sobolev spaces, i.e. we cannot take $\varepsilon = 0$ otherwise**

$$|Du|^{p-1} \in L^{\frac{n}{n-1}} \qquad \text{false in general (fundamental solution)}$$

The map $V(Du)$

- **Theorem 3**

$$|Du|^{\frac{p-2}{2}} Du = V(Du) \in W^{\frac{p(1-\varepsilon)}{2(p-1)}, \frac{2(p-1)}{p}} \qquad \text{for every } \varepsilon > 0$$

- **Also optimal, i.e. we cannot take $\varepsilon = 0$ otherwise**

$$|V(Du)|^{\frac{2}{p}} \approx |Du| \in L^{\frac{n(p-1)}{n-1}}$$

  which is false in general (fundamental solution)

- **For $p = 2$ both Theorems 2 and 3 reduce to Theorem 1**

BV type behavior of Du

- **For $p = 2$ consider $\triangle u = \mu$**
- **Switching (which is forbidden for measure or L^1-data) $\triangle u$** with $D^2 u$, you would have that Du is BV. Nevertheless something survives....
- **Corollary** In the general case $p \neq 2$ consider the singular set

$$\Sigma_u := \left\{ x \in \Omega : \liminf_{\rho \searrow 0} -\!\!\!\!\!\!\int_{B(x,\rho)} |Du(y) - (Du)_{x,\rho}|^{p-1} \, dy > 0 \right.$$

$$\left. \text{or} \qquad \limsup_{\rho \searrow 0} |(Du)_{x,\rho}| = \infty \right\}$$

Then its Hausdorff dimension satisfies

$$\dim(\Sigma_u) \leq n - 1$$

an estimate which is independent of p

Pointwise behavior of Du

- **This follows from a simple potential theory fact. Define**

$$\Sigma_v := \left\{ x \in \Omega : \liminf_{\rho \searrow 0} \fint_{B(x,\rho)} |v(y) - (v)_{x,\rho}|^\gamma \, dy > 0 \right.$$
$$\left. \text{or} \qquad \limsup_{\rho \searrow 0} |(v)_{x,\rho}| = \infty \right\}$$

- **Then**

$$v \in W^{s,\gamma} \Longrightarrow \dim(\Sigma_v) \le n - s\gamma$$

provided $s\gamma < n$

Open problems

- **Prove that**

$$|Du|^{p-2}Du \in W^{1-\varepsilon,1} \qquad \text{for every } \varepsilon > 0$$

- **Limiting case**

$$\|D[|Du_k|^{p-2}Du_k]\|_{\mathcal{M}^1} \leq c$$

where u_k are the approximating solutions.

- **Warning: even for the approximating solutions we must give $D[|Du|^{p-2}Du]$ a sense. Recall that only $D[|Du|^{\frac{p-2}{2}}Du]$ is known to be in L^2.**

What about the (non-linear) Green's function

- **Solution to $\triangle_p u = \delta$ goes as**

$$\frac{1}{|x|^{\frac{n-p}{p-1}}}$$

so that

$$|Du| \approx \frac{1}{|x|^{\frac{n-1}{p-1}}}$$

and

$$|Du|^{p-1} \approx \frac{1}{|x|^{n-1}}$$

finally

$$\left|D[|Du|^{p-1}]\right| \approx \frac{1}{|x|^n} \in \mathcal{M}^1$$

Further weak Schauder estimates

- **Assume again**

$$|a(x,z)-a(x_0,z)| \leq L|x-x_0|^{\alpha}(1+|z|)^{p-1} \qquad \alpha \in (0,1]$$

If

$$\text{div } a(x,Du) = f \in L^{\gamma}$$

we have

$$\|Du(\cdot+h)-Du(\cdot)\|_{L^{\gamma(p-1)}} \leq c|h|^{\frac{1-\varepsilon}{p-1}}$$

that is $Du \in N^{\frac{\alpha-\varepsilon}{p-1},\gamma(p-1)}$

- **And recall that** in the case $f=0$ classical Schauder theory gives

$$\|Du(\cdot+h)-Du(\cdot)\|_{L^{\infty}} \leq c|h|^{\frac{1-\varepsilon}{p-1}}$$

that is $Du \in C^{0,\alpha} \equiv N^{\alpha,\infty}$

Separation of scales

- **Differentiability persists, provided being red in the proper norm**, Lebesgue or Marcinkiecz norms must be used to quantify the difference quotients.
- **Another example** of this phenomenon is given by the solution to

$$\triangle_p u = 1$$

then

$$Du \approx |x|^{\frac{1}{p-1}}$$

- **Therefore**

$$Du \in C^{0,\frac{1}{p-1}} \equiv N^{\frac{1}{p-1},\infty}$$

i.e. the right hand side datum of the equation determines the integrability range of the solution, but does not affect the differentiability one

