Helsinki University of Technology
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Licentiate's Thesis

On the stability of quasiminimizers of the p–Dirichlet integral

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Euklidisessa avaruudessa p–Dirichlet'n integraalin

$$\int |Du|^p dx, \qquad 1 < p < \infty,$$

minimoiminen on ekvivalenttia p–harmonisen Laplacen yhtälön

$$\operatorname{div}(|Du|^{p-2} Du) = 0$$

ratkaisemisen kanssa. Yleisessä metrisessä avaruudessa ongelmaa voidaan lähestyä ainoastaan variaatiolaskennan keinoin, sillä p–Laplacen yhtälön vastinetta ei tunneta. Sobolevin avaruudet voidaan kuitenkin määritellä metrisessä avaruudessa ilman osittaisderivaatan käsitettä. Näitä avaruuksia kutsutaan Newtonin avaruuksiksi.

Tässä lisensiaatintutkimuksessa keskitymme p–Dirichlet'n integraalin minimoijien sijaan kvasiminimoijiin, jotka minimoivat p–Dirichlet'n integraalin kerroinvakiota vaille. Alunperin kvasiminimoijia käytettiin työkaluina minimoijien säännöllisyyden tutkimisessa, mutta sittemmin niitä on tutkittu itsessään mielenkiintoisena funktioluokkana.

Todistamme stabiilisuustuloksen muuttuvan eksponentin p suhteen p–Dirichlet'n integraalin kvasiminimoijille tuplaavassa metrisessä mitta–avaruudessa. Lisäksi todistamme, että kvasiminimoijien ylägradientit ovat globaalisti integroituvia korkeampaan potenssiin kuin lähtökohtaisesti tiedetään. Käänteisen Hölderin ja Poincarén epäyhtälöiden sekä kapasiteettitiheysehdon itseparantuvuudet ovat merkittäviä tulosten todistamisessa.

Tutkimus perustuu yhteiseen artikkeliin Anna Zatorska–Goldsteinin kanssa, ja sen päätulokset ovat tietämämme mukaan uusia metrisen avaruuden tapauksessa. Työn tarkoituksena on olla mahdollisimman yhtenäinen. Muutamaa yleisesti metristen avaruuksien tilanteessa tunnettua lausetta lukuunottamatta kaikki päätulosten todistamiseen tarvittavat työkalut esitellään todistuksineen.

Sivumäärä: 50	Avainsanat:	Caccioppolin epäyhtälö, Gehringin lemma, kapasiteetti, kvasiminimoija, metrinen avaruus, Newtonin avaruus, p–paksuus, Poincarén epäyhtälö, stabiilisuus, tuplaava mitta

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Minimizing the p–Dirichlet integral

$$\int |Du|^p dx, \qquad 1 < p < \infty,$$

is equivalent to solving the p–harmonic equation

$$\operatorname{div}(|Du|^{p-2}Du) = 0,$$

which is a nonlinear generalization of the classical Laplace equation. In a general metric measure space only the variational approach to the problem is available, since it is not clear what is the counterpart of the p–Laplace equation. However, the Sobolev spaces on a metric measure space, called Newtonian spaces, can be defined without the notion of partial derivatives.

Instead of minimizers of the p–Dirichlet integral, in this Licentiate's Thesis we consider its quasiminimizers, that is, Newtonian functions that minimize the p–Dirichlet integral up to a multiplicative constant. Originally they were used as tools in studying regularity of minimizers but have then become an interesting research subject themselves.

We prove a stability result, with respect to the varying exponent p, for a family of quasiminimizers of the p–Dirichlet energy functional on a doubling metric measure space. In addition we prove global higher integrability for upper gradients of quasiminimizers. The self–improving properties of the reverse Hölder inequality, the Poincaré inequality and the capacity thickness condition play an important role in the proofs of the main theorems.

The Thesis is based on a joint article with Anna Zatorska–Goldstein and the main results are to our knowledge new in the metric setting. The idea of this work is to be as self–contained as possible. Except for a few generally known theorems, all needed tools are presented with proofs.

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Preface

In these occasions it is common to thank the supervisor and the instructor of the thesis. In my case this is a pleasant and easy job. Olavi Nevanlinna and Juha Kinnunen were always there for my various questions and laughed rarely at them. A third important person is Anna Zatorska–Goldstein, whose company both mathematically and on free time is a joy. She also read my manuscript carefully and gave valuable comments. And you other folks, you know who you are, hugs.

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Chapter 1

Introduction

One of the most important elliptic variational problems studied in Euclidean spaces is to minimize the p–energy functional, in other words the p–Dirichlet integral

$$\int_\Omega |Du|^p dx$$

in a subset Ω of $\mathbb{R}^n$ among all functions $u\colon \Omega \to \mathbb{R}$ that satisfy the required boundary conditions. This is equivalent to solving the p–harmonic equation

$$\operatorname{div}(|Du|^{p-2}Du) = 0.$$

In a general metric measure space it is not clear what the counterpart for the p–harmonic equation is. However, in such a space, the variational approach to p–harmonic functions is available; it is possible to define p-harmonic functions as minimizers of the Dirichlet integral. The basic reason is that the Sobolev spaces on a metric measure space can be defined without the notion of partial derivatives; see e.g. Hajłasz [10], Shanmugalingam [35]. Direct methods in the calculus of variations are also available and one can prove existence results for the Dirichlet problem; see e.g. A. Björn, J. Björn and Shanmugalingam [3], Shanmugalingam [36].

Quasiminimizers are a class of functions closely related to p–harmonic functions. A function u is called a quasiminimizer if it minimizes the Dirichlet functional up to a multiplicative constant K, that is

$$\int_\Omega |Du|^p dx \leq K \int_\Omega |Dv|^p dx$$

among all functions v which have the same boundary values. The notion of quasiminimizers was introduced by Giaquinta and Giusti in [9] as a tool for unified treatment of variational integrals, elliptic equations and systems, obstacle problems and quasiregular mappings. In the setting of metric spaces, the approach via quasiminimizers is particularly useful, as the Euler equation for the p–Dirichlet energy integral does not need to exist.

In this thesis we study the behaviour of a sequence of p–Dirichlet integral quasiminimizers as p varies. We work in a metric measure space (X, d, μ), where the measure μ is doubling. Moreover, we assume the space to be complete, locally linearly convex and to support a weak $(1, p)$–Poincaré inequality. In this setting, we consider a sequence (u_i) where $u_i \colon \Omega \to \mathbb{R}$ is a K–quasiminimizer of the p_i–Dirichlet integral in an open bounded subset Ω of X. All u_i have the same boundary data. Under the condition that the complement of Ω satisfies a capacity thickness assumption, we prove that if p_i converge to p and u_i converge to u almost everywhere with respect to μ, then u is a K–quasiminimizer of the p–energy integral in Ω with the same boundary data.

Note that since quasiminimizers do not provide unique solutions to the Dirichlet problem, even if p does not vary, they may not converge in general.

In the setting of the main theorem, by using the Gehring lemma, we prove the global higher integrability of upper gradients of quasiminimizers. More precisely, if u is a quasiminimizer of the p–energy integral in Ω with boundary data w, then there exists $\delta > 0$ such that $g_u \in L^{p+\delta}(\Omega)$ and

$$\left(\int_\Omega g_u^{p+\delta} \, \mathrm{d}\mu \right)^{1/(p+\delta)} \le c \left(\int_\Omega g_u^{p} \, \mathrm{d}\mu \right)^{1/p} + c \left(\int_\Omega g_w^{p+\delta} \, \mathrm{d}\mu \right)^{1/(p+\delta)},$$

where c depends only on p and on the constants related to the space and to the domain Ω. As above, Ω is an open bounded subset of X and its complement satisfies a capacity thickness assumption.

