Weighted norm inequalities and Rubio de Francia extrapolation

José María Martell

Instituto de Ciencias Matemáticas
CSIC-UAM-UC3M-UCM
Spain

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Part I

Muckenhoupt Weights and Weighted Norm Inequalities

References: Weighted norm inequalities

- J. Duoandikoetexea, *Fourier analysis*, Graduate Studies in Mathematics 29, American Mathematical Society, Providence, RI, 2001.
- J. García-Cuerva & J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies 116, North-Holland Publishing Co., Amsterdam, 1985.
- L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, Inc., Upper Saddle River, 2004.

Introduction

Weights and Extrapolation

How much information is contained in the following inequalities?

1. $$\int_{\mathbb{R}^n} |Tf(x)|^2\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} |f(x)|^2\, w(x)\, dx, \quad \forall\, w \in A_2$$

2. $$\int_{\mathbb{R}^n} |Tf(x)|^{p_0}\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} |f(x)|^{p_0}\, w(x)\, dx, \quad \forall\, w \in A_{p_0}$$
   ($1 < p_0 < \infty$ is fixed)

3. $$\int_{\mathbb{R}^n} |Tf(x)|^2\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} |Sf(x)|^2\, w(x)\, dx, \quad \forall\, w \in A_\infty$$

4. $$\int_{\mathbb{R}^n} |Tf(x)|^{p_0}\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} |f(x)|^{p_0}\, w(x)\, dx, \quad \forall\, w \in A_\infty$$
   ($0 < p_0 < \infty$ is fixed)

Section 1

Muckenhoupt Weights

Muckenhoupt weights

- Weights $w \geq 0$ a.e., $w \in L^1_{\text{loc}}(\mathbb{R}^n)$
- $L^p(w) = L^p(w(x)\,dx) \rightsquigarrow \|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p\, w(x)\,dx\right)^{\frac{1}{p}}$
- $L^{p,\infty}(w) \rightsquigarrow \|f\|_{L^{p,\infty}(w)} = \sup_{\lambda>0} \lambda\, w\{x \in \mathbb{R}^n : |f(x)| > \lambda\}^{\frac{1}{p}}$

Muckenhoupt's problem

- Characterize weights w so that $M : L^p(w) \longrightarrow L^p(w)$

$$\int_{\mathbb{R}^n} Mf(x)^p\, w(x)\,dx \lesssim \int_{\mathbb{R}^n} |f(x)|^p\, w(x)\,dx$$

- Characterize weights w so that $M : L^p(w) \longrightarrow L^{p,\infty}(w)$

$$w\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \lesssim \frac{1}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p\, w(x)\,dx$$

Muckenhoupt's problem: Weak-type

Proposition

Let $1 \le p < \infty$. $M : L^p(w) \longrightarrow L^{p,\infty}(w)$ *if and only if*

$$(A_p) \qquad \left(\fint_Q w\,dx\right)\left(\fint_Q w^{1-p'}\,dx\right)^{p-1} \le C, \qquad p>1$$

$$(A_1) \qquad \fint_Q w\,dx \le C\,w(y), \qquad a.e.\,y \in Q, \qquad p=1$$

Scheme of the proof

- $\Longrightarrow$ $\begin{cases} \bullet\ p>1 \rightsquigarrow f = w^{1-p'}\chi_Q, \quad \lambda = \fint_Q f = \fint_Q w^{1-p'}\,dx \\ \bullet\ p=1 \rightsquigarrow f=\chi_S, \quad w\,\chi_S \approx \inf_Q w, \quad \lambda = \fint_Q f = \dfrac{|S|}{|Q|} \end{cases}$
- $\Longleftarrow$ Hölder, Vitali.

Muckenhoupt weights: Properties

Definition

- $w \in A_p \qquad \left(\fint_Q w\,dx\right)\left(\fint_Q w^{1-p'}\,dx\right)^{p-1} \le C$
- $w \in A_1 \qquad \fint_Q w\,dx \le C\,w(y), \qquad \text{a.e. } y \in Q$
- $A_\infty = \bigcup_{p\ge 1} A_p$

Properties

- $A_1 \subset A_p \subset A_q$, $1 < p < q$
- $w \in A_p \quad \Longleftrightarrow \quad w^{1-p'} \in A_{p'}$
- $w_1,\, w_2 \in A_1 \quad \Longrightarrow \quad w_1\, w_2^{1-p} \in A_p$ Reverse Factorization

Muckenhoupt weights: Examples

$$(A_p) \quad \left(\fint_Q w\,dx\right)\left(\fint_Q w^{1-p'}\,dx\right)^{p-1} \le C$$

$$(A_1) \quad \fint_Q w\,dx \le C\,w(y), \text{ a.e. } y \in Q \;\equiv\; Mw(x) \le C\,w(x) \text{ a.e. } x \in \mathbb{R}^n$$

Examples

- $w(x) = 1 \in A_1$
- $w(x) = |x|^\alpha \in A_p \iff \begin{cases} -n < \alpha \le 0 & p = 1 \\ -n < \alpha < n\,(p-1) & p > 1 \end{cases}$
- $w(x) = Mf(x)^\delta \in A_1$, for all $0 < \delta < 1$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $Mf < \infty$

  Coifman: $w \in A_1 \implies w(x) \approx Mf(x)^\delta$

A_1 weights: The Rubio de Francia Algorithm

Constructing A_1 weights

Let $0 \le u \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Find $U \ge 0$ such that

1. $0 \le u(x) \le U(x)$ a.e. $x \in \mathbb{R}^n$
2. $\|U\|_p \lesssim \|u\|_p$
3. $U \in A_1$, that is, $MU(x) \lesssim U(x)$ a.e. $x \in \mathbb{R}^n$

The Rubio de Francia Algorithm

- $U = Mu$ WRONG!!! $M\chi_{Q_0}(x) \approx (1+|x|)^{-n} \notin A_1$
- $U = M(u^r)^{\frac{1}{r}}$, $1 < r < p$
- $U = \mathcal{R}u = \displaystyle\sum_{k=0}^{\infty} \frac{M^k u}{2^k \|M\|_{L^p}^k}$

$$\begin{cases} \bullet\ 0 \le u(x) \le \mathcal{R}u(x) \\ \bullet\ \|\mathcal{R}u\|_p \le 2\|u\|_p \\ \bullet\ M(\mathcal{R}u)(x) \le 2\|M\|_p\,\mathcal{R}u(x) \end{cases}$$

Muckenhoupt's problem: Strong-type and Reverse Hölder

- $w \in A_q \equiv \left.\begin{array}{l} M : L^q(w) \longrightarrow L^{q,\infty}(w) \\ M : L^\infty(w) \longrightarrow L^\infty(w) \end{array}\right\} \rightsquigarrow \begin{array}{c} M : L^r(w) \longrightarrow L^r(w) \\ q < r < \infty \end{array}$

Can we move (a little) to the left? YES

Theorem (Reverse Hölder Inequality)

Given $w \in A_p$, *there exists* $\epsilon > 0$ *such that*

$$(RH_{1+\epsilon}) \qquad \left(\fint_Q w(x)^{1+\epsilon}\, dx\right)^{\frac{1}{1+\epsilon}} \le C \fint_Q w(x)\, dx$$

Consequently,

- $w^{1+\delta} \in A_p$ *for some* $\delta > 0$
- $w \in A_q$ *for some* $1 < q < p$

Theorem (Muckenhoupt's Theorem)

Let $1 < p < \infty$. $\qquad M : L^p(w) \longrightarrow L^p(w) \iff w \in A_p$

Muckenhoupt Weights: Properties

- P. Jones' Factorization

$$w \in A_p,\ 1 < p < \infty \quad \Longleftrightarrow \quad w = w_1\, w_2^{1-p} \text{ with } w_1, w_2 \in A_1$$

- $A_\infty = \bigcup_{p\geq 1} A_p$ can characterized by
  - $w \in RH_{1+\epsilon}$ for some $\epsilon > 0$
  - $\exists\, \delta > 0$ such that $\dfrac{w(S)}{w(Q)} \leq C \left(\dfrac{|S|}{|Q|}\right)^{\delta}$, $S \subset Q$
  - $\exists\, 0 < \alpha, \beta < 1$ such that $S \subset Q$, $\dfrac{|S|}{|Q|} < \alpha \implies \dfrac{w(S)}{w(Q)} < \beta$
  - $\left(\fint_Q w\, dx\right) \exp\left(\fint_Q \log w^{-1}\, dx\right) \leq C$

Extrapolation at first glance

Theorem (Rubio de Francia; García-Cuerva)

Let $0 < p_0 < \infty$. *Assume that* T *satisfies*

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0}\, w(x)\, dx \le C_w \int_{\mathbb{R}^n} |f(x)|^{p_0}\, w(x)\, dx, \qquad \forall w \in A_1.$$

Then T *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $p > p_0$.

Proof. Let $r = p/p_0 > 1$. By duality, $\exists\, h \ge 0$, $\|h\|_{r'} = 1$ such that

$$\|Tf\|_p^{p_0} = \big\| |Tf|^{p_0} \big\|_r = \int_{\mathbb{R}^n} |Tf|^{p_0}\, h\, dx \le \int_{\mathbb{R}^n} |Tf|^{p_0}\, \mathcal{R}h\, dx$$

$$\lesssim \int_{\mathbb{R}^n} |f|^{p_0}\, \mathcal{R}h\, dx \le \|f\|_p^{p_0}\, \|\mathcal{R}h\|_{r'} \lesssim \|f\|_p^{p_0}$$

$$\mathcal{R}h = \sum_{k=0}^{\infty} \frac{M^k h}{2^k\, \|M\|_{L^{r'}}^k}$$

Other Maximal operators

$$M_{\mathcal{Q}}f(x) = \sup_{Q\in\mathcal{Q}, Q\ni x} \fint_Q |f(y)|\,dy, \qquad \mathcal{Q} = \{\text{cubes in } \mathbb{R}^n\}$$

$$M_{\mathcal{D}}f(x) = \sup_{Q\in\mathcal{D}, Q\ni x} \fint_Q |f(y)|\,dy, \qquad \mathcal{D} = \{\text{dyadic cubes in } \mathbb{R}^n\}$$

$$M_{\mathcal{R}}f(x) = \sup_{R\in\mathcal{R}, R\ni x} \fint_R |f(y)|\,dy \qquad \mathcal{R} = \{\text{Rectangles in } \mathbb{R}^n\}$$

$$M_{\mathcal{Z}}f(x) = \sup_{R\in\mathcal{Z}, R\ni x} \fint_R |f(y)|\,dy \qquad \mathcal{Z} = \{\text{Rectangles } (s, t, s\,t) \text{ in } \mathbb{R}^3\}$$