- **While for measure data problems** we have seen

$$Du \in N^{\frac{1-\varepsilon}{p-1},p-1}$$

Differentiability of Du for function data

- **This time we consider**

$$-\text{div}\ a(x, Du) = f \in L^\gamma \qquad 1 < \gamma < (p^*)' = \frac{np}{np - n + p}$$

- **Theorem 4**

$$Du \in W^{\frac{1-\varepsilon}{p-1}, \gamma(p-1)} \qquad \forall\, \varepsilon > 0$$

- **In particular, for** $p = 2$ we have

$$Du \in W^{1-\varepsilon, \gamma} \qquad \forall\, \varepsilon > 0$$

As a consequence we have only

$$|Du|^{p-1} \in L^{\frac{n\gamma}{n-\gamma} - \varepsilon} \qquad \forall\, \varepsilon > 0$$

but not the sharp result $|Du|^{p-1} \in L^{\frac{n\gamma}{n-\gamma}}$

Pointwise behavior of Du

- **Corollary** Consider the singular set

$$\Sigma_u := \left\{ x \in \Omega : \liminf_{\rho \searrow 0} \fint_{B(x,\rho)} |Du(y) - (Du)_{x,\rho}|^{\gamma(p-1)}\, dy > 0 \right.$$

$$\left. \text{or} \qquad \limsup_{\rho \searrow 0} |(Du)_{x,\rho}| = \infty \right\}$$

Then its Hausdorff dimension satisfies

$$\dim(\Sigma_u) \leq n - \gamma$$

Open problems

- **For solutions to**

$$-\text{div}\, a(x, Du) = f \in L^{\gamma} \qquad 1 < \gamma < (p^*)' = \frac{np}{np - n + p}$$

- **Prove**

$$Du \in W^{\frac{1}{p-1}, \gamma(p-1)}$$

- **In particular, for** $p = 2$ prove that

$$Du \in W^{1,\gamma}$$

- **By standard linear theory this is the case for**

$$\triangle u = f$$

Part 3: Finer and different scales - Density conditions

Morrey spaces

- **A density condition**

$$|\mu|(B_R) \lesssim R^{n-\theta} \qquad \theta \in [0,n]$$

- **Norm for measures**

$$\|\mu\|_{L^{1,\theta}(\Omega)} := \sup_{B_R \subset \Omega} R^{\theta-n}|\mu|(B_R)$$

- **Functions**

$$\int_{B_R} |v|^\gamma \, dx \lesssim R^{n-\theta}$$

$$\|v\|^\gamma_{L^{\gamma,\theta}(\Omega)} := \sup_{B_R \subset \Omega} R^{\theta-n} \int_{B_R} |v|^\gamma \, dx$$

Be careful with Morrey spaces

- **In fact we have**

$$L^{\gamma,n} \equiv L^{\gamma} \qquad L^{\gamma,0} \equiv L^{\infty}$$

- **but, for instance**

$$L^{1,\theta} \not\subset L \log L$$

  whenever θ is close to zero. Therefore this is an integrability scale orthogonal to the Lebesgue one.

- **Also: Morrey spaces are neither rearrangement invariant spaces (obvious) nor interpolation spaces (Stein, Zygmund, Blasco, Ruiz, Vega)**

Morrey spaces

- **Morrey-Marcinkiewicz spaces (Stampacchia, Adams, Lewis)**

$$\sup_{\lambda>0} \lambda^{\gamma} |B_R \cap \{|v| > \lambda\}| \lesssim R^{n-\theta}$$

and set

$$\begin{aligned} \|v\|^{\gamma}_{\mathcal{M}^{\gamma,\theta}(\Omega)} &:= \sup_{B_R \subset \Omega} \sup_{\lambda>0} \lambda^{\gamma} R^{\theta-n} |B_R \cap \{|v| > \lambda\}| \\ &:= \sup_{B_R \subset \Omega} R^{\theta-n} \|v\|^{\gamma}_{\mathcal{M}(B_R)} \end{aligned}$$