Assuming that the space satisfies a weak $(1, q)$–Poincaré inequality for some $q < p$, where p is the natural exponent associated with the studied problem, is a standard, yet non–trivial, assumption in the metric setting. However, as shown by Keith and Zhong [16], the Poincaré inequality has a self improving property. In quite general spaces a weak $(1, p)$–Poincaré implies a weak $(1, q)$–Poincaré for some $q < p$. The same holds also for a p–fatness condition, that is a capacity thickness property of a set. We refer the reader to Sections 2.3 and 2.9, respectively. Consequently, together with the Gehring lemma stating that the reverse Hölder inequality is an open ended condition, we make use of three self–improving results.

The Gehring lemma in the context of a metric space has been studied among others by Maasalo [31] and Zatorska–Goldstein [39]. These references contain each a slightly different version of the lemma, and here the version of [39] responds directly to our needs.

Quasiminimizers have already several years been an active research subject in the setting of doubling metric measure space with a Poincaré inequality. As examples we mention A. Björn and Marola [4], J. Björn [5], Kinnunen and Martio [22], Kinnunen and Shanmugalingam [23]. It has been proved by Kinnunen and Shanmugalingam that quasiminimizers are (locally) Hölder continuous, see [23]. In [22], Kinnunen and Martio have studied the nonlinear potential theory for quasiminimizers. The boundary continuity for quasiminimizers on a bounded set Ω with fixed boundary data has been examined by J. Björn [5]. There exist also previous stability results. It was shown by Kinnunen and Martio that the class of (local) quasisuperminimizers, for fixed p, is closed under monotone convergence, provided that the limit function is bounded [22].

The backround of studying stability of the p–Dirichlet integral is naturally in Euclidean spaces. Li and Martio have examined a quasilinear elliptic operator and in [27] they prove a convergence result for the solutions of an obstacle problem with varying p in a bounded subset Ω of $\mathbb{R}^n$. Later they have proved a similar result for a double obstacle problem, see [28]. The stability question we consider in the thesis is closely related to the work of Lindqvist, [29]. It discusses the convergence of solutions to the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$$

with varying p and the problem is solved for a bounded subset of $\mathbb{R}^n$. In the Euclidean setting this is equivalent to studying minimizers to the corresponding variational problem, whereas in our case only the variational approach is possible. Both Li and Martio's obstacle problem results require a measure or a capacity thickness condition on the complement of Ω. A question on regularity of the set Ω is raised also in [30], where Lindqvist studies the stability with respect to p of the p–harmonic eigenvalue problem. This indicates that regularity of the set we need to assume does not arise from the metric setting.

The thesis is based on a paper written jointly with Anna Zatorska–Goldstein, [32] and it is organized as follows: in Chapter 2 we fix the general setup and present basic definitions and facts about analytic tools used in the metric setting. A reader familiar with metric measure spaces may omit this part. Chapter 3 presents results, such as Sobolev–Poincaré type inequalities, needed in the subsequent chapters. Chapter 4 contains the proof of the higher integrability result and Chapter 5 the proof of the stability result.

The authors of the original paper worked closely together at research as well as during the writing stage of the paper. In [32], Chapters 2 and 3 are considerably abridged. In the thesis they are presented in more detail and proofs are included. Parts of Chapters 4 and 5 have been explicated as well.

Chapter 2

Definitions and general setting

Our notation is standard. We assume that a ball in a metric space (X, d) comes always with a centre and a radius, i.e.

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

with $0 < r < \infty$. We denote the average of a locally integrable function in a ball B by

$$u_B = ⨍_B u d\mu = \frac{1}{\mu(B)} \int_B u d\mu,$$

and when there is no possibility for confusion, we denote with σB a ball with the same center as B but σ times its radius. We say that a set U is *compactly contained* in V and write $U \subset\subset V$ if the closure of U is a compact subset of V.

A reader familiar with doubling metric measure spaces and Newtonian spaces may omit this chapter and see the general setup at the end of it.

2.1 Paths and modulus

Let $I \subset \mathbb{R}$ be an interval. A path $\gamma : I \to X$ is a continuous mapping, and if the interval is closed, $I = [a, b]$, we define the length of γ by

$$\ell(\gamma) = \sup \sum_{i=i}^{k} d(\gamma(t_i), \gamma(t_{i-1})),$$

where the supremum is taken over all finite partitions $a = t_0 \le t_1 \le \ldots \le t_k = b$. If I is not closed, then we set

$$\ell(\gamma) = \sup \ell(\gamma_{|J}),$$

where the supremum is taken over all closed subintervals J of I and $\gamma_{|J}$ is the restriction of γ to J. We say that a path $\gamma : I \to X$ is rectifiable if its lentgh is finite. Similarly, we say that a path is locally rectifiable if its restrictions to each closed subinterval of I are rectifiable.

Now let Γ be a family of paths in X and $1 \leq p < \infty$. The p–modulus of Γ is defined as

$$\operatorname{Mod}_p(\Gamma) = \inf \int_X g^p d\mu,$$

where the infimum is taken over all non–negative Borel functions $g : X \to [0, \infty]$ satisfying

$$\int_\gamma g ds \geq 1$$

for all locally rectifiable $\gamma \in \Gamma$.

If a property fails only for a set of paths that is of zero p–modulus we say that the property holds for p–almost all paths.

For the definition of the path integral in metric spaces or further information on paths or modulus see for example [10] or [38].

2.2 Upper gradients

Let u be a real valued function on X. A non–negative Borel measurable function g on X is said to be an *upper gradient* of u if for all rectifiable paths γ joining points x and y in X we have

$$|u(x) - u(y)| \leq \int_\gamma g ds. \tag{2.1}$$

The definition implies immediately, that an upper gradient is not unique and that $g \equiv \infty$ is always an upper gradient. For instance in $\mathbb{R}^n$ with the standard metric $g = |Du|$ is an upper gradient of a smooth function u.

If the inequality (2.1) fails only for a set of paths that is of zero p–modulus, then g is said to be a *p–weak upper gradient* of u. Without loss of generality we limit ourselves to consider only p–weak upper gradients. This is justified by the fact that the L^p–closure of the set of L^p–integrable upper gradients of a function u is exactly the set of all L^p–integrable p–weak upper gradients, see [26]. Instead of p–weak upper gradient we sometimes write weak upper gradient for short.

The motivation to work with weak upper gradients is that they collaborate better with a measure. Namely, if a p–weak upper gradient of a function u coincides μ–almost everywhere with a Borel function, then the latter is also a p–weak upper gradient of u.

We recall that if $1 < p < \infty$, every function u that has a p–integrable p–weak upper gradient has a *minimal p–integrable p–weak upper gradient* denoted g_u. Here g_u is minimal in the sense that

$$\|g_u\|_{L^p(X)} = \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all p–weak upper gradients of u. Moreover, if g is an upper gradient of u, then $g_u \leq g$ μ–almost everywhere in X.

It is important to notice that for every $c \in \mathbb{R}$ the minimal p–weak upper gradient satisfies $g_u = 0$ μ–almost everywhere on the set

$$\{x \in X \colon u(x) = c\}.$$

The notion of an upper gradient is interesting only in spaces were there is abundance of rectifiable paths. This can be guaranteed for example by requiring that the space admits a Poincaré inequality.

2.3 Poincaré inequality

We say that the space supports a weak $(1, q)$–Poincaré inequality if there exist $c > 0$ and $\tau \geq 1$ such that

$$\fint_B |u - u_B| \,\mathrm{d}\mu \leq cr \left(\fint_{\tau B} g^q \,\mathrm{d}\mu \right)^{1/q}$$

for all balls $B = B(x, r)$ in X and all pairs $\{u, g\}$ where u is a locxally integrable function and g is a q–weak upper gradient of u. The above inequality is called weak since we allow a larger ball on the right–hand side.

