Aim of this minicourse is to give an elementary introduction to Colombeau theory of algebras of generalized functions. Starting from the original work of Jean François Colombeau in the 80’s we introduce the Colombeau algebra $\mathcal{G}(\Omega)$, on an open subset $\Omega$ of $\mathbb{R}^n$, as a framework where to define a “reasonable” product of distributions. After having explained the meaning of “reasonable” in this context, we prove that the product obtained by linear embedding of $D'(\Omega)$ into $\mathcal{G}(\Omega)$ extends the classical multiplication of smooth functions. Due to its differential-algebraic properties Colombeau theory is particularly suited to answer a wealth of questions on solutions to linear and nonlinear partial differential equations involving non-smooth coefficients and strongly singular data.

In the second part of the minicourse we study some PDE problems in the Colombeau context. In particular we prove that the Cauchy problem for an hyperbolic equation with singular coefficients and singular initial data is well-posed in the Colombeau context and we investigate the qualitative properties of the Colombeau solution.

The third and last part of the minicourse is devoted to some recent research topics in Colombeau theory: generalized Fourier integral operators and functional analytic methods for algebras of generalized functions.
Monday May 18 at 10:15-12:00 and 14:15-16:00 (Hall J): Colombeau theory and multiplication of distributions

Our elementary introduction to Colombeau theory starts with some short overview on multiplication of distributions. Main idea in the Colombeau construction is the sequential approach to distributions: to each distribution \( u \) is associated a net \((u_\varepsilon)_\varepsilon\) of smooth functions obtained via convolution with a mollifier. The Colombeau algebra \( \mathcal{G}(\Omega) \) is a factor space and each element is an equivalence class \( u = [(u_\varepsilon)_\varepsilon] \) of nets of smooth functions satisfying some moderateness estimates with respect to the parameter \( \varepsilon \in (0,1] \).

Particular attention will be given to two subalgebras of \( \mathcal{G}(\Omega) \): the ring \( \tilde{\mathbb{C}} \) of complex generalized numbers and the subalgebra \( \mathcal{G}^\infty(\Omega) \) of regular generalized functions. Since \( \mathcal{G}^\infty(\Omega) \cap D'(\Omega) = \mathcal{C}^\infty(\Omega) \) one can develop in the Colombeau context a regularity theory which extends the usual notion of \( \mathcal{C}^\infty \)-regularity in the distributional setting.

References: [1, 9, 12]

Tuesday May 19 at 10:15-12:00 and 14:15-16:00 (Hall J): Colombeau theory and differential operators with singular coefficients

We begin with the general definition of a partial differential operator with Colombeau coefficients and we discuss solvability in the Colombeau context when the coefficients are constant. We then focus on some specific examples generated by classical operators with singular coefficients via embedding of the coefficients into the Colombeau algebra. In particular we prove that the Cauchy problem for an hyperbolic equation with singular coefficients and singular initial data is well-posed in the Colombeau setting. The investigation of the regularity properties of the solution requires some basic notions of microlocal analysis in the Colombeau framework.

References: [10, 11, 12]

Wednesday May 20 at 10:15-12:00 and 14:15-16:00 (Hall J): Different research topics in Colombeau theory

Motivated by the results on hyperbolic equations with singular coefficients, a theory of generalized Fourier integral operators acting on Colombeau algebras, where both the phase function and amplitude are objects of Colombeau type, has been recently initiated in [5, 7]. This provides a useful extension of Hörmander’s theory of Fourier integral operators. In fact, the standard theory heavily relies on the smoothness of the coefficients (e.g. in applications to hyperbolic equations) whereas this generalized theory, which can deal with the non-differentiable and even distributional setting, is a powerful tool for a variety of problems where the standard theory unfortunately fails on the technical level. For example, appearance of light cones in crystals in conical refraction is a very important problem for applications, and the phase functions are not smooth because the underlying problem is weakly hyperbolic. Here, the development and application of generalized FIO methods, combined with a Colombeau version of symplectic geometry, is surely extremely interesting and important. In this final part of the minicourse we will see some basic ideas concerning the theory of generalized Fourier integral operators with a particular attention for the special case of generalized pseudodifferential operators and their application to elliptic equations.

The general problem of existence and qualitative properties of solutions for partial differential equations in the Colombeau setting has been recently approached via functional analytic methods. This means that a topological theory of Colombeau algebras has been developed providing a duality theory as well as some relevant theorems of functional analysis: closed graph theorem, open mapping theorem, Riesz representation theorem for Hilbert \( \tilde{\mathbb{C}} \)-modules, etc. Some of these topological and functional analytic aspects will be discussed in the last lecture of the minicourse.

References: [2, 3, 4, 6, 8]
References