Lorentz spaces

- g **belongs to** $L(\gamma, q)(\Omega)$ **iff**

$$\|g\|_{L(\gamma,q)(\Omega)} := \left[\gamma \int_0^\infty (\lambda^\gamma |\{x \in \Omega \ : \ |g(x)| > \lambda\}|)^{\frac{q}{\gamma}} \ \frac{d\lambda}{\lambda}\right]^{\frac{1}{q}} < \infty$$

where

$$1 \leq \gamma < \infty \qquad\qquad 0 < q \leq \infty$$

For $q = \infty$ we have

$$L(\gamma, \infty) = \mathcal{M}^\gamma$$

- **The second index tunes the first one and for**

$$0 < q < t < r \leq \infty$$

we have

$$L^r \equiv L(r,r) \subset L(t,q) \subset L(t,t) \subset L(t,r) \subset L(q,q) \equiv L^q$$

Lorentz spaces are not bizarre

- **Recall that**

$$\frac{1}{|x|^{\frac{n}{\gamma}}} \in \mathcal{M}^\gamma \equiv L(\gamma,\infty)$$

Then improving the integrability in a logarithmic factor gives

$$\frac{1}{|x|^{\frac{n}{\gamma}} \log^\beta |x|} \in L(\gamma,q) \Longleftrightarrow q > \frac{1}{\beta}$$

so we conclude that Lorentz spaces are even less fine than we would like!!!

- **Characterization, vertical dyadic splitting**

$$g \leq \sum_{k\in\mathbb{Z}} 2^k \chi_{A_k} \quad \text{and} \quad \|2^k|A_k|^{\frac{1}{\gamma}}\|_{l^q(\mathbb{Z})} \approx_{\gamma,q} \|g\|_{L(\gamma,q)(\Omega)}$$

Lorentz-Morrey spaces

- g **belongs to** $L^{\theta}(\gamma, q)(\Omega)$ **iff**

$$\|g\|_{L(\gamma,q)(B_R)} \lesssim R^{\frac{n-\theta}{\gamma}}$$

whenever

$$1 \leq \gamma < \infty \qquad 0 < q < \infty \qquad \theta \in [0, n]$$

- **Morreyzation of the Lorentz norm**

$$\|g\|_{L^{\theta}(\gamma,q)(\Omega)} := \sup_{B_R \subseteq \Omega} R^{\frac{\theta-n}{\gamma}} \|g\|_{L(\gamma,q)(B_R)}$$

Adams' theorem (Duke Math. J. 75)

- **Morrey regularization**

$$f \in L^{\gamma,\theta} \Longrightarrow I_\beta(f) \in L^{\frac{\theta\gamma}{\theta-\gamma\beta},\theta} \qquad \gamma\beta < \theta$$

- **The case** $\gamma = 1$

$$\mu, f \in L^{1,\theta} \Longrightarrow I_\beta(\mu, f) \in \mathcal{M}^{\frac{\theta}{\theta-\beta},\theta} \qquad \beta < \theta$$

- **Borderline**

$$f \in L^{1,\theta} \cap L \log L \Longrightarrow I_\beta(f) \in L^{\frac{\theta}{\theta-\beta}} \qquad \beta < \theta$$

Adams' theorem (Duke Math. J. 75)

- **Corollary, for** $\triangle u = f$ (take $\beta = 1$) it holds that

$$f \in L^{\gamma,\theta} \Longrightarrow Du \in L^{\frac{\theta\gamma}{\theta-\gamma},\theta} \qquad 1 < \gamma < \theta$$

- $L^{1,\theta}$**-data**

$$f \in L^{1,\theta} \Longrightarrow Du \in \mathcal{M}^{\frac{\theta}{\theta-1},\theta}$$

- **Borderline**

$$f \in L^{1,\theta} \cap L \log L \Longrightarrow Du \in L^{\frac{\theta}{\theta-1}}$$

Significant case

- **Another theorem of Adams**

$$|\mu|(B_R) \lesssim R^{n-\theta}, \qquad \theta < p \Longrightarrow \mu \in W^{-1,p'}$$

- **Therefore**

$$Du \in L^p \qquad \text{maximal regularity}$$

holds for solutions to div $(c(x,u)|Du|^{p-2}Du) = \mu$

- **Significant case**

$$|\mu|(B_R) \lesssim R^{n-\theta} \qquad p \leq \theta \leq n$$

The non-linear case div $a(x, u, Du) = \mu, f$

- **When $\theta = n$**

$$f \in L^{\gamma} \qquad \text{and} \qquad \gamma < \frac{np}{np - n + p} = (p^*)'$$

- **When $\theta < n$, we take**

$$f \in L^{\gamma,\theta} \qquad \text{and} \qquad \gamma \leq \frac{\theta p}{\theta p - \theta + p} =: p(\theta)$$

This is *initially* suggested by the fact when dealing with Morrey spaces θ plays the role of the dimension n.