The Poincaré inequality may not immediately seem to be a geometric property of a space, but in closer examination it turns out to be one. Loosely we could say that the validity of a $(1, q)$–Poincaré inequality guarantees the existence of short rectifiable curves. For examples and necessary conditions for the validity of $(1, q)$–Poincaré inequalities see [11].

A result of [11] shows that in a doubling metric measure space a weak $(1, q)$–Poincaré inequality implies a weak (t, q)–Poincaré inequality for some $t > q$ and possibly a new τ i.e. there exist $c' > 0$ and $\tau' \geq 1$ such that for all balls B in X

$$\left(\fint_B |u - u_B|^t \, \mathrm{d}\mu \right)^{1/t} \leq c' r \left(\fint_{\tau' B} g^q \, \mathrm{d}\mu \right)^{1/q}, \tag{2.2}$$

where

$$\begin{cases} 1 \leq t \leq Qq/(Q-q) & \text{if} \quad q < Q, \\ 1 \leq t & \text{if} \quad q \geq Q, \end{cases}$$

and $Q = \log_2 c_d$.

Let $1 < p < \infty$. Observe, that if X supports a weak $(1, p)$–Poincaré, then it supports a weak $(1, p')$–Poincaré for all $p' > p$ by Hölder's inequality. The converse is not true in general. However, in a complete doubling metric measure space supporting a weak $(1, p)$–Poincaré inequality, there exists $1 < q < p$ such that the space admits a weak $(1, q)$–Poincaré inequality by a result in [16]. Increasing q if necessary we may additionally assume that $p \in (q, q^*)$, where $q^* = qQ/(Q-q) < \infty$. Here q^* can be interpreted as the metric space counterpart of the Sobolev conjugate of q.

2.4 Newtonian spaces

Newtonian spaces are the metric counterpart of Sobolev spaces. Let $1 \leq p < \infty$. We define the space $\widetilde{N}^{1,p}(X)$ to be the collection of all p–integrable functions u on X that have a p–integrable p–weak upper gradient g on X. This space is equipped with the seminorm

$$||u||_{\widetilde{N}^{1,p}(X)} = ||u||_{L^p(X)} + \inf ||g||_{L^p(X)},$$

where the infimum is taken over all p–weak upper gradients of u. Note that by the remark in Section 2.2 if $p > 1$, the norm in $\widetilde{N}^{1,p}(X)$ is precisely the sum of the L^p–norm of the function and of the L^p–norm of the minimal upper gradient.

We define an equivalence relation in $\widetilde{N}^{1,p}(X)$ by saying that $u \sim v$ if

$$||u - v||_{\widetilde{N}^{1,p}(X)} = 0.$$

The *Newtonian space* $N^{1,p}(X)$ is then defined to be the space $\tilde{N}^{1,p}(X)/\sim$ with the norm

$$||u||_{N^{1,p}(X)} = ||u||_{\tilde{N}^{1,p}(X)}.$$

The normed space $(N^{1,p}(X), \|\cdot\|_{N^{1,p}(X)})$ is a Banach space, and as it is common we call $u \in N^{1,p}(X)$ functions instead of speaking of equivalence classes $[u]$.

It should be noted, that if u is a function in $\widetilde{N}^{1,p}(X)$ and v is a function that agrees with u μ–almost everywhere on X, it does not necessarily follow that v is also in $\widetilde{N}^{1,p}(X)$. However, if v is known to be in $\widetilde{N}^{1,p}(X)$, then it belongs to the same $N^{1,p}$–equivalence class with u, see [35], [36]. The set where the functions in the same equivalence class of $N^{1,p}$ do not coincide is smaller than only a measure zero set . The finer measure we need to determine the size of this set is *capacity*, see the next section.

We conclude by listing some basic properties of Newtonian spaces and of upper gradients of Newtonian functions. First of all, if u and v are in $N^{1,p}(X)$, then $\min(u, v)$ and $\max(u, v)$ are in $N^{1,p}(X)$ as well. Moreover, if g_u and g_v are the minimal p–weak upper gradients of u and v, respectively, then $g_u\chi_E + g_v\chi_{X\setminus E}$ with

$$E = \{x \in X : u(x) < v(x)\}$$

is a p–weak upper gradient of $\min(u, v)$, see for example [2] or [33]. Finally, if $u \in N^{1,p}(X)$ and η is a bounded Lipschitz continuous function, then $\eta u \in N^{1,p}(X)$ and

$$g_{\eta u} \leq |\eta| g_u + |u| g_\eta.$$

2.5 Capacity

The *p–capacity* of a set $E \subset X$ is defined by

$$C_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E. We say that a property holds p–quasieverywhere (p–q.e.) if the set of points for which the property fails is of zero p–capacity. It follows straight from the definition, that $\mu(E) \leq C_p(E)$. Thus, if there is a set of zero p–capacity, it is also of measure zero. Notice that the converse is not generally true.

There is a natural connection between the p–capacity and the p–modulus. Let E be a subset of X and Γ_E denote the family of all rectifiable paths that intersect E. It is possible to show that $C_p(E) = 0$ implies $\operatorname{Mod}_p(\Gamma_E) = 0$. From this it follows that if u is a function in $N^{1,p}(X)$ and v is a function that agrees with u p–quasieverywhere on X, v belongs to the same equivalence class in $N^{1,p}(X)$.

Let Ω be a bounded subset of X and let $E \subset\subset \Omega$. We define the *relative p–capacity* of E with respect to Ω by

$$\operatorname{cap}_p(E, \Omega) = \inf_u \int_\Omega g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N^{1,p}(\Omega)$ such that $u \geq 1$ on E and $u = 0$ on $X \setminus \Omega$ p–quasieverywhere. Lemma 3.6 in Chapter 3 shows that in a doubling metric measure space admitting a weak Poincaré inequality, the measure and the capacities are comparable in some sense on bounded sets.

The p–capacity enjoys the following properties. The capacity of an empty set is zero, i.e. $C_p(\emptyset) = 0$. The p–capacity is monotone and subadditive, that is

$$C_p(E_1) \leq C_p(E_2)$$

whenever $E_1 \subset E_2$ and

$$C_p\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq \sum_{i=1}^{\infty} C_p(E_i)$$

for every countable family $\{E_i\}_{i=1}^{\infty}$ of subsets of X. Moreover, the capacity of an arbitrary set E can be determined as the infimum of capacities of open sets containing E. In other words the p–capacity is outer capacity. For compact sets $\{V_i\}_{i=1}^{\infty}$ such that $X \supset V_1 \supset V_2 \supset \ldots$ it holds true that

$$C_p\Big(\bigcap_{i=1}^{\infty} V_i\Big) = \lim_{i\to\infty} C_p(V_i).$$

Finally, for a countable family of open subsets of X such that $U_1 \subset U_2 \subset \ldots \subset X$ we have

$$C_p\Big(\bigcup_{i=1}^{\infty} U_i\Big) = \lim_{i\to\infty} C_p(U_i).$$

These properties are proved for example in [19]. The notion of p–capacity makes it possible to define Newtonian spaces with zero boundary values.

2.6 Newtonian spaces with zero boundary values

Let Ω be an arbitrary subset of X. We define $N_0^{1,p}(\Omega)$ to be the set of functions $u \in N^{1,p}(\Omega)$ that are zero on $X \setminus \Omega$ p–quasieverywhere. The space $N_0^{1,p}(\Omega)$ is equipped with the norm

$$||u||_{N_0^{1,p}(\Omega)} = ||u||_{N^{1,p}(\Omega)}$$

and it is a Banach space [35]. It is possible to define Newtonian spaces with zero boundary values in several different ways. In general these approaches give different spaces but it can be shown that for a wide class of metric spaces the definitions agree. Let us present two other definitions based on Lipschitz functions.