Muckenhoupt Bases

Definitions

- Basis: $\mathcal{B}$ collection of open sets $B \subset \mathbb{R}^n$
- Maximal operator: $M_{\mathcal{B}} f(x) = \sup_{B\in\mathcal{B}, B\ni x} \fint_B |f(y)|\,dy, \quad x \in \bigcup_{B\in\mathcal{B}} B$
- Weight: $0 < w(B) < \infty$ for every $B \in \mathcal{B}$
- Muckenhoupt weights: $A_{\infty,\mathcal{B}} = \bigcup_{p\ge 1} A_{p,\mathcal{B}}$
  - $w \in A_{p,\mathcal{B}} \qquad \left(\fint_B w\,dx\right)\left(\fint_B w^{1-p'}\,dx\right)^{p-1} \le C$
  - $w \in A_{1,\mathcal{B}} \qquad M_{\mathcal{B}} w(x) \le C\,w(x), \quad \text{a.e. } x \in \mathbb{R}^n$
- Muckenhoupt Basis: $M_{\mathcal{B}} : L^p(w) \to L^p(w)$, $\forall\, w \in A_{p,\mathcal{B}}$, $1 < p < \infty$

Muckenhoupt Bases

- $\mathcal{B}$ Muckenhoupt basis $\rightsquigarrow M_{\mathcal{B}} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$, $1 < p < \infty$
    - $M_{\mathcal{B}}$ may fail to be of weak-type $(1,1)$

Properties

- $A_{1,\mathcal{B}} \subset A_{p,\mathcal{B}} \subset A_{q,\mathcal{B}}$, $1 < p < q$
- $w \in A_{p,\mathcal{B}} \iff w^{1-p'} \in A_{p',\mathcal{B}}$
- $w_1,\, w_2 \in A_{1,\mathcal{B}} \implies w_1\, w_2^{1-p} \in A_{p,\mathcal{B}}$ Reverse Factorization

  The converse is true [Jawerth]

Properties that may fail: [Gurka, et al.] [Soria]

- Reverse Hölder inequality
- $w \in A_{p,\mathcal{B}} \implies w^{1+\delta} \in A_{p,\mathcal{B}}$ or $w \in A_{p-\epsilon,\mathcal{B}}$
- $(M_{\mathcal{B}} f)^{\delta} \in A_{1,\mathcal{B}}$, $0 < \delta < 1$

Extrapolation at first glance: Muckenhoupt Bases

Theorem (Jawerth)

Let $0 < p_0 < \infty$. Assume that T satisfies

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0}\, w(x)\, dx \le C_w \int_{\mathbb{R}^n} |f(x)|^{p_0}\, w(x)\, dx, \quad \forall w \in A_{1,\mathcal{B}}.$$

Then T is bounded on $L^p(\mathbb{R}^n)$ for all $p > p_0$.

Proof. Let $r = p/p_0 > 1$. By duality, $\exists\, h \ge 0$, $\|h\|_{r'} = 1$ such that

$$\|Tf\|_p^{p_0} = \big\||Tf|^{p_0}\big\|_r = \int_{\mathbb{R}^n} |Tf|^{p_0}\, h\, dx \le \int_{\mathbb{R}^n} |Tf|^{p_0}\, \mathcal{R}h\, dx$$

$$\lesssim \int_{\mathbb{R}^n} |f|^{p_0}\, \mathcal{R}h\, dx \le \|f\|_p^{p_0}\, \|\mathcal{R}h\|_{r'} \lesssim \|f\|_p^{p_0}$$

$$\mathcal{R}h = \sum_{k=0}^{\infty} \frac{M_{\mathcal{B}}^k h}{2^k\, \|M_{\mathcal{B}}\|_{L^{r'}}^k}$$

Section 2

Weighted norm inequalities

Calderón-Zygmund operators

- Hilbert transform $$Hf(x) = \int_{\mathbb{R}} \frac{1}{x-y} f(y)\,dy$$
- Riesz transforms $$R_j f(x) = \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,dy$$

Calderón-Zygmund operators

- $T : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$
- $Tf(x) = \int_{\mathbb{R}^n} K(x,y)\, f(y)\,dy, \quad x \in \mathbb{R}^n \setminus \operatorname{supp} f, \qquad f \in L^\infty_c$
- K is smooth: for $|x-y| > 2\,|x-x'|$,
$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \le C\,\frac{|x-x'|^\delta}{|x-y|^{n+\delta}}$$

Calderón-Zygmund Theory

Theorem

- T *bounded on* $L^p(\mathbb{R}^n)$, $1 < p < \infty$ $(\textit{weak } p = 1)$
- T *bounded on* $L^p(w)$, $1 < p < \infty$, $w \in A_p$ $(\textit{weak } p = 1)$

Proof

1. L^2 boundedness ✓
2. Calderón-Zygmund decomposition $\rightsquigarrow$ $1 \le p < 2$
3. Duality $\rightsquigarrow$ $2 < p < \infty$
4. Different approaches $\rightsquigarrow$ $L^p(w)$

Weighted norm inequalities for CZO: Approach I

Theorem (Hunt, Muckenhoupt, Wheeden; Coifman, Fefferman)

- $T : L^p(w) \longrightarrow L^p(w)$, $1 < p < \infty$, $w \in A_p$
- $T : L^1(w) \longrightarrow L^{1,\infty}(w)$, $w \in A_1$

Proof I: Coifman, Fefferman

- $|Tf| \lesssim T_* f + |f|$ with $T_* f(x) = \sup_{\epsilon>0} \left| \int_{|x-y|>\epsilon} K(x,y)\, f(y)\, dy \right|$
- Good-λ: For every $w \in A_\infty$, $\lambda > 0$ and $0 < \gamma < \gamma_0$
$$w\{|T_* f| > 3\,\lambda, Mf \le \gamma\,\lambda\} \lesssim \gamma^\delta\, w\{|T_* f| > \lambda,\}$$
- $\|T_* f\|_{L^p(w)} \lesssim \|Mf\|_{L^p(w)}$, $0 < p < \infty$, $w \in A_\infty$
- $\|T_* f\|_{L^{1,\infty}(w)} \lesssim \|Mf\|_{L^{1,\infty}(w)}$, $w \in A_\infty$

Weighted norm inequalities for CZO: Approach II

Theorem

- $T : L^p(w) \longrightarrow L^p(w)$, $1 < p < \infty$, $w \in A_p$
- $T : L^1(w) \longrightarrow L^{1,\infty}(w)$, $w \in A_1$

Proof II: Journé

- $M^{\#} f(x) = \sup_{Q \ni x} \fint_Q |f(y) - f_Q| \, dy$
- $M^{\#}(Tf)(x) \lesssim M_s f(x) = M(|f|^s)(x)^{\frac{1}{s}}$, $1 < s < \infty$
- Fefferman-Stein: $\|Mf\|_{L^p(w)} \lesssim \|M^{\#} f\|_{L^p(w)}$, $0 < p < \infty$, $w \in A_\infty$
- $\|Tf\|_{L^p(w)} \lesssim \|M_s f\|_{L^p(w)}$, $0 < p < \infty$, $w \in A_\infty$
- Calderón-Zygmund decomposition for $p = 1$ and $w \in A_1$

Weighted norm inequalities for CZO: Approach III

Theorem

- $T : L^p(w) \longrightarrow L^p(w)$, $1 < p < \infty$, $w \in A_p$
- $T : L^1(w) \longrightarrow L^{1,\infty}(w)$, $w \in A_1$

Proof III: Álvarez, Pérez

- $M_\delta^\# f(x) = M^\#(|f|^\delta)(x)^{\frac{1}{\delta}}$
- $M_\delta^\#(Tf)(x) \lesssim Mf(x)$, $0 < \delta < 1$
- Fefferman-Stein: $\|Mf\|_{L^p(w)} \lesssim \|M^\# f\|_{L^p(w)}$, $0 < p < \infty$, $w \in A_\infty$
- $\|Tf\|_{L^p(w)} \lesssim \|Mf\|_{L^p(w)}$, $0 < p < \infty$, $w \in A_\infty$
- Fefferman-Stein: $\|Mf\|_{L^{1,\infty}(w)} \lesssim \|M^\# f\|_{L^{1,\infty}(w)}$, $w \in A_\infty$
- $\|Tf\|_{L^{1,\infty}(w)} \lesssim \|Mf\|_{L^{1,\infty}(w)}$, $w \in A_\infty$

Coifman's Inequality

- If T is a CZO then $\|Tf\|_{L^p(w)} \lesssim \|Mf\|_{L^p(w)}$, $0 < p < \infty$, $w \in A_\infty$

  Proof without good-λ for $0 < p < 1$, $w \in A_1$ [Cruz-Uribe, Martell, Pérez]

Other Examples

- Mf and $M^{\#}f$ [Fefferman, Stein]
- Tf with kernel L^r-smooth and $M_{r'}f$ $(1 < r < \infty)$

  [Rubio de Francia, Ruiz, Torrea; Watson; Martell, Pérez, Trujillo]
- $\mathcal{C}f$ and $M_{L\,(\log L)\,(\log\log\log L)}f$ [Grafakos, Martell, Soria]
- Fractional integrals: $I_\alpha f$ and $M_\alpha f$ [Muckenhoupt, Wheeden]

  Proof without good-λ for $p = 1$ and $w \in A_\infty$ [CMP]
- f and $S_d f$ [Chang, Wilson, Wolff]

  Proof without good-λ for $p = 2$ and $w \in A_\infty$ [CMP]

Other examples

- If an operator T satisfies

$$\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} Mf(x)^p\, w(x)\, dx, \quad 0 < p < \infty, \quad w \in A_\infty$$

Then $T : L^p(w) \longrightarrow L^p(w)$, $w \in A_p$, $1 < p < \infty$

What else can we say about T? Does T behave like M?

- Given two operators T, S and $0 < p_0 < \infty$ assume that

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0}\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} |Sf(x)|^{p_0}\, w(x)\, dx, \quad \forall\, w \in A_\infty$$

What can we say about T? Does T behave like S?

Part II

Extrapolation I: A_p weights

References: Extrapolation

- D. Cruz-Uribe, J. M. Martell & C. Pérez, *Extrapolation from A_∞ weights and applications*, J. Funct. Anal. 213 (2004), 412–439.
- G. Curbera, J. García-Cuerva, J. M. Martell & C. Pérez, *Extrapolation with weights, rearrangement-invariant function spaces, modular inequalities and applications to singular integrals*, Adv. Math. 203 (2006), 256–318.
- D. Cruz-Uribe, J. M. Martell & C. Pérez, *Extensions of Rubio de Francia's extrapolation theorem*, Proceedings of the 7th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial 2004), Collect. Math. 2006, 195-231.
- D. Cruz-Uribe, J. M. Martell & C. Pérez, *Weights, Extrapolation and the Theory of Rubio de Francia*, in preparation.