Non-linear Adams' theorem

- **For** div $a(x,u,Du) = \mu, f$ it holds that

$$f \in L^{\gamma,\theta} \Longrightarrow |Du|^{p-1} \in L^{\frac{\theta\gamma}{\theta-\gamma},\theta} \qquad 1 < \gamma \leq p(\theta)$$

- $L^{1,\theta}$**-case**

$$\mu, f \in L^{1,\theta} \Longrightarrow |Du|^{p-1} \in \mathcal{M}^{\frac{\theta}{\theta-1},\theta}$$

- **Bordeline**

$$\mu, f \in L^{1,\theta} \cap L \log L \Longrightarrow |Du|^{p-1} \in L^{\frac{\theta}{\theta-1}}$$

$$\mu, f \in L \log L^{\theta} \Longrightarrow |Du|^{p-1} \in L^{\frac{\theta}{\theta-1},\theta}$$

Morrey-Gehring regularity

- **Results for**

$$\gamma > \frac{\theta p}{\theta p - \theta + p} = p(\theta)$$

- **For solutions to div $a(x, u, Du) = f$ it holds that if**

$$\gamma > \frac{\theta p}{\theta p - \theta + p} \qquad \text{and} \qquad p < \theta \leq n$$

then

$$f \in L^{\gamma,\theta} \Longrightarrow Du \in L^{h,\theta} \qquad h > p$$

A theorem of Adams & Lewis (Studia Math. 82)

- **Optimal**

$$f \in L^{\theta}(\gamma, q) \Longrightarrow I_{\beta}(f) \in L^{\theta}\left(\frac{\theta\gamma}{\theta - \beta\gamma}, \frac{\theta q}{\theta - \beta\gamma}\right) \qquad \gamma\beta < \theta$$

- **The standard case $\theta = n$, via interpolation**

$$f \in L(\gamma, \theta) \Longrightarrow I_{\beta}(f) \in L\left(\frac{n\gamma}{n - \beta\gamma}, q\right) \qquad \gamma\beta < \theta$$

A discontinuity phenomenon

- **A discontinuity phenomenon**

$$L^{\theta}\left(\frac{\theta\gamma}{\theta-\beta\gamma},\frac{\theta q}{\theta-\beta\gamma}\right)\subset L\left(\frac{\theta\gamma}{\theta-\beta\gamma},\frac{\theta q}{\theta-\beta\gamma}\right)\subset L\left(\frac{n\gamma}{n-\beta\gamma},q\right)$$

- **but**

$$L\left(\frac{n\gamma}{n-\beta\gamma},q\right)\subset L\left(\frac{n\gamma}{n-\beta\gamma},\frac{nq}{n-\beta\gamma}\right)$$

- **Therefore while**

$$\lim_{\theta\to n} L^{\theta}(\gamma,q)=L(\gamma,q)$$

we do not recover the sharp embedding result form the Lorentz-Morrey one letting $\theta \to n$

The non-linear Adams & Lewis theorem

- **For solutions to**

$$\text{div}\ a(x, Du) = f \in L^{\theta}(\gamma, q)$$

- **we have**

$$|Du|^{p-1} \in L^{\theta}\left(\frac{\theta\gamma}{\theta-\gamma}, \frac{\theta q}{\theta-\gamma}\right) \qquad \text{locally in } \Omega\ .$$

- **For $p = 2$ we recover the linear result**
- **Taking $\gamma = q$ gives back Adams' theorem**

The local estimate

- **Optimal local estimates are available**

$$
\begin{aligned}
\| |Du|^{p-1} \|_{L^{\theta}\left(\frac{\theta\gamma}{\theta-\gamma}, \frac{\theta q}{\theta-\gamma}\right)(B_{R/2})} & \\
\leq cR^{\frac{\theta-\gamma}{\gamma}-n} \| (|Du| + s)^{p-1} \|_{L^1(B_R)} & \\
+ c\|f\|_{L^{\theta}(\gamma,q)(B_R)} &
\end{aligned}
$$

Back to the case $\theta = n$ - Pure Lorentz spaces

- **For solutions to**

$$\text{div } a(x, Du) = f \in L(\gamma, q) \qquad 1 < \gamma \leq \frac{np}{np - n + p} = (p^*)'$$

- **we have**

$$|Du|^{p-1} \in L\left(\frac{n\gamma}{n-\gamma}, q\right) \qquad \text{locally in } \Omega\ .$$