Define $\mathrm{Lip}_0^{1,p}(\Omega)$ to be the collection of all Lipschitz functions in $N^{1,p}(X)$ that vanish on $X \setminus \Omega$ and let $\mathrm{Lip}_{C,0}^{1,p}(\Omega)$ be the collection of functions in $\mathrm{Lip}_0^{1,p}(\Omega)$ that have compact support in Ω. Let $H_0^{1,p}(\Omega)$ be the closure of $\mathrm{Lip}_0^{1,p}(\Omega)$ in the norm of $N^{1,p}(X)$ and $H_{C,0}^{1,p}(\Omega)$ be the closure of $\mathrm{Lip}_{C,0}^{1,p}(\Omega)$ in the norm of $N^{1,p}(X)$. If X is proper, doubling metric measure space supporting a $(1,p)$–Poincaré inequality and Ω is an open subset of X, then

$$H_{C,0}^{1,p}(\Omega) = H_0^{1,p}(\Omega) = N_0^{1,p}(\Omega).$$

The subject is discussed and the equality is proved in [36].

Notice that if $E \subset X$ is bounded and $1 < q \leq p < \infty$, then $N_0^{1,p}(E) \subset N_0^{1,q}(E)$. Probably the easiest way to see this is to consider Newtonian function as a limit of compactly supported Lipschitz functions and then use the Hölder inequality.

2.7 Quasiminimizers

Let Ω be an open subset of X. Let $w \in N^{1,p}(\Omega)$. We say that $u \in N^{1,p}(\Omega)$ is a quasiminimizer of the p–energy integral in Ω with boundary data w, if $u - w \in N_0^{1,p}(\Omega)$ and there exists a constant $K > 0$ such that for all open $\Omega' \subset\subset \Omega$ and all $\phi \in N_0^{1,p}(\Omega')$ we have

$$\int_{\overline{\Omega'}} g_u^p \, \mathrm{d}\mu \leq K \int_{\overline{\Omega'}} g_{u+\phi}^p \, \mathrm{d}\mu \, . \tag{2.3}$$

This is not the only possible way to define quasiminimizers. For example the integral in (2.3) can be taken over Ω' instead of its closure. Also requiring that Ω' is compactly contained in Ω is not necessary.

As for test functions, it is possible to use compactly supported Lipschitz functions or $\phi \in N^{1,p}(\Omega)$ such that $\operatorname{supp}\phi \subset\subset \Omega$ instead of $N_0^{1,p}(\Omega)$–functions. In these cases $u \in N^{1,p}(\Omega)$ is a quasiminimizer, if

$$\int_{\{\phi \neq 0\}} g_u^p \,\mathrm{d}\mu \leq K \int_{\{\phi \neq 0\}} g_{u+\phi}^p \,\mathrm{d}\mu \tag{2.4}$$

or

$$\int_{\operatorname{supp}\phi} g_u^p \,\mathrm{d}\mu \leq K \int_{\operatorname{supp}\phi} g_{u+\phi}^p \,\mathrm{d}\mu \tag{2.5}$$

for all test functions. All these definitions are equivalent. For further discussion and the equivalence proof see [1]. We will use different definitions without further notice.

If the equations (2.3), (2.4) and (2.5) hold only for all non–negative or non–positive test functions, $u \in N^{1,p}(\Omega)$ is called a quasisuperminimizer or a quasisubminimizer, respectively.

Quasiminimizers have several difficulties. Most importantly, they do not provide unique solutions to the Dirichlet problem. They do not obey the comparison principle nor have linear structure even in $\mathbb{R}^n$ if the corresponding Euler equation is linear. Also being a quasiminimizer is not a local property as being a weak solution of the Euler equation is. Quasiminimizers do, however, form an extremely general and flexible class of functions in calculus of variations. Originally quasiminimizers served as tools for unified treatment of minimizers of variational integrals. Nevertheless, they have an interesting theory themselves.

2.8 Local linear convexity

The local linear convexity, in brief LLC–property, of X means that there exist constants $C > 0$ and $r_0 > 0$ such that for all balls B in X with radius at most r_0, every pair of points in the annulus $2B \setminus \overline{B}$ can be connected by a curve lying in the annulus $2CB \setminus C^{-1}\overline{B}$.

Although the definition is simple in some sense, it may be hard to see what restrictions it imposes on a space. One possibility is to compare it to the Poincaré inequality. What do they have in common if anything? In Section 2.3 we stated that the validity of a Poincaré inequality is in fact in close relation to the geometry of the space. It is possible to construct examples of spaces that admit a Poincaré inequality but are not locally linearly convex. Thus Poincaré does not always imply LLC. Nonetheless, this is true in a complete metric space with a doubling measure that satisfies with some $s > 1$

$$\frac{\mu(B)}{\mu(\sigma B)} \leq c\Big(\frac{1}{\sigma}\Big)^s$$

for all balls. With these assumptions a $(1,p)$–Poincaré implies LLC for all $p \leq s$, [24]. Hence we have the feeling that LLC does not restrict our choice of spaces much more than requiring a Poincaré inequality.

2.9 p–fatness

We say that the set $E \subset X$ is uniformly p–fat if there exist constants $c_f > 0$ and $r_0 > 0$ such that for all $x \in E$ and $0 < r < r_0$

$$\operatorname{cap}_p(E \cap B(x,r); B(x,2r)) \geq c_f \operatorname{cap}_p(B(x,r); B(x,2r)).$$

The p–fatness is a non–geometric way to impose some regularity on a set. Especially, the above inequality does not give any information on the boundary of a set. However in $\mathbb{R}^n$ all Lipschitz domains or domains satisfying the exterior cone condition are uniformly p–fat for every p, $1 < p < \infty$. Also the complement of every domain $\Omega \neq \mathbb{R}^n$ is uniformly p–fat if $p > n$. The p–fatness is weaker than the corresponding measure thickness property, as Lemma 3.7 shows.

If X is a proper, LLC, doubling metric measure space supporting a $(1,q)$–Poincaré inequality for some $1 < q < p$ and Ω is an open and bounded subset of X such that $\operatorname{cap}_p(X \setminus \Omega) > 0$ and $X \setminus \Omega$ is uniformly p–fat, then Theorem 1.2 in [6] states that $X \setminus \Omega$ is also uniformly p_0–fat for some $p_0 < p$.

General setup

Throughout this work we assume (X, μ, d) to be a complete metric measure space equipped with a Borel regular measure μ such that the measure of every non–empty open bounded set is positive and finite. We will additionally assume that the measure is *doubling*, i.e. there exists a constant $c_d > 0$ such that for every ball B in X

$$\mu(2B) \leq c_d \mu(B).$$

We refer to this property calling (X, d, μ) or briefly X a doubling metric measure space. We point out that the concept of *doubling metric space* is different from *doubling (metric) measure space.* A doubling metric space has the following property: there exists a fixed number N such that every ball of radius $r > 0$ can be covered by at most N balls with radius $r/2$. This property is weaker than carrying a doubling measure; a space supporting a doubling measure is always doubling as a metric space.

Observe, that a complete doubling metric measure space is always *proper*, that is its closed and bounded subsets are compact. Finally, we will assume that X is a locally linearly convex space (LLC) and carries a weak $(1,p)$–Poincaré inequality for some $1 < p < \infty$.

If not otherwise mentioned, all constants depend only on the constants of the space X, i.e. the doubling constant and the constants of the Poincaré inequality. When a subset Ω of X is considered, we allow dependence on its characteristical constants that are clear in each context. Constants may also depend on the quasiminimalizing constant K.

Chapter 3

Preliminary results

In this chapter we have collected some basic facts concerning properties of capacity, Newtonian spaces and Sobolev–Poincaré type inequalities in the metric setting.