`http://www.uam.es/chema.martell`

Section 3

Extrapolation on Lebesgue spaces

Extrapolation at first glance

Theorem (Rubio de Francia; García-Cuerva)

Let $0 < p_0 < \infty$. *Assume that* T *satisfies*

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0}\, w(x)\, dx \le C_w \int_{\mathbb{R}^n} |f(x)|^{p_0}\, w(x)\, dx, \qquad \forall w \in A_1.$$

Then T *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $p > p_0$.

Proof. Let $r = p/p_0 > 1$. By duality, $\exists\, h \ge 0$, $\|h\|_{r'} = 1$ such that

$$\|Tf\|_p^{p_0} = \big\||Tf|^{p_0}\big\|_r = \int_{\mathbb{R}^n} |Tf|^{p_0}\, h\, dx \le \int_{\mathbb{R}^n} |Tf|^{p_0}\, \mathcal{R}h\, dx$$

$$\lesssim \int_{\mathbb{R}^n} |f|^{p_0}\, \mathcal{R}h\, dx \le \|f\|_p^{p_0}\, \|\mathcal{R}h\|_{r'} \lesssim \|f\|_p^{p_0}$$

$$\mathcal{R}h = \sum_{k=0}^{\infty} \frac{M^k h}{2^k\, \|M\|_{L^{r'}}^k}$$

The Rubio de Francia Extrapolation Theorem

Theorem (Rubio de Francia; García-Cuerva)

Let $1 \le p_0 < \infty$. *Assume that* T *satisfies*

$$(\star) \qquad \int_{\mathbb{R}^n} |Tf(x)|^{p_0}\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} |f(x)|^{p_0}\, w(x)\, dx, \qquad \forall\, w \in A_{p_0}$$

Then $T : L^p(w) \longrightarrow L^p(w)$, $w \in A_p$, $1 < p < \infty$

Remark: $p = 1$ not true in general (even weak-type)

Example: M, M^2, ...

New simple proof: The Rubio de Francia algorithms

- Fix $1 < p < \infty$ and $w \in A_p$
- $M : L^p(w) \longrightarrow L^p(w)$

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h}{2^k \, \|M\|^k_{L^p(w)}}$$ $h \in L^p(w)$	1. $0 \le |h| \le \mathcal{R}h$ 2. $\|\mathcal{R}h\|_{L^p(w)} \le 2\, \|h\|_{L^p(w)}$ 3. $\mathcal{R}h \in A_1$ $M(\mathcal{R}h) \le 2\, \|M\|\, \mathcal{R}h$

- $M'f(x) := \dfrac{M(f\,w)(x)}{w(x)} : L^{p'}(w) \longrightarrow L^{p'}(w) \qquad w^{1-p'} \in A_{p'}$

$$\mathcal{R}'h(x) = \sum_{k=0}^{\infty} \frac{(M')^k h}{2^k \, \|M'\|^k_{L^{p'}(w)}}$$ $h \in L^{p'}(w)$	4. $0 \le |h| \le \mathcal{R}'h$ 5. $\|\mathcal{R}'h\|_{L^{p'}(w)} \le 2\, \|h\|_{L^{p'}(w)}$ 6. $\mathcal{R}'h \cdot w \in A_1$ $M'(\mathcal{R}'h) \le 2\, \|M'\|\, \mathcal{R}'h$

New simple proof

$$\|Tf\|_{L^p(w)} = \Big(\exists\, 0 \le h \in L^{p'}(w) \text{ with } \|h\|_{L^{p'}(w)} = 1\Big)$$

$$= \int_{\mathbb{R}^n} |Tf|\, h\, w\, dx \overset{4}{\le} \int_{\mathbb{R}^n} |Tf|\, \mathcal{R}f^{-\frac{1}{p_0'}}\, \mathcal{R}f^{\frac{1}{p_0'}}\, \mathcal{R}'h\, w\, dx$$

$$\le \Big(\int_{\mathbb{R}^n} |Tf|^{p_0}\, \mathcal{R}f^{1-p_0}\, \mathcal{R}'h\, w\, dx\Big)^{\frac{1}{p_0}} \Big(\int_{\mathbb{R}^n} \mathcal{R}f\, \mathcal{R}'h\, w\, dx\Big)^{\frac{1}{p_0'}}$$

Reverse Factorization + 3 + 6 $\rightsquigarrow$ $\mathcal{R}f^{1-p_0}\,(\mathcal{R}'h\, w) \in A_{p_0}$

$$\overset{(\star)}{\lesssim} \Big(\int_{\mathbb{R}^n} |f|^{p_0}\, \mathcal{R}f^{1-p_0}\, \mathcal{R}'h\, w\, dx\Big)^{\frac{1}{p_0}} \Big(\int_{\mathbb{R}^n} \mathcal{R}f\, \mathcal{R}'h\, w\, dx\Big)^{\frac{1}{p_0'}}$$

$$\overset{1}{\le} \int_{\mathbb{R}^n} \mathcal{R}f\, \mathcal{R}'h\, w\, dx \le \|\mathcal{R}f\|_{L^p(w)}\, \|\mathcal{R}'h\|_{L^{p'}(w)}$$

$$\overset{2+4}{\lesssim} \|f\|_{L^p(w)}\, \|h\|_{L^{p'}(w)} = \|f\|_{L^p(w)}$$

New simple proof

- **Ingredients**
  - $L^{p'}(w)$ is the dual of $L^p(w)$; Hölder's inequality

    $L^{p'}(w)$ and $L^p(w)$ are associate spaces

  - M sublinear, positive, bounded on $L^p(w)$ if $w \in A_p$
  - M' sublinear, positive, bounded on $L^{p'}(w)$ if $w \in A_p$

    $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$

  - Reverse Factorization: $w_1, w_2 \in A_1 \quad \Longrightarrow \quad w_1\, w_2^{1-p} \in A_p$

- **We have NOT used** any property of T

  We can replace Tf by F and the proof goes through

  Rescaling: For all $w \in A_1$

  $$\|Tf\|_{L^2(w)} \lesssim \|f\|_{L^2(w)} \quad \equiv \quad \big\||Tf|^2\big\|_{L^1(w)} \lesssim \big\||f|^2\big\|_{L^1(w)}$$

Muckenhoupt Bases

Definitions

- Basis: $\mathcal{B}$ collection of open sets $B \subset \mathbb{R}^n$
- Maximal operator: $M_{\mathcal{B}} f(x) = \sup_{B \in \mathcal{B}, B \ni x} \fint_B |f(y)|\, dy, \qquad x \in \bigcup_{B \in \mathcal{B}} B$
- Weight: $0 < w(B) < \infty$ for every $B \in \mathcal{B}$
- Muckenhoupt weights: $A_{\infty,\mathcal{B}} = \bigcup_{p \geq 1} A_{p,\mathcal{B}}$
  - $w \in A_{p,\mathcal{B}} \qquad \left(\fint_B w\, dx\right) \left(\fint_B w^{1-p'}\, dx\right)^{p-1} \leq C$
  - $w \in A_{1,\mathcal{B}} \qquad M_{\mathcal{B}} w(x) \leq C\, w(x), \quad$ a.e. $x \in \mathbb{R}^n$
- Muckenhoupt Basis: $M_{\mathcal{B}} : L^p(w) \to L^p(w)$, $\forall\, w \in A_{p,\mathcal{B}}$, $1 < p < \infty$

Extensions of the Extrapolation: Muckenhoupt Bases

- $\mathcal{B}$ Muckenhoupt basis
- $w \in A_{\infty,\mathcal{B}}$
- $M'_{\mathcal{B}} f(x) = \dfrac{M_{\mathcal{B}}(f\,w)(x)}{w(x)}, \quad x \in \bigcup_{B\in\mathcal{B}} B$

Proposition

If $\mathcal{B}$ is a Muckenhoupt basis and $1 < p < \infty$,

- $M_{\mathcal{B}}$ is sublinear, positive and bounded on $L^p(w)$ for $w \in A_{p,\mathcal{B}}$
- $M'_{\mathcal{B}}$ is sublinear, positive and bounded on $L^{p'}(w)$ for $w \in A_{p,\mathcal{B}}$
- If $w_1,\, w_2 \in A_{1,\mathcal{B}}$ then $w_1\, w_2^{1-p} \in A_{p,\mathcal{B}}$ Reverse Factorization

Extensions of the Extrapolation: Elimination of the operator

- $\mathcal{F} \subset \big\{(f,g) : f, g \geq 0 \text{ measurable}\big\}$

  Example: $\mathcal{F} = \big\{(|Tf|, |f|) : \ f \in L_0^\infty\big\}$ or C_0^∞ or L^2, ...

- Notation: Given $0 < p < \infty$ and $w \in A_{r,\mathcal{B}}$:

$$(\star) \qquad \int_{\mathbb{R}^n} f^p \, w \, dx \lesssim \int_{\mathbb{R}^n} g^p \, w \, dx, \qquad (f,g) \in \mathcal{F},$$

holds for all $(f,g) \in \mathcal{F}$ with left-hand side finite

Extension of the Rubio de Francia Extrapolation

Theorem

Let $\mathcal{B}$ *be a Muckenhoupt basis and* $1 \le p_0 < \infty$. *Assume that for every* $w \in A_{p_0,\mathcal{B}}$

$$(\star) \qquad \int_{\mathbb{R}^n} f(x)^{p_0}\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)\, dx, \qquad (f,g) \in \mathcal{F}$$

Then, for all $1 < p < \infty$, *and for all* $w \in A_{p,\mathcal{B}}$

$$\int_{\mathbb{R}^n} f(x)^{p}\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} g(x)^{p}\, w(x)\, dx, \qquad (f,g) \in \mathcal{F}$$

Proof: The Rubio de Francia algorithms

- Fix $1 < p < \infty$ and $w \in A_{p,\mathcal{B}}$
- $M_{\mathcal{B}} : L^p(w) \longrightarrow L^p(w)$

$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M_{\mathcal{B}}^k h}{2^k \, \|M_{\mathcal{B}}\|_{L^p(w)}^k}$ $0 \le h \in L^p(w)$	① $0 \le h \le \mathcal{R}h$ ② $\|\mathcal{R}h\|_{L^p(w)} \le 2 \, \|h\|_{L^p(w)}$ ③ $\mathcal{R}h \in A_{1,\mathcal{B}}$ $M_{\mathcal{B}}(\mathcal{R}h) \le 2 \, \|M_{\mathcal{B}}\| \, \mathcal{R}h$