- **This result was known in the case** $1 < \gamma < (p^*)'$, proved by Kilpeläinen & Li for $q = \infty$, and later by Alvino & Ferone & Trombetti for $q > 0$
- **In the borderline case** $\gamma = (p^*)'$ we have

$$Du \in L(p, q(p-1))$$

and we are around the maximal regularity

Back to measure data

- **Concentration of measures $\Longrightarrow$ Faster blow-up of solutions**

$$|\mu|(B_R) \lesssim R^{n-\theta} \Longrightarrow |Du|^{p-1} \in \mathcal{M}^{\frac{\theta}{\theta-1},\theta}$$

- **In the borderline case $\theta = p$**

$$Du \in \mathcal{M}^{p,p} \subset \mathcal{M}^{p}$$

- **For $\theta = n$ (no real Morrey spaces) this covers Boccardo-Gallöuet results**

$$|Du|^{p-1} \in \mathcal{M}^{\frac{n}{n-1},n} = \mathcal{M}^{\frac{n}{n-1}}$$

- **For $p = n$, which forces $\theta = n$. In the case of equations this covers the results of Dolzmann-Hungerühler-Müller and Kilpeläinen-Shanmugalingam-Zhong**, who proved, for $p = n$, that

$$Du \in \mathcal{M}^{n}$$

Connections with removability of singularities

- **A theorem of Serrin (Acta Math. 64)**

$$\text{div } a(x, Du) = 0 \qquad \in \Omega \setminus C$$

where C is a compact set with negligible θ-capacity, and $p < \theta \leq n$. If

$$|u|^{p-1} \in L^{\frac{\theta}{\theta-p}+\delta}$$

and in particular if

$$|Du|^{p-1} \in L^{\frac{\theta}{\theta-1}+\delta}$$

then

$$\text{div } a(x, Du) = 0 \qquad \text{in } \Omega$$

An idea of the proof

- **In the linear case we have**

$$|Du| \lesssim I_1(|f|)$$

  and conclusion follows by the Riesz potential behavior

- **In the non-linear case we instead have, up to controllable corrections**

$$||Du|^{p-1} > \lambda| \lesssim |I_1(|f|) > \lambda|$$

  and once again the conclusion follows by the Riesz potential behavior

- We are back to the linear case

Good λ-inequality principle

- **To implement the previous fact we actually have something like**

$$
\begin{aligned}
|\{|Du|^{p-1} \geq T\lambda\}| \quad &\leq \quad \frac{1}{T^{\beta}}|\{|Du|^{p-1} \geq \lambda\}| \\
&\qquad +|\{I_1(|f|) \geq \varepsilon\lambda\}|
\end{aligned}
$$

- **In a similar way**

$$
\begin{aligned}
|\{|u|^{p-1} \geq T\lambda\}| \quad &\leq \quad \frac{1}{T^{\beta_0}}|\{|u|^{p-1} \geq \lambda\}| \\
&\qquad +|\{I_p(|f|) \geq \varepsilon\lambda\}|
\end{aligned}
$$

Higher derivatives in Morrey spaces

- **Classical result**

$$\triangle u = f \in L^{\gamma,\theta} \qquad \gamma > 1$$

then

$$D^2 u \in L^{\gamma,\theta}$$

- **It extends to much more general elliptic problems – see results of Lieberman (J. Funct. Anal. 03)**

Besov-Morrey spaces

- **Classical definition. For**

$$0 < s < 1 \qquad \gamma \geq 1 \qquad \theta \in [0, n]$$

$$v \in W^{s,\gamma,\theta}$$

iff

$$\int_{B_R} \int_{B_R} \frac{|v(x) - v(y)|^\gamma}{|x - y|^{n+s\gamma}} \, dx \, dy \lesssim R^{n-\theta}$$

- **Intuitively**

$$\int_{B_R} |D^s v(x)|^\gamma \, dx \lesssim R^{n-\theta}$$

Non-linear results

- **For solutions u to**

$$\text{div } a(x, Du) = \mu \in L^{\gamma,\theta} \qquad 1 \leq \gamma < (p^*)'$$

it holds that

$$Du \in W^{\frac{1-\varepsilon}{p-1},\gamma(p-1),\theta} \qquad \forall\ \varepsilon > 0$$

- **In particular, for $p = 2$**

$$Du \in W^{1-\varepsilon,\gamma,\theta}$$

Last open problem

- **In the case $\gamma > 1$ prove that**

$$Du \in W^{\frac{1}{p-1},\gamma(p-1),\theta}$$

- **In particular, for $p = 2$**

$$Du \in W^{1,\gamma,\theta}$$

as for the linear case

$$\triangle u = f$$