3.1 Results for upper gradients

Let us begin with a convergence result for path integrals. The proof will be omitted, but we may refer for example to [35].

Lemma 3.1. *Let (f_i) be a sequence of non–negative Borel measurable functions such that f_i converges to f in $L^p(X)$. There exists a subsequence (f_{i_k}) such that*

$$\lim_{k\to\infty}\int_\gamma f_{i_k}\,ds = \int_\gamma f\,ds$$

for p–almost all paths γ.

We continue with an upper gradient lemma. Its proof follows the same way as the proof of Lemma 2.4. in [22].

Lemma 3.2. *Suppose that $u, v \in N^{1,p}(X)$ and that η is a Lipschitz continuous function in X with $0 \le \eta \le 1$. Let g_u, g_v and g_η be the minimal p–weak upper gradients of u, v and η, respectively. Define $w = u + \eta(v-u)$. Then*

$$g_w \le (1-\eta)g_u + \eta g_v + |v-u|g_\eta$$

μ–almost everywhere in X.

Proof. Newtonian functions are absolutely continuous on p–almost all paths. Therefore we can take γ to be the arc–length parametrization of a rectifiable path on which u, v and η are absolutely continuous. Define then the function $h : [0, \ell(\gamma)] \to (-\infty, \infty)$ by

$$h(s) = (u \circ \gamma)(s) + (\eta \circ \gamma)(s)\big((v \circ \gamma)(s) - (u \circ \gamma)(s)\big).$$

Now h is absolutely continuous and hence differentiable almost everywhere with respect to the one–dimensional Lebesgue measure. Therefore for almost every $s \in [0, \ell(\gamma)]$, we have

$$\begin{aligned} h'(s) =&(u \circ \gamma)'(s)\big(1 - (\eta \circ \gamma)(s)\big) + (\eta \circ \gamma)'(s)\big((v \circ \gamma)(s) - (u \circ \gamma)(s)\big) \\ &+ (\eta \circ \gamma)(s)(v \circ \gamma)'(s). \end{aligned}$$

Next we show that

$$|(u \circ \gamma)'(s)| \leq g_u(\gamma(s)),$$
$$|(v \circ \gamma)'(s)| \leq g_v(\gamma(s))$$

and

$$|(\eta \circ \gamma)'(s)| \leq g_\eta(\gamma(s))$$

for almost every $s \in [0, \ell(\gamma)]$. Indeed, take for example u. Remember that $(u \circ \gamma)$ is absolutely continuous, so that if $s' \in (0, \ell(\gamma))$ we have

$$\begin{aligned} |(u{\circ}\gamma)'(s)| &= \lim_{s' \to s} \left| \frac{(u(\gamma)(s')) - (u(\gamma)(s))}{s' - s} \right| \\ &\leq \lim_{s' \to s} \frac{1}{|s' - s|} \int_s^{s'} g_u(\gamma(t))dt = g_u(\gamma(s)) \end{aligned}$$

for almost every $s \in [0, \ell(\gamma)]$ by the Lebesgue theorem. This is in fact true for all p–weak upper gradients of u for p–almost all paths and not only for g_u.

With this we obtain

$$\begin{aligned} |h'(s)| \leq& g_u(\gamma(s))\big(1 - \eta(\gamma(s))\big) + g_\eta(\gamma(s))\big|v(\gamma(s)) - u(\gamma(s))\big| \\ &+ \eta(\gamma(s) g_v(\gamma(s)) \end{aligned}$$

for almost every $s \in [0, \ell(\gamma)]$. This proves the claim. □

Lemma 3.3 is from [36] and its proof is from [15].

Lemma 3.3. *Let X be a metric measure space and $1 < p < \infty$. If (u_i) is a sequence of functions in $N^{1,p}(X)$ such that u_i converges weakly to u in L^p and a sequence of upper gradients (g_i) converges weakly to g in*

L^p, then g is a weak upper gradient of u after modifying u on a set of measure zero. Moreover, there are convex combination sequences

$$\tilde{u}_i = \sum_{k=i}^{n_i} \lambda_{ki} u_k$$

and

$$\tilde{g}_i = \sum_{k=i}^{n_i} \lambda_{ki} g_k,$$

with

$$\sum_{k=i}^{n_i} \lambda_{ki} = 1, \qquad \lambda_{ki} > 0,$$

so that $\tilde{u}_i$ converges to u in L^p and $\tilde{g}_i$ converges to g in L^p. Furthermore, $u \in N^{1,p}(X)$.

Proof. Mazur's lemma states that in a normed space for each weakly convergent sequence there exists a sequence of convex combinations that converges in the norm to the same limit. Applying this on (u_i) and (g_i) simultaneously we get a sequence of convex combinations of u_i's and g_i's, denoted $(\tilde{u}_i)$ and $(\tilde{g}_i)$, respectively. We show that $\tilde{g}_i$ is a weak upper gradient of $\tilde{u}_i$. Take x, y in X. Then

$$|\tilde{u}_i(x) - \tilde{u}_i(y)| \leq \sum_{k=i}^{n_i} \lambda_{ki} |u_k(x) - u_k(y)| \leq \sum_{k=i}^{n_i} \lambda_{ki} \int_\gamma g_k ds = \int_\gamma \tilde{g}_i ds$$

for p–almost all paths joining x and y.

To see that g is a weak upper gradient of u we use Lemma 3.1. Now $(\tilde{g}_i)$ is a sequence of non–negative Borel functions that converges in L^p. Then there exists a subsequence, still denoted $(\tilde{g}_i)$, such that

$$\lim_{i\to\infty} \int_\gamma \tilde{g}_i ds = \int_\gamma g ds$$

for p–almost all paths. From a sequence converging in L^p we can extract a subsequence that converges μ–almost everywhere. Hence for μ–almost all x and y in X and p–almost all paths joining them we get

$$|u(x) - u(y)| = \lim_{i\to\infty} |\tilde{u}_i(x) - \tilde{u}_i(y)| \leq \lim_{i\to\infty} \int_\gamma \tilde{g}_i ds = \int_\gamma g ds,$$

as we wished. Unfortunately, we should not be satisfied with this. If we look carefully back, we see that u is a weak limit of L^p–functions and thus defined only outside of a set of measure zero instead of a set of zero p–capacity. However, $\tilde{u}_i$ are defined p–quasieverywhere, and we are able to redefine u on a set of measure zero by setting

$$u(x) = \frac{1}{2}\Big(\limsup_{i\to\infty} \tilde{u}_i(x) - \liminf_{i\to\infty} \tilde{u}_i(x)\Big)$$

whenever it makes sense, i.e. at least one of the limits is different from infinity. Since both limits exist p-quasieverywhere, it can only happen in a set of zero p–capacity. This completes the proof. □

3.2 Sobolev inequalities

We now turn to Sobolev–Poincaré type inequalities in the metric setting. The following proposition is a capacity version of the Sobolev–Poincaré inequality. The proof is a straightforward generalization of the Euclidean case, nevertheless we present it here for the reader's convenience. One can also see [5] for a proof of this Sobolev–Poincaré inequality.

Proposition 3.4. *Let X be a doubling metric measure space admitting a weak $(1,q)$–Poincaré inequality and $u \in N^{1,q}(X)$. Then there exists $c > 0$ such that for all balls B in X and $S = \{x \in \frac{1}{2}B\colon u(x) = 0\}$ the inequality*

$$\left(\fint_B |u|^t \,\mathrm{d}\mu \right)^{1/t} \le \left(\frac{c}{\operatorname{cap}_q(S,B)} \int_{\tau' B} g^q \,\mathrm{d}\mu \right)^{1/q} \tag{3.1}$$

holds for t and τ' as in (2.2).