- $M'_{\mathcal{B}} f(x) := \frac{M(f\,w)(x)}{w(x)} : L^{p'}(w) \longrightarrow L^{p'}(w)$

$\mathcal{R}'h(x) = \sum_{k=0}^{\infty} \frac{(M'_{\mathcal{B}})^k h}{2^k \, \|M'_{\mathcal{B}}\|_{L^{p'}(w)}^k}$ $0 \le h \in L^{p'}(w)$	④ $0 \le h \le \mathcal{R}'h$ ⑤ $\|\mathcal{R}'h\|_{L^{p'}(w)} \le 2 \, \|h\|_{L^{p'}(w)}$ ⑥ $\mathcal{R}'h \cdot w \in A_1$ $M'_{\mathcal{B}}(\mathcal{R}'h) \le 2 \, \|M'_{\mathcal{B}}\| \, \mathcal{R}'h$

Proof

$$\|f\|_{L^p(w)} = \Big(\exists\, 0 \le h \in L^{p'}(w) \text{ with } \|h\|_{L^{p'}(w)} = 1\Big)$$

$$= \int_{\mathbb{R}^n} f\, h\, w\, dx \overset{④}{\le} \int_{\mathbb{R}^n} f\, \mathcal{R}g^{-\frac{1}{p_0'}}\, \mathcal{R}g^{\frac{1}{p_0'}}\, \mathcal{R}'h\, w\, dx$$

$$\le \Big(\int_{\mathbb{R}^n} f^{p_0}\, \mathcal{R}g^{1-p_0}\, \mathcal{R}'h\, w\, dx\Big)^{\frac{1}{p_0}} \Big(\int_{\mathbb{R}^n} \mathcal{R}g\, \mathcal{R}'h\, w\, dx\Big)^{\frac{1}{p_0'}}$$

Reverse Factorization + ③ + ⑥ $\rightsquigarrow$ $\mathcal{R}g^{1-p_0}\,(\mathcal{R}'h\, w) \in A_{p_0,\mathcal{B}}$

$$\overset{(\star)}{\lesssim} \Big(\int_{\mathbb{R}^n} g^{p_0}\, \mathcal{R}g^{1-p_0}\, \mathcal{R}'h\, w\, dx\Big)^{\frac{1}{p_0}} \Big(\int_{\mathbb{R}^n} \mathcal{R}g\, \mathcal{R}'h\, w\, dx\Big)^{\frac{1}{p_0'}}$$

$$\overset{①}{\le} \int_{\mathbb{R}^n} \mathcal{R}g\, \mathcal{R}'h\, w\, dx \le \|\mathcal{R}g\|_{L^p(w)}\, \|\mathcal{R}'h\|_{L^{p'}(w)}$$

$$\overset{②+④}{\lesssim} \|g\|_{L^p(w)}\, \|h\|_{L^{p'}(w)} = \|g\|_{L^p(w)}$$

Consequences: Vector-valued Inequalities

Corollary

Let $\mathcal{B}$ be a Muckenhoupt basis and $1 \le p_0 < \infty$. Assume that for every $w \in A_{p_0,\mathcal{B}}$

$$(\star) \qquad \int_{\mathbb{R}^n} f(x)^{p_0}\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)\, dx, \qquad (f,g) \in \mathcal{F}$$

Then, for all $1 < p < \infty$, and for all $w \in A_{p,\mathcal{B}}$

$$\|f\|_{L^p(w)} \lesssim \|g\|_{L^p(w)}, \qquad (f,g) \in \mathcal{F}$$

Furthermore, for all $1 < p, q < \infty$, and for all $w \in A_{p,\mathcal{B}}$

$$\Big\|\Big(\sum_j f_j^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w)} \lesssim \Big\|\Big(\sum_j g_j^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w)}, \quad \{(f_j, g_j)\}_j \in \mathcal{F}$$

Vector-valued Inequalities: Proof

- Fix $1 < q < \infty$
- $\mathcal{F}_q = \Big\{ (F, G) = \Big(\big(\sum_j f_j^q\big)^{\frac{1}{q}}, \big(\sum_j g_j^q\big)^{\frac{1}{q}} \Big) : \quad \{(f_j, g_j)\}_j \subset \mathcal{F} \Big\}$
- For all $w \in A_{q,\mathcal{B}}$ and $(F, G) \in \mathcal{F}_q$

$$(\star\star) \qquad \|F\|_{L^q(w)}^q = \sum_j \int_{\mathbb{R}^n} f_j^q \, w \, dx \overset{(\star)}{\lesssim} \sum_j \int_{\mathbb{R}^n} g_j^q \, w \, dx = \|G\|_{L^q(w)}^q$$

- Apply Extrapolation to $\mathcal{F}_q$ from $(\star\star)$ $(p_0 = q)$: for all $w \in A_{p,\mathcal{B}}$

$$\Big\| \Big(\sum_j f_j^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)} = \|F\|_{L^p(w)} \lesssim \|G\|_{L^p(w)} = \Big\| \Big(\sum_j g_j^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)}$$

Consequences: Weak-type extrapolation

Corollary

Let $\mathcal{B}$ *be a Muckenhoupt basis and* $1 \le p_0 < \infty$. *Assume that for every* $w \in A_{p_0,\mathcal{B}}$

$$(\star) \qquad \|f\|_{L^{p_0,\infty}(w)} \lesssim \|g\|_{L^{p_0}(w)}, \qquad (f,g) \in \mathcal{F}$$

Then, for all $1 < p < \infty$, *and for all* $w \in A_{p,\mathcal{B}}$

$$\|f\|_{L^{p,\infty}(w)} \lesssim \|g\|_{L^{p}(w)}, \qquad (f,g) \in \mathcal{F}$$

Weak-type extrapolation: Proof. [Grafakos, Martell]

- $\mathcal{F}_{\text{weak}} = \Big\{ (f_\lambda, g) = \big(\lambda\, \chi_{\{f>\lambda\}}, g\big) : \quad (f,g) \in \mathcal{F}, \quad \lambda > 0 \Big\}$

- For all $w \in A_{p_0,\mathcal{B}}$ and $(f_\lambda, g) \in \mathcal{F}_{\text{weak}}$

$$(\star\star) \qquad \|f_\lambda\|_{L^{p_0}(w)} = \lambda\, w\{f > \lambda\}^{\frac{1}{p_0}} \le \|f\|_{L^{p_0,\infty}(w)} \overset{(\star)}{\lesssim} \|g\|_{L^{p_0}(w)}$$

- Apply Extrapolation to $\mathcal{F}_{\text{weak}}$ from $(\star\star)$: for all $w \in A_{p,\mathcal{B}}$, $\lambda > 0$

$$\lambda\, w\{f > \lambda\}^{\frac{1}{p}} = \|f_\lambda\|_{L^p(w)} \lesssim \|g\|_{L^p(w)}$$

Consequences: Rescaling

Corollary

Let $\mathcal{B}$ *be a Muckenhoupt basis and* $0 < r \le p_0 < \infty$. *Assume that for every* $w \in A_{p_0/r,\mathcal{B}}$

$$(\star) \qquad \|f\|_{L^{p_0}(w)} \lesssim \|g\|_{L^{p_0}(w)}, \qquad (f,g) \in \mathcal{F}$$

Then, for all $r < p < \infty$, *and for all* $w \in A_{p/r,\mathcal{B}}$

$$\|f\|_{L^{p}(w)} \lesssim \|g\|_{L^{p}(w)}, \qquad (f,g) \in \mathcal{F}$$

Proof. $\mathcal{F}_r = \big\{(f^r, g^r) : \quad (f,g) \in \mathcal{F}\big\}$

Extrapolation for one-sided weights

- One-sided Hardy-Littlewood maximal functions in $\mathbb{R}$

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)|\, dy, \qquad M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)|\, dy$$

- One-sided weights

$$(A_p^+) \qquad \left(\frac{1}{h} \int_{x-h}^{x} w\, dx \right) \left(\frac{1}{h} \int_x^{x+h} w^{1-p'}\, dx \right)^{p-1} \le C, \quad p > 1$$

$$(A_1^+) \qquad M^- w(x) \le C\, w(x), \qquad \text{a.e. } x \in \mathbb{R}^n$$

- $M^+ : L^p(w) \longrightarrow L^p(w) \quad \Longleftrightarrow \quad w \in A_p^+$ (weak-type for $p = 1$)
- Analogously M^-, A_p^-
- $w \in A_p^+ \quad \Longleftrightarrow \quad w^{1-p'} \in A_p^-$
- Reverse Factorization: $w_1 \in A_1^+,\ w_2 \in A_1^- \quad \Longrightarrow \quad w_1\, w_2^{1-p} \in A_p^+$

Extrapolation for one-sided weights

Theorem

Let $1 \le p_0 < \infty$ *and assume that for every* $w \in A_{p_0}^+$

$$(\star) \qquad \|f\|_{L^{p_0}(w)} \lesssim \|g\|_{L^{p_0}(w)} \qquad (f,g) \in \mathcal{F}$$

Then, for all $1 < p < \infty$, *and for all* $w \in A_p^+$

$$\|f\|_{L^p(w)} \lesssim \|g\|_{L^p(w)}, \qquad (f,g) \in \mathcal{F}$$

Remark: $M \quad \rightsquigarrow \quad M^+$ and $M' \quad \rightsquigarrow \quad M^-$

Skip the proof

Proof: The Rubio de Francia algorithms

- Fix $1 < p < \infty$ and $w \in A_p^+$
- $M^+ : L^p(w) \longrightarrow L^p(w)$

$\mathcal{R}^+ h(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(M^+)^k h}{2^k \, \|M^+\|_{L^p(w)}^k}$ $0 \le h \in L^p(w)$	❶ $0 \le h \le \mathcal{R}^+ h$ ❷ $\|\mathcal{R}^+ h\|_{L^p(w)} \lesssim \|h\|_{L^p(w)}$ ❸ $\mathcal{R}^+ h \in A_1^-$ $M^+(\mathcal{R}^+ h) \lesssim \mathcal{R}^+ h$

- $(M^-)' f(x) := \dfrac{M^-(f\,w)(x)}{w(x)} : L^{p'}(w) \longrightarrow L^{p'}(w)$ since $w^{1-p'} \in A_p^-$