Proof. If $u_B = 0$ then the assertion follows from the (t,q)–Poincaré inequality (2.2) and (3.2). We may thus assume that $u_B = 1$. Take a Lipschitz cut–off function η such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $\frac{1}{2}B$, $\operatorname{supp}\eta \subset B$ and $g_\eta \le \frac{c}{r}$. Then $\phi = -\eta(u - u_B) \in N_0^{1,p}(B)$ and $\phi = 1$ on S. Therefore

$$\operatorname{cap}_q(S,B) \le \int_B g_\phi^q \,\mathrm{d}\mu \,.$$

Since $g_\phi \le |u - u_B| g_\eta + \eta g_u$, we have

$$\operatorname{cap}_q(S,B) \le \frac{c}{r^q} \int_B |u - u_B|^q \,\mathrm{d}\mu + c \int_B g_u^q \,\mathrm{d}\mu \,.$$

The space X admits the weak (q,q)–Poincaré inequality, so we obtain

$$\operatorname{cap}_q(S,B) \le c \int_{\tau' B} g_u^q \,\mathrm{d}\mu,$$

and therefore

$$1 = u_B \le \left(\frac{c}{\operatorname{cap}_q(S,B)} \int_{\tau' B} g_u^q \,\mathrm{d}\mu \right)^{1/q}.$$

We can now estimate

$$\begin{aligned}\left(\fint_B |u|^t \,\mathrm{d}\mu \right)^{1/t} &\le c\left(\fint_B |u - u_B|^t \,\mathrm{d}\mu \right)^{1/t} + c u_B \\ &\le \left(\frac{c}{\operatorname{cap}_q(S,B)} \int_{\tau' B} g_u^q \,\mathrm{d}\mu \right)^{1/q},\end{aligned}$$

by the (t,q)–Poincaré inequality and (3.2). $\square$

The next lemma is a Sobolev type inequality for Newtonian functions with zero boundary values.

Lemma 3.5. *Let $1 < p < \infty$ and X be a doubling metric measure space supporting a weak $(1,q)$–Poincaré inequality for some $1 < q < p$. Moreover, let $u \in N_0^{1,p}(B)$ and the radius r of B at most* $\operatorname{diam}(X)/3$. *Then*

$$\left(\fint_B |u|^t d\mu \right)^{1/t} \le cr \left(\fint_B g_u^q d\mu \right)^{1/q},$$

where t is defined in (2.2).

Proof. The arguments are as in the proof of Lemma 2.1 in [23]. Let $B = B(x_0, r)$.We start by noting that

$$\left(\fint_{2B} |u|^t d\mu \right)^{1/t} \le \left(\fint_{2B} |u - u_{2B}|^t d\mu \right)^{1/t} + |u_{2B}|,$$

where t is as in (2.2). By the Hölder inequality and the fact that u vanishes on $2B \setminus B$, we have

$$|u_{2B}| \le \fint_{2B} |u| \chi_B \,\mathrm{d}\mu \le \left(\fint_{2B} |u| \,\mathrm{d}\mu \right) \left(\frac{\mu(B)}{\mu(2B)} \right)^{1-1/q}.$$

By assumption $r < \operatorname{diam}(X)/3$, so that the set $X \setminus (3/2)B$ in nonempty. Moreover, the sphere $\partial B(x_0, 3r/2)$ is nonempty. If this was not the case, it would be possible to give an example of a function that violates the Poincaré inequality.Thus we can take a point x' on the sphere and have $B' = B(x', r/2) \subset 2B \setminus B$. In addition $2B \subset 8B'$ and hence,

$$\begin{aligned}\frac{\mu(B)}{\mu(2B)} &= \frac{\mu(2B) - \mu(2B \setminus B)}{\mu(2B)} \le 1 - \frac{\mu(B')}{\mu(2B)} \\ &\le 1 - \frac{\mu(B')}{\mu(8B')} = \gamma < 1.\end{aligned}$$

Collecting the three estimates we get

$$\left(\fint_{2B}|u|^t d\mu\right)^{1/t} \le \left(\fint_{2B}|u-u_{2B}|^t d\mu\right)^{1/t} + \left(\fint_{2B}|u|\,\mathrm{d}\mu\right)\left(\frac{\mu(B)}{\mu(2B)}\right)^{1-1/q}$$

and furthermore

$$(1-\gamma^{1-1/t})\left(\fint_{2B}|u|^t d\mu\right)^{1/t} \le \left(\fint_{2B}|u-u_{2B}|^t d\mu\right)^{1/t}.$$

From this, by the (t,q)–Poincaré inequality, we get

$$\left(\fint_{2B}|u|^t d\mu\right)^{1/t} \le cr\left(\fint_{2\tau' B} g_u^q d\mu\right)^{1/q},$$

since $(1-\gamma^{1-1/t})>0$. When we note that u vanishes outside B, we finally obtain

$$\left(\fint_{B}|u|^t d\mu\right)^{1/t} \le cr\left(\fint_{B} g_u^q d\mu\right)^{1/q},$$

where the doubling property of μ assures that c depends only on the doubling constant and the constant in (2.2). $\square$

3.3 Capacity and fatness

The next lemma provides us with an estimate for the capacity of a ball and shows that capacities cap_p and C_p are equivalent on a bounded set. The reader may also see [5] for the proof.

Lemma 3.6. *Let X be a doubling metric measure space admitting a weak $(1,q)$–Poincaré inequality and let $E \subset B = B(x_0,r)$ with $0 < r <$ $\mathrm{diam}(X)/6$. There exists $c>0$ such that*

$$\frac{\mu(E)}{cr^q} \le \mathrm{cap}_q(E,2B) \le \frac{c\mu(B)}{r^q} \tag{3.2}$$

and

$$\frac{C_q(E)}{c(1+r^q)} \le \mathrm{cap}_q(E,2B) \le 2^{q-1}\left(1+\frac{1}{r^q}\right)C_q(E).$$

Proof. Let u be admissible in the definition of $\mathrm{cap}_q(E,2B)$, i.e. $u \in N_0^{1,p}(2B)$ such that $u \ge 1$ on E. Then

$$\mu(E) = \int_E d\mu \le \int_E |u|^q d\mu \le \int_{2B}|u|^q d\mu \le cr^q\int_{2B} g_u^q d\mu,$$

where the last inequality is due to Lemma 3.5 with $t = q$. However, u is also admissible in the definition of $C_q(E)$, so that

$$C_q(E) \le \int_X |u|^q d\mu + \int_X g_u^q d\mu \le c(1 + r^q) \int_{2B} g_u^q d\mu$$

again by Lemma 3.5 and the fact that u and thus g_u vanish outside $2B$. Taking infimum over all admissible u in these two inequalities implies the result.

Let then u be admissible in $C_q(E)$, and choose a Lipschitz function η such that $0 \le \eta \le 1$, $\eta = 1$ on B, $g_\eta \le 1/r$ and $\eta = 0$ outside $2B$. Then η and $u\eta$ are admissible in $\mathrm{cap}_q(E, 2B)$, and we obtain

$$\mathrm{cap}_q(E, 2B) \le \int_{2B} g_\eta^q d\mu \le \frac{\mu(2B)}{r^q} \le c\frac{\mu(B)}{r^q}$$

by the doubling property. In addition,

$$\begin{aligned}
\mathrm{cap}_q(E, 2B) &\le \int_{2B} g_{u\eta}^q d\mu \le \int_{2B} (\eta g_u + |u| g_\eta)^q d\mu \\
&\le \frac{2^{q-1}}{r^q} \int_{2B} |u|^q d\mu + 2^{q-1} \int_{2B} g_u^q d\mu \\
&\le 2^{q-1}(1 + \frac{1}{r^q})\Big(\int_X |u|^q d\mu + \int_X g_u^q d\mu \Big).
\end{aligned}$$

We take infimum over all admissible u and the proof is complete. □

To finish this section we show that the measure density is a stronger condition than the capacity density. A non–empty set $E \subset X$ is *uniformly μ–thick*, if there are constants $c > 0$ and $0 < r_0 \le 1$ such that

$$\mu(B(x, r) \cap E) \ge c\mu(B(x, r))$$

for every $x \in E$ and $0 < r < r_0$.

Proposition 3.7. *Suppose that (X, d, μ) is a doubling metric measure space that supports a weak $(1, p)$–Poincaré inequality for some $1 < p < \infty$. Let $E \subset X$ be uniformly μ–thick. Then E is uniformly p–fat.*

Proof. Let $E \subset X$ be μ–thick and take an arbitrary $x \in E$. Then

$$\mathrm{cap}_p(E \cap B(x, r); B(x, 2r)) \ge \frac{1}{cr^p}\mu(E \cap B(x, r)) \ge \frac{c}{r^p}\mu(B(x, r))$$

by Lemma 3.6 with $E = E \cap B(x,r)$ and by μ–thickness of E. This holds for $0 < r < r_0$. Using Lemma 3.6 once more, only this time with $E = B(x,r)$ we get

$$\frac{c}{r^p}\mu(B(x,r)) \geq c\,\mathrm{cap}_p(B(x,r); B(x,2r)) \tag{3.3}$$

and the proof is complete. □

3.4 Hardy inequality

Next we present some useful results concerning Newtonian spaces with zero boundary values. Proposition 3.11 provides a characterisation for $N_0^{1,p}$–functions by means of the Hardy inequality. Lemma 3.12 gives a sufficient condition for a sequence of $N_0^{1,p}$–functions to converge to a $N_0^{1,p}$–function. Finally, Lemma 3.13 shows that $N_0^{1,p}$ can be presented as an intersection of $N^{1,p}$ and $N_0^{1,q}$ with $q < p$.

In the proof of Proposition 3.11 we need a Whitney decomposition. For completeness we present it here with a proof.

Proposition 3.8 (Whitney decomposition). *In a doubling metric measure space (X, d, μ) there exist constants $\sigma, M > 0$ such that for each open and bounded $E \subset X$ there exist disjoint balls $\{B_i\}_{i=1}^{\infty}$ such that*

- $E = \bigcup_{j=1}^{\infty} 5B_j$,
- *Each point of E belongs to at most M balls $20B_i$,*
- $\sigma B_j \cap (X \setminus E) \neq \emptyset$ *for all* $i = 1, 2, \ldots$.

Proof. For each $x \in E$ set $r_x = \delta_x/100$, where $\delta_x = \mathrm{dist}(x, X \setminus E)$. Then

$$E = \bigcup_{x \in E} B(x, r_x).$$

Since E is bounded, the Vitali covering theorem implies that there exist disjoint balls $\{B(x_i, r_{x_i})\}_{i=1}^{\infty}$, that we denote B_i, such that

$$E = \bigcup_{i=1}^{\infty} 5B_i.$$

The Vitali covering theorem requires only that (X, d) is doubling and that the radii of the balls are uniformly bounded, see for example [13].

Notice, that for example $101B_i \cap (X \setminus E) \neq \emptyset$ for all $i = 1, 2, \ldots$, so that the first and the last claim hold true.

Denote $r_{x_i} = r_i$ for brevity. Let us assume that $y \in E$ belongs to M balls $20B_i$. We show that this leads to a contradiction. Observe, that it is true in a doubling space (X, d, μ) that there exists a positive number N_λ such that if $r > 0$ and B_i are disjoint balls with radii $r_i \geq r/\lambda$ contained in $B(x, r)$ for some $x \in X$, then the number of the balls is at most N_λ.

We can see that $20B_i \subset B(y, 40r_i)$. On the other hand for all $1 \leq i \leq M$ it is true that

$$\operatorname{dist}(x_i, X \setminus E) \leq \operatorname{dist}(y, X \setminus E) + d(x_i, y) \leq \operatorname{dist}(y, X \setminus E) + 20r_i.$$

This implies

$$\delta_y \geq \delta_{x_i} - 20r_i = 100r_i - 20r_i > r_i,$$

which means that $B(y, 40r_i) \subset B(y, 40\delta_y)$. Thus the balls $\{B_i\}_{i=1}^M$ belong to a fixed ball $B(y, 40\delta_y)$. Remember that they are disjoint. Moreover, for all $1 \leq i \leq M$ we have

$$\delta_y \leq d(y, x_i) + \delta_{x_i} \leq r_i + 100r_i,$$

so that $r_i \geq \delta_y/101$. Now $B(y, 40\delta_y)$ contains M disjoint balls with radius greater than $\delta_y/101$. This leads to contradiction with large values of M since the space is doubling. □

The following theorem provides a sufficient condition for a function to belong to a Newtonian space with zero boundary values. We show that the Euclidean proof in [40] can be adapted to the metric setting. For an alternative proof of see [17].

Theorem 3.9. *Let (X, d, μ) be a doubling metric measure space and assume that X admits a weak $(1, p)$–Poincaré inequality, $1 < p < \infty$. Let $\Omega \subset X$ be open. Then $u \in N^{1,p}(\Omega)$ is in $N_0^{1,p}(\Omega)$ if*

$$\frac{|u(x)|}{\operatorname{dist}(x, X \setminus \Omega)}$$

is in $L^p(\Omega)$.

Proof. Let us consider positive and negative parts of u separately. Set $u^+ = \max(u, 0)$ and set $u_i(x) = \min(u^+(x), i\delta(x))$, $i = 1, 2, \ldots$, where

$$\delta(x) = \begin{cases} \operatorname{dist}(x, X \setminus \Omega), & x \in \Omega, \\ 0, & x \in X \setminus \Omega. \end{cases}$$

Now $u^+ \in N^{1,p}(\Omega)$ and $i\delta$ is an i–Lipschitz function that is zero on the boundary of Ω. Furthermore, δ is zero outside of Ω, so that $i\delta \in N_0^{1,p}(\Omega)$. Thus $u_i \in N_0^{1,p}(\Omega)$.

Let $\widetilde{u_i}$ be the zero extension of u_i to $N^{1,p}(X)$, i.e.

$$\widetilde{u_i}(x) = \begin{cases} u_i(x), & x \in \Omega, \\ 0, & x \in X \setminus \Omega. \end{cases}$$

We show that $(\widetilde{u_i})$ is a bounded sequence in $L^p(X)$. Indeed,

$$\int_X |\widetilde{u_i}|^p d\mu = \int_\Omega |u_i|^p d\mu \leq \int_\Omega |u^+|^p d\mu \leq \int_\Omega |u|^p d\mu \leq c,$$

where c does not depend on i. Since $L^p(X)$ is reflexive, there is $v \in L^p(X)$ and a subsequence, still denoted $(\widetilde{u_i})$, such that

$$\widetilde{u_i} \rightharpoonup v, \qquad \text{when } i \to \infty$$

in $L^p(X)$. Observe, that by a remark in Section 2.2,

$$g_{u^+}\chi_{\{u^+ < i\delta\}} + g_{i\delta}\chi_{\{u^+ \geq i\delta\}}$$

is an upper gradient of u_i. Furthermore,

$$\begin{aligned} \int_X g_{\widetilde{u_i}}^p d\mu = \int_\Omega g_{\widetilde{u_i}}^p d\mu &= \int_\Omega g_{u_i}^p d\mu \\ &\leq \int_{\{x\in\Omega:\ u^+(x)<i\delta(x)\}} g_{u^+}^p d\mu + \int_{\{x\in\Omega:\ u^+(x)\geq i\delta(x)\}} g_{i\delta}^p d\mu \\ &\leq \int_\Omega g_{u^+}^p d\mu + i^p \int_{\{x\in\Omega:\ u^+(x)\geq i\delta(x)\}} d\mu \\ &\leq \int_\Omega g_u^p d\mu + i^p \int_\Omega \left(\frac{|u(x)|}{i\delta(x)}\right)^p d\mu \\ &= \int_\Omega g_u^p d\mu + \int_\Omega \left(\frac{|u(x)|}{\delta(x)}\right)^p d\mu \leq c, \end{aligned} \tag{3.4}$$

since u is in $N^{1,p}(\Omega)$ and $|u|/\delta$ is in $L^p(\Omega)$. Notice, that for a Lipschitz function its Lipschitz constant is always an upper gradient.