$\mathcal{R}^- h(x) = \displaystyle\sum_{k=0}^{\infty} \frac{((M^-)')^k h}{2^k \, \|(M^-)'\|_{L^{p'}(w)}^k}$ $0 \le h \in L^{p'}(w)$	❹ $0 \le h \le \mathcal{R}' h$ ❺ $\|\mathcal{R}^- h\|_{L^{p'}(w)} \lesssim \|h\|_{L^{p'}(w)}$ ❻ $\mathcal{R}^- h \cdot w \in A_1^+$ $(M^-)'(\mathcal{R}' h) \lesssim \mathcal{R}' h$

Proof

$$\|f\|_{L^p(w)} = \left(\exists\, 0 \le h \in L^{p'}(w) \text{ with } \|h\|_{L^{p'}(w)} = 1\right)$$

$$= \int_{\mathbb{R}} f\,h\,w\,dx \overset{④}{\le} \int_{\mathbb{R}} f\,\mathcal{R}^+g^{-\frac{1}{p_0'}}\,\mathcal{R}^+g^{\frac{1}{p_0'}}\,\mathcal{R}^-h\,w\,dx$$

$$\le \left(\int_{\mathbb{R}} f^{p_0}\,\mathcal{R}^+g^{1-p_0}\,\mathcal{R}^-h\,w\,dx\right)^{\frac{1}{p_0}} \left(\int_{\mathbb{R}} \mathcal{R}^+g\,\mathcal{R}^-h\,w\,dx\right)^{\frac{1}{p_0'}}$$

Reverse Factorization + ③ + ⑥ $\rightsquigarrow$ $\mathcal{R}^+g^{1-p_0}\,(\mathcal{R}^-h\,w) \in A_{p_0}^+$

$$\overset{(\star)}{\lesssim} \left(\int_{\mathbb{R}} g^{p_0}\,\mathcal{R}^+g^{1-p_0}\,\mathcal{R}^-h\,w\,dx\right)^{\frac{1}{p_0}} \left(\int_{\mathbb{R}} \mathcal{R}^+g\,\mathcal{R}^-h\,w\,dx\right)^{\frac{1}{p_0'}}$$

$$\overset{①}{\le} \int_{\mathbb{R}} \mathcal{R}^+g\,\mathcal{R}^-h\,w\,dx \le \|\mathcal{R}^+g\|_{L^p(w)}\,\|\mathcal{R}^-h\|_{L^{p'}(w)}$$

$$\overset{②+④}{\lesssim} \|g\|_{L^p(w)}\,\|h\|_{L^{p'}(w)} = \|g\|_{L^p(w)}$$

Section 4

Extrapolation on Function Spaces

Introduction

Theorem

Fix $1 \le p_0 < \infty$. *If*

$$\|f\|_{L^{p_0}(w)} \lesssim \|g\|_{L^{p_0}(w)}, \qquad (f,g) \in \mathcal{F}, \quad \forall\, w \in A_{p_0}.$$

Then, for all $1 < p < \infty$,

$$\|f\|_{L^{p}(w)} \lesssim \|g\|_{L^{p}(w)}, \qquad (f,g) \in \mathcal{F}, \quad \forall\, w \in A_{p}.$$

- Can we prove estimates in other "Banach function spaces"?
  - $\|f\|_{L^{p,\infty}(w)} \lesssim \|g\|_{L^{p,\infty}(w)}$, $\forall\, w \in A_p$?
  - $L^p\,(\log L)^\alpha(w)$; $\mathbb{X}(w)$?
- Can we prove estimates in "modular spaces"?
  - $\displaystyle\int_{\mathbb{R}^n} \phi(f)\, w\, dx \lesssim \int_{\mathbb{R}^n} \phi(g)\, w\, dx, \quad \forall\, w \in A_{?}$?
  - $\phi(t) = t^p \rightsquigarrow L^p$; $\quad \phi(t) = t^p\,(\log t)^\alpha$; $\quad \phi(t) \approx \max\{t^p, t^q\}$

Extrapolation on Banach Function Spaces

- $\mathcal{M}$ measurable functions
- Banach function norm: $\rho : \mathcal{M} \longrightarrow [0, \infty]$
  - $\rho(f) = 0 \iff f = 0$ μ-a.e.
  - $\rho(f+g) \le \rho(f) + \rho(g), \qquad \rho(a\, f) = |a|\, \rho(f)$
  - $0 \le f \le g \implies \rho(f) \le \rho(g)$
  - $0 \le f_n \nearrow f \implies \rho(f_n) \nearrow \rho(f).$
  - $|E| < \infty \implies \rho(\chi_E) < \infty, \qquad \int_E |f|\, dx \le C_E\, \rho(f)$
- Banach Functions Space:

$$\mathbb{X} = \mathbb{X}(\rho) = \left\{ f \in \mathcal{M} : \ \|f\|_{\mathbb{X}} = \rho(f) < \infty \right\}$$

Associate Spaces

- $\mathbb{X} = \mathbb{X}(\rho)$ a Banach Function Space
- Associate space: $\mathbb{X}' = \mathbb{X}(\rho')$,

$$\rho'(f) = \sup\Big\{ \int_{\mathbb{R}^n} |f\,g|\,dx : \; g \in \mathcal{M}, \; \rho(g) \le 1 \Big\}.$$

- Generalized Hölder's inequality

$$\int_{\mathbb{R}^n} |f\,g|\,dx \le \|f\|_{\mathbb{X}}\,\|g\|_{\mathbb{X}'}, \qquad f \in \mathbb{X}, \quad g \in \mathbb{X}'$$

- "Duality"

$$\|f\|_{\mathbb{X}} = \sup\Big\{ \Big| \int_{\mathbb{R}^n} f\,g\,dx \Big| : \; g \in \mathbb{X}', \; \|g\|_{\mathbb{X}'} \le 1 \Big\},$$

- Rescaling: $0 < r < \infty$

$$\mathbb{X}^r = \{ f \in \mathcal{M} : \; |f|^r \in \mathbb{X} \}, \qquad \|f\|_{\mathbb{X}^r} = \big\| |f|^r \big\|_{\mathbb{X}}^{\frac{1}{r}}$$

Rearrangement Invariant Banach Function Spaces

- Distribution function: $\mu_f(\lambda) = \big|\{x \in \mathbb{R}^n : \ |f(x)| > \lambda\}\big|$
- Decreasing rearrangement: $f^*(t) = \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}$
- $\mathbb{X}$ rearrangement invariant: $\mu_f = \mu_g \quad \Longrightarrow \quad \|f\|_{\mathbb{X}} = \|g\|_{\mathbb{X}}$
- Luxemburg's representation theorem

$$\|f\|_{\mathbb{X}} = \|f^*\|_{\overline{\mathbb{X}}}, \qquad \overline{\mathbb{X}} \text{ r.i. BFS over } (\mathbb{R}^+, dt)$$

- Weighted spaces: $\mathbb{X}(w) \quad \rightsquigarrow \quad \|f\|_{\mathbb{X}(w)} = \|f^*_w\|_{\overline{\mathbb{X}}}$

Boyd Indices

- Boyd indices: $1 \le p_{\mathbb{X}} \le q_{\mathbb{X}} \le \infty$

  Dilation operator, scale of interpolation, $\overline{\mathbb{X}}$

- $p_{\mathbb{X}'} = (q_{\mathbb{X}})', \quad q_{\mathbb{X}'} = (p_{\mathbb{X}})'; \qquad p_{\mathbb{X}^r} = r \cdot p_{\mathbb{X}}, \quad q_{\mathbb{X}^r} = r \cdot q_{\mathbb{X}}$

- Lorentz-Shimogaki: $M : \mathbb{X} \longrightarrow \mathbb{X} \iff p_{\mathbb{X}} > 1$

- Boyd: $H : \mathbb{X} \longrightarrow \mathbb{X} \iff 1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$

- $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty \iff M : \mathbb{X} \longrightarrow \mathbb{X}, \quad M : \mathbb{X}' \longrightarrow \mathbb{X}'$

Examples

- Lebesgue spaces: $\mathbb{X} = L^p \quad \rightsquigarrow \quad p_{\mathbb{X}} = q_{\mathbb{X}} = p, \qquad (L^p)^r = L^{p\,r}$
- Lorentz spaces: $\mathbb{X} = L^{p,q} \quad \rightsquigarrow \quad p_{\mathbb{X}} = q_{\mathbb{X}} = p, \qquad (L^{p,q})^r = L^{p\,r,q\,r}$

$$\|f\|_{L^{p,q}} = \Big(\int_0^\infty f^*(s)^q\, s^{\frac{q}{p}}\,\frac{ds}{s}\Big)^{\frac{1}{q}}, \qquad \|f\|_{L^{p,\infty}} = \sup_{0<s<\infty} f^*(s)\, s^{\frac{1}{p}}$$

- Orlicz spaces: $\mathbb{X} = L^\psi$, ψ is a Young function

  increasing, continuous, convex, $\psi(0) = 0$

$$\|f\|_{L^\psi} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \psi\Big(\frac{|f(x)|}{\lambda}\Big)\, dx \le 1\Big\}.$$

  Boyd indices can be calculated from ψ (Dilation indices)

  - $\psi(t) = t^p(1+\log^+ t)^\alpha \rightsquigarrow L^\psi = L^p(\log L)^\alpha$

$$p_{\mathbb{X}} = q_{\mathbb{X}} = p, \qquad \mathbb{X}^r = L^{p\,r}(\log L)^\alpha.$$

- $\mathbb{X} = L^p \cap L^q$ or $\mathbb{X} = L^p + L^q \quad \rightsquigarrow \quad p_{\mathbb{X}} = \min\{p,q\}$, $q_{\mathbb{X}} = \max\{p,q\}$

Extrapolation on r.i. Banach function spaces

Theorem

Let $1 \le p_0 < \infty$. *Assume that for every* $w \in A_{p_0}$

$$(\star) \qquad \int_{\mathbb{R}^n} f(x)^{p_0}\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)\, dx, \qquad (f,g) \in \mathcal{F}$$

Then, if $\mathbb{X}$ *is an r.i. BFS with* $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$, $\forall\, w \in A_{p_{\mathbb{X}}}$

$$\|f\|_{\mathbb{X}(w)} \lesssim \|g\|_{\mathbb{X}(w)}, \qquad (f,g) \in \mathcal{F}$$

Furthermore, for every $1 < q < \infty$

$$\Big\| \Big(\sum_j f_j^q\Big)^{\frac{1}{q}} \Big\|_{\mathbb{X}(w)} \lesssim \Big\| \Big(\sum_j g_j^q\Big)^{\frac{1}{q}} \Big\|_{\mathbb{X}(w)}, \qquad \{(f_j, g_j)\}_j \in \mathcal{F}$$

Proposition

Let $\mathbb{X}$ *be a r.i. BFS with* $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$*, and* $w \in A_{p_{\mathbb{X}}}$*. Then,*

- $M : \mathbb{X}(w) \longrightarrow \mathbb{X}(w)$
- $M' : \mathbb{X}'(w) \longrightarrow \mathbb{X}'(w)$

Proof (Boyd Interpolation Theorem)

Take $1 < p < p_{\mathbb{X}} \le q_{\mathbb{X}} < q < \infty$ with $w \in A_p$ (and $w \in A_q$)

- $$\left.\begin{array}{l} M : L^p(w) \longrightarrow L^p(w) \\ M : L^q(w) \longrightarrow L^q(w) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} M : \mathbb{Y}(w) \longrightarrow \mathbb{Y}(w) \\ p < p_{\mathbb{Y}} \le q_{\mathbb{Y}} < q \end{array}\right.$$
- Apply it to $\mathbb{Y} = \mathbb{X}$ as $p < p_{\mathbb{X}} \le q_{\mathbb{X}} < q$
- $$\left.\begin{array}{l} M' : L^{p'}(w) \longrightarrow L^{p'}(w) \\ M : L^{q'}(w) \longrightarrow L^{q'}(w) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} M' : \mathbb{Y}(w) \longrightarrow \mathbb{Y}(w) \\ q' < p_{\mathbb{Y}} \le q_{\mathbb{Y}} < p' \end{array}\right.$$
- Apply it to $\mathbb{Y} = \mathbb{X}'$ as $q' < p_{\mathbb{X}'} \le q_{\mathbb{X}'} < p' \;\equiv\; p < p_{\mathbb{X}} \le q_{\mathbb{X}} < q$

Proof: The Rubio de Francia algorithms

- $M : \mathbb{X}(w) \longrightarrow \mathbb{X}(w)$

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h}{2^k \, \|M\|^k_{\mathbb{X}(w)}}$$ $$0 \le h \in \mathbb{X}(w)$$	1. $0 \le h \le \mathcal{R}h$ 2. $\|\mathcal{R}h\|_{\mathbb{X}(w)} \le 2\, \|h\|_{\mathbb{X}(w)}$ 3. $\mathcal{R}h \in A_1$ $M(\mathcal{R}h) \le 2\, \|M\|\, \mathcal{R}h$

- $M' : \mathbb{X}'(w) \longrightarrow \mathbb{X}'(w)$

$$\mathcal{R}'h(x) = \sum_{k=0}^{\infty} \frac{(M')^k h}{2^k \, \|M'\|^k_{\mathbb{X}'(w)}}$$ $$0 \le h \in \mathbb{X}'(w)$$	4. $0 \le h \le \mathcal{R}'h$ 5. $\|\mathcal{R}'h\|_{\mathbb{X}'(w)} \le 2\, \|h\|_{\mathbb{X}'(w)}$ 6. $\mathcal{R}'h \cdot w \in A_1$ $M'(\mathcal{R}'h) \le 2\, \|M'\|\, \mathcal{R}'h$

Proof

$$\|f\|_{\mathbb{X}(w)} = \Big(\exists\, 0 \le h \in \mathbb{X}'(w) \text{ with } \|h\|_{\mathbb{X}'(w)} = 1\Big)$$

$$= \int_{\mathbb{R}^n} f\, h\, w\, dx \overset{4}{\le} \int_{\mathbb{R}^n} f\, \mathcal{R}g^{-\frac{1}{p'}}\, \mathcal{R}g^{\frac{1}{p'}}\, \mathcal{R}'h\, w\, dx$$

$$\le \Big(\int_{\mathbb{R}^n} f^p\, \mathcal{R}g^{1-p}\, \mathcal{R}'h\, w\, dx\Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}^n} \mathcal{R}g\, \mathcal{R}'h\, w\, dx\Big)^{\frac{1}{p'}}$$

Reverse Factorization + 3 + 6 $\rightsquigarrow$ $\mathcal{R}g^{1-p}\,(\mathcal{R}'h\, w) \in A_p$

$$\overset{(\star)}{\lesssim} \Big(\int_{\mathbb{R}^n} g^p\, \mathcal{R}g^{1-p}\, \mathcal{R}'h\, w\, dx\Big)^{\frac{1}{p_0}} \Big(\int_{\mathbb{R}^n} \mathcal{R}g\, \mathcal{R}'h\, w\, dx\Big)^{\frac{1}{p_0'}}$$

$$\overset{1}{\le} \int_{\mathbb{R}^n} \mathcal{R}g\, \mathcal{R}'h\, w\, dx \le \|\mathcal{R}g\|_{\mathbb{X}(w)}\, \|\mathcal{R}'h\|_{\mathbb{X}'(w)}$$

$$\overset{2+4}{\lesssim} \|g\|_{\mathbb{X}(w)}\, \|h\|_{\mathbb{X}'(w)} = \|g\|_{\mathbb{X}(w)}$$

Extrapolation on Modular Spaces

- Young functions: $\phi : [0,\infty) \longrightarrow [0,\infty)$, increasing, convex

$$\lim_{t\to 0^+} \phi(t)/t = 0 \qquad\qquad \lim_{t\to\infty} \phi(t)/t = \infty$$

- Δ_2: $\phi(2\,t) \lesssim \phi(t)$, $t \ge 0$

- Complementary function: $\overline{\phi}(s) = \sup_{t>0}\{s\,t - \phi(t)\}$, $s \ge 0$

$$\phi^{-1}(t)\,\overline{\phi}^{-1}(t) \approx t, \qquad t \ge 0.$$

- Young's inequality: $s\,t \le \phi(s) + \overline{\phi}(t)$, $\quad s, t \ge 0$

Dilation Indices

- Dilation indices: $1 \le i_\phi \le I_\phi \le \infty$
- $i_\phi = \min\{r_0, r_\infty\}$ and $I_\phi = \max\{r_0, r_\infty\}$ provided there exist

$$r_0 = \lim_{t\to 0} t\,\phi'(t)/\phi(t) \qquad\qquad r_\infty = \lim_{t\to\infty} t\,\phi'(t)/\phi(t)$$

- $i_{\overline{\phi}} = (I_\phi)', \quad I_{\overline{\phi}} = (i_\phi)'$
- $\phi \in \Delta_2 \quad \Longleftrightarrow \quad I_\phi < \infty$
- Kerman-Torchinski:

$$\int_{\mathbb{R}^n} \phi\big(Mf(x)\big)\,dx \le C \int_{\mathbb{R}^n} \phi\big(C\,|f(x)|\big)\,dx \quad \Longleftrightarrow \quad i_\phi > 1$$

- $1 < i_\phi \le I_\phi < \infty \quad \Longleftrightarrow \quad \phi, \overline{\phi} \in \Delta_2 \quad \Longleftrightarrow$

$$\int_{\mathbb{R}^n} \phi(Mf)\,dx \lesssim \int_{\mathbb{R}^n} \phi(|f|)\,dx \qquad \int_{\mathbb{R}^n} \overline{\phi}(Mf)\,dx \lesssim \int_{\mathbb{R}^n} \overline{\phi}(|f|)\,dx$$

Examples

- $\phi(t) = t^p/p \quad \rightsquigarrow \quad \overline{\phi}(t) = t^{p'}/p', \qquad i_\phi = I_\phi = r_0 = r_\infty = p$

- $\phi(t) \approx t^p(1 + \log^+ t)^\alpha \quad \rightsquigarrow \quad \overline{\phi}(t) \approx t^{p'}\,(1 + \log^+ t)^{\alpha\,(p'-1)}$

  $$i_\phi = I_\phi = r_0 = r_\infty = p$$

- $\phi(t) \approx \max\{t^p, t^q\} \quad \rightsquigarrow \quad \overline{\phi}(t) \approx \min\{t^{p'}, t^{q'}\},$

  $$r_0 = \min\{p,q\}, \quad r_\infty = \max\{p,q\}$$

  $$i_\phi = \min\{p,q\}, \quad I_\phi = \max\{p,q\}$$

- $\phi(t) \approx \begin{cases} t^p & t \le 1 \\ e^t & t \ge 1 \end{cases} \quad \rightsquigarrow \quad \overline{\phi}(t) \approx \begin{cases} t^{p'} & t \le 1 \\ t\,(1 + \log t) & t \ge 1 \end{cases}$

  $$r_0 = p, \quad r_\infty = \infty, \qquad i_\phi = p, \quad I_\phi = \infty \quad (\phi \notin \Delta_2,\ \overline{\phi} \in \Delta_2)$$

Extrapolation on modular spaces

Theorem

Let $1 \le p_0 < \infty$. *Assume that for every* $w \in A_{p_0}$

$$(\star) \qquad \|f\|_{L^{p_0}(w)} \lesssim \|g\|_{L^{p_0}(w)} \qquad (f,g) \in \mathcal{F}$$

Then, if ϕ *is a Young functions with* $1 < i_\phi \le I_\phi < \infty$, $\forall\, w \in A_{i_\phi}$

$$\int_{\mathbb{R}^n} \phi(f(x))\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} \phi(g(x))\, w(x)\, dx$$

Furthermore, $\sup_{\lambda>0} \phi(\lambda)\, w\{f > \lambda\} \lesssim \sup_{\lambda>0} \phi(\lambda)\, w\{g > \lambda\}$

Remark: $\|\phi(f)\|_{L^{1,\infty}(w)} = \sup_{\lambda>0} \phi(\lambda)\, w\{f > \lambda\}$

Remark: Vector-valued inequalities

Part III

Extrapolation II: A_∞ weights

References: Extrapolation

- D. Cruz-Uribe, J. M. Martell & C. Pérez, *Extrapolation from A_∞ weights and applications*, J. Funct. Anal. 213 (2004), 412–439.
- G. Curbera, J. García-Cuerva, J. M. Martell & C. Pérez, *Extrapolation with weights, rearrangement-invariant function spaces, modular inequalities and applications to singular integrals*, Adv. Math. 203 (2006), 256–318.
- D. Cruz-Uribe, J. M. Martell & C. Pérez, *Extensions of Rubio de Francia's extrapolation theorem*, Proceedings of the 7th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial 2004), Collect. Math. 2006, 195-231.
- D. Cruz-Uribe, J. M. Martell & C. Pérez, *Weights, Extrapolation and the Theory of Rubio de Francia*, in preparation.

`http://www.uam.es/chema.martell`

Section 5

Extrapolation for A_∞ weights

Other examples

- If an operator T satisfies

$$\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} Mf(x)^p\, w(x)\, dx, \quad 0 < p < \infty, \quad w \in A_\infty$$

Then $T : L^p(w) \longrightarrow L^p(w)$, $w \in A_p$, $1 < p < \infty$

What else can we say about T?   Does T behave like M?

- Given two operators T, S and $0 < p_0 < \infty$ assume that

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0}\, w(x)\, dx \lesssim \int_{\mathbb{R}^n} |Sf(x)|^{p_0}\, w(x)\, dx, \quad \forall\, w \in A_\infty$$

What can we say about T?   Does T behave like S?

Extrapolation for A_∞ weights

Theorem

Let $0 < p_0 < \infty$. *Assume that for every* $w \in A_\infty$

$$(\star) \qquad \|f\|_{L^{p_0}(w)} \lesssim \|g\|_{L^{p_0}(w)} \qquad (f,g) \in \mathcal{F}$$

Then, for all $0 < p < \infty$, *and for all* $w \in A_\infty$

$$\|f\|_{L^{p}(w)} \lesssim \|g\|_{L^{p}(w)} \qquad (f,g) \in \mathcal{F}$$

Furthermore, for all $0 < p, q < \infty$ *and for all* $w \in A_\infty$

$$\Big\| \Big(\sum_j f_j^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)} \lesssim \Big\| \Big(\sum_j g_j^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)}, \qquad \{(f_j, g_j)\}_j \in \mathcal{F}$$

Proof

- $(\star)$ implies that for any $1 \le r < \infty$ and $w \in A_r$

$$(\star\star) \qquad \|f^{\frac{p_0}{r}}\|_{L^r(w)} \lesssim \|g^{\frac{p_0}{r}}\|_{L^r(w)}, \qquad (f,g) \in \mathcal{F}$$

- $(f^{\frac{p_0}{r}}, g^{\frac{p_0}{r}}) \quad \rightsquigarrow \quad \mathcal{F}_{p_0/r}$
- Apply extrapolation to $\mathcal{F}_{p_0/r}$ from $(\star\star)$: for all $1 < q < \infty$, $w \in A_q$

$$(\star\star\star) \qquad \|f^{\frac{p_0}{r}}\|_{L^q(w)} \lesssim \|g^{\frac{p_0}{r}}\|_{L^q(w)}, \qquad (f,g) \in \mathcal{F}$$

- Fix $0 < p < \infty$ and $w \in A_\infty \quad \rightsquigarrow \quad \exists\, q > \max\{1, p/p_0\}$, $w \in A_q$
- Pick $r = p_0\, p/q > 1$ and use $(\star\star\star)$

Extrapolation for A_∞ weights on Banach function spaces

Theorem

Let $0 < p_0 < \infty$. *Assume that for every* $w \in A_\infty$

$$(\star) \qquad \|f\|_{L^{p_0}(w)} \lesssim \|g\|_{L^{p_0}(w)} \qquad (f,g) \in \mathcal{F}$$

Then, for all $\mathbb{X}$ *r.i. BFS with* $q_{\mathbb{X}} < \infty$, $0 < p < \infty$, *and* $w \in A_\infty$

$$\|f\|_{\mathbb{X}^p(w)} \lesssim \|g\|_{\mathbb{X}^p(w)} \qquad (f,g) \in \mathcal{F}$$

Furthermore, for all $0 < p, q < \infty$ *and for all* $w \in A_\infty$

$$\Big\| \Big(\sum_j f_j^q \Big)^{\frac{1}{q}} \Big\|_{\mathbb{X}^p(w)} \lesssim \Big\| \Big(\sum_j g_j^q \Big)^{\frac{1}{q}} \Big\|_{\mathbb{X}^p(w)}, \qquad \{(f_j, g_j)\}_j \in \mathcal{F}$$

- $\|f\|_{L^{p,\infty}(w)} \lesssim \|g\|_{L^{p,\infty}(w)}$, $0 < p < \infty$, $w \in A_\infty$
- $\|f\|_{L^{p,q}(w)} \lesssim \|g\|_{L^{p,q}(w)}$, $0 < p, q < \infty$, $w \in A_\infty$

Proof

- $(\star)$ implies that for any $1 \le r < \infty$ and $w \in A_r$

$$(\star\star) \qquad \big\| f^{\frac{p_0}{r}} \big\|_{L^r(w)} \lesssim \big\| g^{\frac{p_0}{r}} \big\|_{L^r(w)}, \qquad (f,g) \in \mathcal{F}$$

- For all $\mathbb{Y}$ with $1 < p_{\mathbb{Y}} \le q_{\mathbb{Y}} < \infty$, $w \in A_{p_{\mathbb{Y}}}$

$$(\star\star\star) \qquad \big\| f^{\frac{p_0}{r}} \big\|_{\mathbb{Y}(w)} \lesssim \big\| g^{\frac{p_0}{r}} \big\|_{\mathbb{Y}(w)}, \qquad (f,g) \in \mathcal{F}$$

- Fix $\mathbb{X}$, $0 < p < \infty$, $w \in A_\infty$ $\rightsquigarrow$ $\exists\, q > \max\left\{ p_{\mathbb{X}}, \dfrac{p\, p_{\mathbb{X}}}{p_0} \right\}$, $w \in A_q$
- Pick $\mathbb{Y} = \mathbb{X}^{q/p_{\mathbb{X}}}$, r.i. BFS with $p_{\mathbb{Y}} = q > 1$, $q_{\mathbb{Y}} = q_{\mathbb{X}}\, q/p_{\mathbb{X}} < \infty$
- Pick $r = p_0\, q/(p\, p_{\mathbb{X}}) > 1$ and use $(\star\star\star)$

Extrapolation for A_∞ weights on modular spaces

Theorem

Let $0 < p_0 < \infty$. *Assume that for every* $w \in A_\infty$

$$(\star) \qquad \|f\|_{L^{p_0}(w)} \lesssim \|g\|_{L^{p_0}(w)} \qquad (f,g) \in \mathcal{F}$$

If ϕ *is a Young function with* $I_\phi < \infty$, *for all* $0 < p, q < \infty$, $w \in A_\infty$

$$\int_{\mathbb{R}^n} \phi(f^q)^p \, w \, dx \lesssim \int_{\mathbb{R}^n} \phi(g^q)^p \, w \, dx \qquad (f,g) \in \mathcal{F}$$

Furthermore, for all $\mathbb{X}$ *r.i. BFS with* $q_{\mathbb{X}} < \infty$, $0 < p, q < \infty$, $w \in A_\infty$

$$\|\phi(f^q)\|_{\mathbb{X}^p(w)} \lesssim \|\phi(g^q)\|_{\mathbb{X}^p(w)} \qquad (f,g) \in \mathcal{F}$$

$$\sup_{\lambda>0} \phi(\lambda^q)^p \, w\{f > \lambda\} \lesssim \sup_{\lambda>0} \phi(\lambda^q)^p \, w\{f > \lambda\} \qquad (f,g) \in \mathcal{F}$$

Section 6

Applications

Coifman's Inequality: Extensions of Boyd and Lorentz-Shimogaki

- Coifman's inequality: T is a CZO

$$\|Tf\|_{L^p(w)} \lesssim \|Mf\|_{L^p(w)}, \qquad 0 < p < \infty, \quad w \in A_\infty$$

- Extrapolation: $\mathbb{X}$ r.i. BFS, $q_{\mathbb{X}} < \infty$, for all $w \in A_\infty$, $0 < q < \infty$

$$\|Tf\|_{\mathbb{X}(w)} \lesssim \|Mf\|_{\mathbb{X}(w)},$$

$$\Big\| \Big(\sum_j |Tf_j|^q \Big)^{\frac{1}{q}} \Big\|_{\mathbb{X}(w)} \lesssim \Big\| \Big(\sum_j (Mf_j)^q \Big)^{\frac{1}{q}} \Big\|_{\mathbb{X}(w)},$$

Theorem (Lorentz-Shimogaki; Boyd)

Let $\mathbb{X}$ be a r.i. BFS.

- $M : \mathbb{X} \longrightarrow \mathbb{X} \iff p_{\mathbb{X}} > 1$
- $H : \mathbb{X} \longrightarrow \mathbb{X} \iff 1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$

Coifman's Inequality: Extensions of Boyd and Lorentz-Shimogaki

Theorem

Let $\mathbb{X}$ be a r.i. BFS and T be a CZO.

- *If* $1 < p_{\mathbb{X}} \le \infty$, $M : \mathbb{X}(w) \longrightarrow \mathbb{X}(w)$, $\forall\, w \in A_{p_{\mathbb{X}}}$
- *If* $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$, $T : \mathbb{X}(w) \longrightarrow \mathbb{X}(w)$, $\forall\, w \in A_{p_{\mathbb{X}}}$
- *If* $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$, *for all* $1 < q < \infty$ *and all* $w \in A_{p_{\mathbb{X}}}$

$$\Big\| \Big(\sum_j (M f_j)^q \Big)^{\frac{1}{q}} \Big\|_{\mathbb{X}(w)} \lesssim \Big\| \Big(\sum_j |f_j|^q \Big)^{\frac{1}{q}} \Big\|_{\mathbb{X}(w)},$$

$$\Big\| \Big(\sum_j |T f_j|^q \Big)^{\frac{1}{q}} \Big\|_{\mathbb{X}(w)} \lesssim \Big\| \Big(\sum_j |f_j|^q \Big)^{\frac{1}{q}} \Big\|_{\mathbb{X}(w)},$$

Remark: It suffices to assume that T satisfies

$$\|Tf\|_{L^{p_0}(w)} \lesssim \|Mf\|_{L^{p_0}(w)}, \qquad \forall\, w \in A_\infty, \quad \text{some } 0 < p_0 < \infty$$

Proof

1. If $1 < p_{\mathbb{X}} \le \infty$, $M : \mathbb{X}(w) \longrightarrow \mathbb{X}(w)$, $\forall\, w \in A_{p_{\mathbb{X}}}$
2. If $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$, $T : \mathbb{X}(w) \longrightarrow \mathbb{X}(w)$, $\forall\, w \in A_{p_{\mathbb{X}}}$

$$\|Tf\|_{\mathbb{X}(w)} \lesssim \|Mf\|_{\mathbb{X}(w)} \qquad \text{Coifman: } q_{\mathbb{X}} < \infty, w \in A_\infty$$

$$\lesssim \|f\|_{\mathbb{X}(w)} \qquad (1): p_{\mathbb{X}} > 1, w \in A_{p_{\mathbb{X}}}$$

3. Vector-valued for T and M: let $1 < q < \infty$

$$\big\| \|\{Tf_j\}\|_{\ell^q} \big\|_{\mathbb{X}(w)} \lesssim \big\| \|\{Mf_j\}\|_{\ell^q} \big\|_{\mathbb{X}(w)} \qquad \text{Coifman: } q_{\mathbb{X}} < \infty, w \in A_\infty$$

$$\lesssim \big\| M\big(\|\{f_j\}\|_{\ell^q}\big) \big\|_{\mathbb{X}(w)} \qquad \text{Coifman: } q_{\mathbb{X}} < \infty, w \in A_\infty$$

$$\lesssim \big\| \|\{f_j\}\|_{\ell^q} \big\|_{\mathbb{X}(w)} \qquad (1): p_{\mathbb{X}} > 1, w \in A_{p_{\mathbb{X}}}$$

Auxiliary result:

$$\big\| \|\{Mf_j\}\|_{\ell^q} \big\|_{L^p(w)} \lesssim \big\| M\big(\|\{f_j\}\|_{\ell^q}\big) \big\|_{L^p(w)}, \quad 0 < p < \infty,\ w \in A_\infty$$

Commutators with CZO

- $b \in \mathrm{BMO}$: $\displaystyle \sup_Q \fint_Q |b(x) - b_Q|\, dx < \infty$
- First order commutator:

$$[b,T]f(x) = b(x)\,Tf(x) - T(b\,f)(x) = \int_{\mathbb{R}^n} (b(x) - b(y))\,K(x,y)\,f(y)\,dy$$

- Pérez: $\big\|[b,T]f\big\|_{L^p(w)} \lesssim \|M^2 f\|_{L^p(w)}, \qquad \forall\, w \in A_\infty, \quad 0 < p < \infty$

- Find end-point estimates for an operator S verifying

$$\big\|Sf\big\|_{L^{p_0}(w)} \lesssim \|M^2 f\|_{L^{p_0}(w)}, \qquad \forall\, w \in A_\infty, \quad \text{some } 0 < p_0 < \infty$$

Commutators with CZO: End-point estimates

- $M^2 f(x) \approx M_{L\log L} f(x) = \sup_{Q\ni x} \|f\|_{L(\log L),Q}$
- Vitali: for all $w \in A_1$

$$w\{x : M_{L\log L} f(x) > \lambda\} \lesssim \int_{\mathbb{R}^n} \varphi\left(\frac{|f(x)|}{\lambda}\right) w(x) dx, \ \ \varphi(t) = t(1+\log^+ t)$$

Corollary

Assume that S satisfies

$$\|Sf\|_{L^{p_0}(w)} \lesssim \|M^2 f\|_{L^{p_0}(w)}, \qquad \forall\, w \in A_\infty, \quad \textit{some } 0 < p_0 < \infty$$

Then, for all $w \in A_1$

$$w\{x : |Sf(x)| > \lambda\} \lesssim \int_{\mathbb{R}^n} \varphi\left(\frac{|f(x)|}{\lambda}\right) w(x)\, dx, \qquad \varphi(t) = t\,(1+\log^+ t)$$

Proof

- $\psi(t) = \dfrac{t^2}{1+\log^+ 1/t^2}$ is a Young function: $i_\psi = I_\psi = 2$
- Extrapolation: for all $w \in A_\infty$, $0 < p, q < \infty$,

$$\sup_{\lambda>0} \psi(\lambda^q)^p\, w\{|Sf| > \lambda\} \lesssim \sup_{\lambda>0} \psi(\lambda^q)^p\, w\{M^2 f > \lambda\}$$

- Pick $p = 1$, $q = 1/2$: $\phi(t) = \psi(t^{1/2}) = \dfrac{t}{1+\log^+ 1/t}$

$$\begin{aligned} w\{|Sf| > 1\} &= \phi(1)\, w\{|Sf| > 1\} \le \sup_{\lambda>0} \phi(\lambda)\, w\{|Sf| > \lambda\} \\ &\lesssim \sup_{\lambda>0} \phi(\lambda)\, w\{M^2 f > \lambda\} \lesssim \sup_{\lambda>0} \phi(\lambda) \int_{\mathbb{R}^n} \varphi\left(\frac{|f|}{\lambda}\right) dx \\ &\lesssim \sup_{\lambda>0} \phi(\lambda)\, \varphi(1/\lambda) \int_{\mathbb{R}^n} \varphi(|f|)\, dx = \int_{\mathbb{R}^n} \varphi(|f|)\, dx \end{aligned}$$

Section 7

Further results

Further results: Variable L^p spaces

- $p : \mathbb{R}^n \longrightarrow (1, \infty)$
- $p_- = \inf p(x) > 1 \qquad p_+ = \sup p(x) < \infty$
- $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} \leq 1 \right\}$
- $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$: M bounded on $L^{p(\cdot)}(\mathbb{R}^n)$
- Diening: $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n) \quad \Longleftrightarrow \quad p'(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$

Further results: Variable L^p spaces

Theorem (Cruz-Uribe, Fiorenza, Martell, Pérez)

Let $1 \le p_0 < \infty$. *Assume that for every* $w \in A_{p_0}$

$$(\star) \qquad \|f\|_{L^{p_0}(w)} \lesssim \|g\|_{L^{p_0}(w)} \qquad (f,g) \in \mathcal{F}$$

If $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ *then*

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)} \qquad (f,g) \in \mathcal{F}$$

Furthermore, for every $1 < q < \infty$

$$\Big\|\Big(\sum_j f_j^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \Big\|\Big(\sum_j g_j^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \quad \{(f_j,g_j)\}_j \in \mathcal{F}$$

Further results: Variable L^p spaces

Corollary (Cruz-Uribe, Fiorenza, Martell, Pérez)

Let $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ *and* T *be a CZO. Then for every* $1 < q < \infty$

$$\|Tf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\Big\|\Big(\sum_j Mf_j^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \Big\|\Big(\sum_j |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\Big\|\Big(\sum_j |Tf_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \Big\|\Big(\sum_j |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

Remark: Rough SIO, smooth maximal operators, commutators, multipliers, square functions, fractional integrals, etc.

The Calderón-Zygmund inequality and Poisson's equation

- $\Omega \subset \mathbb{R}^n$, $n \geq 3$; $\quad p : \Omega \longrightarrow (1, \infty)$ with $1 < p_- \leq p_+ < n/2$,

$$|p(x) - p(y)| \leq \begin{cases} \big(-\log(|x-y|)\big)^{-1} & |x-y| \leq 1/2, \quad x, y \in \Omega \\ \big(\log(e + |x|)\big)^{-1} & |y| \geq |x|, \quad x, y \in \Omega \end{cases}$$

- $\dfrac{1}{p(x)} - \dfrac{1}{q(x)} = \dfrac{2}{n}, \qquad \dfrac{1}{p(x)} - \dfrac{1}{r(x)} = \dfrac{1}{n}$

Corollary (Cruz-Uribe, Fiorenza, Martell, Pérez)

Given $f \in L^{p(\cdot)}(\mathbb{R}^n)$, there exists $u \in L^{q(\cdot)}(\mathbb{R}^n)$ such that

$$\Delta u(x) = f(x), \qquad a.e.\ x \in \Omega,$$

$$\|u\|_{L^{q(\cdot)}(\Omega)} + \|D^1 u\|_{L^{r(\cdot)}(\Omega)} + \|D^2 u\|_{L^{p(\cdot)}(\Omega)} \lesssim \|f\|_{L^{p(\cdot)}(\Omega)}.$$

If Ω is bounded, $\|u\|_{W^{2,p(\cdot)}(\Omega)} \lesssim \|f\|_{L^{p(\cdot)}(\Omega)}$

Wavelet characterization of $L^{p(\cdot)}$ spaces

- ψ orthonormal wavelet: $\{\psi_I : I \in \mathcal{D}\}$ orthonormal basis of $L^2(\mathbb{R})$
- $\mathcal{W}_\psi f = \Big(\sum_{I\in\mathcal{D}} |\langle f, \psi_I\rangle|^2 \, |I|^{-1} \, \chi_I \Big)^{\frac{1}{2}}$
- [García-Cuerva, Martell]: If ψ is regular, for every $1 < p < \infty$

$$\|f\|_{L^p(w)} \lesssim \|\mathcal{W}_\psi f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}, \qquad \forall\, w \in A_p, \quad f \in L^p(w)$$

Corollary

If $p(\cdot) \in \mathfrak{B}$ and ψ is a regular orthonormal wavelet then

$$\|f\|_{L^{p(\cdot)}(\mathbb{R})} \lesssim \|\mathcal{W}_\psi f\|_{L^{p(\cdot)}(\mathbb{R})} \lesssim \|f\|_{L^{p(\cdot)}(\mathbb{R})}$$

for all $f \in L^{p(\cdot)}(\mathbb{R})$

Sawyer's conjecture

- [Sawyer 1985]

$$u\,v\Big\{x\in\mathbb{R}:\frac{M(f\,v)(x)}{v(x)}>\lambda\Big\}\le\frac{C}{\lambda}\int_{\mathbb{R}}|f|\,u\,v\,dx,\qquad \forall\,u,v\in A_1$$

Sawyer's Conjecture — H satisfies the same inequality

- [Cruz-Uribe, Martell, Pérez, Int. Math. Res. Not. 05]

  The inequality holds for:

  - The Hilbert transform H
  - M, M^d and $T\in CZO$ in $\mathbb{R}^n$, $n\ge 1$
  - $u,v\in A_1$ & $u\in A_1$, $v\in A_\infty(u)$

Sawyer's conjecture: Scheme of the proof

- Step 1: $\left\|M^d(f\,v)\cdot v^{-1}\right\|_{L^{1,\infty}(u\,v)} \lesssim C\,\|f\|_{L^{1,\infty}(u\,v)}, \quad u, v \in A_1$
- Step 2: Extrapolation

Theorem

Let $0 < p_0 < \infty$. Assume that for every $w \in A_\infty$

$$(\star) \qquad \|f\|_{L^{p_0}(w)} \lesssim \|g\|_{L^{p_0}(w)} \qquad (f,g) \in \mathcal{F}$$

Then for every $u \in A_1$, $v \in A_\infty$

$$\left\|f\cdot v^{-1}\right\|_{L^{1,\infty}(u\,v)} \leq C\,\left\|g\cdot v^{-1}\right\|_{L^{1,\infty}(u\,v)} \qquad (f,g) \in \mathcal{F}$$

- Step 3: $\mathcal{F} \quad \rightsquigarrow \quad \big(M(f\,v), M^d(f\,v)\big), \quad \big(T(f\,v), M(f\,v)\big)$
- Corollary: Vector-valued inequalities