Now $(g_{\widetilde{u_i}})$ is bounded in $L^p(X)$. Consequently, there is $g \in L^p(X)$ and a subsequence of the subsequence $(\widetilde{u_i})$ such that

$$g_{\widetilde{u_i}} \rightharpoonup g, \qquad \text{when } i \to \infty$$

in $L^p(X)$. By Lemma 3.3 $v \in N^{1,p}(X)$.

On the other hand, for all $x \in X$, we have $\widetilde{u_i}(x) \to \widetilde{u^+}(x)$, where

$$\widetilde{u^+}(x) = \begin{cases} u^+(x), & x \in \Omega \\ 0, & x \in X \setminus \Omega. \end{cases}$$

This implies that $\widetilde{u^+}$ agrees with v μ–almost everywhere, and since $\widetilde{u^+} \in N_0^{1,p}(\Omega)$, $v \in N_0^{1,p}(\Omega)$ as well.

We point out, that we have to know a priori, that both limit functions are Newtonian functions before we are able to conclude that they belong to the same equivalence class in $N^{1,p}$. If we did not know this, for example if only one of them was Newtonian, we would need to show that they agree p–quasieverywhere, see Section 2.4.

The proof follows when we apply the above reasoning to u^-. $\square$

The arguments in the above proof imply a stronger result, namely that the theorem remains valid if $|u|/\delta$ belongs to the weak L^p instead of L^p. This can be see in (3.4).

Let $f\colon X \to \mathbb{R}$ be a locally integrable function. We define the Hardy–Littlewood maximal function as

$$Mf(x) = \sup_{B \ni x} \fint_B |f| d\mu,$$

where the supremum is taken over all balls containing $x \in X$. The following classical Hardy–Littlewood theorem holds also in the metric setting. See [37] for the Euclidean proof that can be generalised to the metric case.

Theorem 3.10. *Let $f \in L^p(X)$ and $1 < p < \infty$. Then there exists a constant c depending only on the doubling constant and p such that*

$$\int_X |Mf|^p d\mu \le c \int_X |f|^p d\mu.$$

The idea for a proof of the following proposition is taken from [6] and [34]. The proposition is from [6].

Proposition 3.11. *Let X be a proper, doubling, LLC metric measure space supporting a weak $(1,q)$–Poincaré inequality for some $1 < q < p$ and suppose that Ω is a bounded domain in X such that $X \setminus \Omega$ is uniformly p–fat. Then there is a constant $c(\Omega, p) > 0$ such that a function $u \in N^{1,p}(X)$ is in $N_0^{1,p}(\Omega)$ if and only if*

$$\int_\Omega \left(\frac{|u(x)|}{\operatorname{dist}(x, X \setminus \Omega)} \right)^p \mathrm{d}\mu \le c \int_\Omega g_u(x)^p \, \mathrm{d}\mu\,. \tag{3.5}$$

Notice, that the constant c in the above proposition formally depends on p. However, if p varies inside a bounded interval, the arguments in the proof of Proposition 3.11 show that the corresponding constants are uniformly bounded. For this reason, since in our case all exponents vary inside a bounded interval (q, q^*) we omit the dependence of the constant on p.

Proof of the proposition 3.11. Suppose first that (3.5) holds. Then by Theorem 3.9 u is in $N_0^{1,p}(\Omega)$ and this concludes the first part of the proof. Observe, that we did not need the fact that $X \setminus \Omega$ is p–fat, nor that the space is LLC.

As mentioned in Section 2.9, if $X \setminus \Omega$ is uniformly p–fat, then $X \setminus \Omega$ is also uniformly p_0–fat for some $p_0 < p$. If $p_0 < q$, we can choose a larger p_0 order to have $q = p_0$ and conclude that X admits the $(1, p_0)$–Poincaré inequality. If $p_0 \geq q$ then the $(1, p_0)$–Poincaré inequality follows from the Hölder inequality.

Let then $u \in N_0^{1,p}(\Omega)$ and $\{B_j\}_{j=1}^{\infty}$ be the Whitney decomposition of Ω. To be more specific,

$$\begin{aligned} &B_i \cap B_j = \emptyset, \qquad i \neq j, \\ &\Omega = \bigcup_{j=1}^{\infty} 5B_j \end{aligned}$$

and

$$\sigma B_j \cap (X \setminus \Omega) \neq \emptyset$$

for all $i = 1, 2, \ldots$ for some positive real number σ that can be chosen to be greater than 5. Set $r_j = \operatorname{dist}(B_j, \partial\Omega) > 0$ and $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. Then $\delta(x) \geq r_j$ for all $x \in B_j$. Now

$$\begin{aligned} \int_{\Omega} \left|\frac{u}{\delta}\right|^p d\mu &\leq \sum_{j=1}^{\infty} r_j^{-p} \int_{5B_j} |u|^p d\mu \leq \sum_{j=1}^{\infty} r_j^{-p} \int_{\sigma B_j} |u|^p d\mu \\ &\leq \sum_{j=1}^{\infty} r_j^{-p} \mu(\sigma B_j) \fint_{\sigma B_j} |u|^p d\mu \\ &\leq c \sum_{j=1}^{\infty} r_j^{-p} \mu(B_j) \left(\fint_{\sigma B_j} |u|^t d\mu \right)^{p/t}, \\ &\leq c \sum_{j=1}^{\infty} r_j^{-p} \mu(B_j) \left(\frac{c}{\operatorname{cap}_{p_0}(S_j, 2\sigma B_j)} \int_{2\tau'\sigma B_j} g_u^{p_0} d\mu \right)^{p/p_0}, \end{aligned} \tag{3.6}$$

by the Hölder inequality and Proposition 3.4 with t chosen so that $t/p > 1$. Here $S_j = \{x \in \sigma B_j \colon u(x) = 0\}$, for $j = 1, 2, \ldots$. We proceed by estimating the capacity of S_j. To this end, set

$$V_j = \{x \in \sigma B_j \setminus \Omega \colon u(x) \neq 0\},$$

for $j = 1, 2, \ldots$. Since Ω is bounded, u is also in $N_0^{1,p_0}(\Omega)$ and thus $C_{p_0}(V_j) = 0$ for all j. Lemma 3.2 implies that $\operatorname{cap}_{p_0}(V_j, 2\sigma B_j) = 0$ as well. Furthermore,

$$S_j \setminus \Omega = (\sigma B_j \setminus \Omega) \setminus V_j$$

and hence

$$\begin{aligned}\operatorname{cap}_{p_0}(S_j, 2\sigma B_j) &\geq \operatorname{cap}_{p_0}(S_j \setminus \Omega, 2\sigma B_j) = \operatorname{cap}_{p_0}\big((\sigma B_j \setminus \Omega) \setminus V_j, 2\sigma B_j\big)\\ &= \operatorname{cap}_{p_0}(\sigma B_j \setminus \Omega, 2\sigma B_j) \geq c\, \operatorname{cap}_{p_0}(\sigma B_j, 2\sigma B_j),\end{aligned}$$

where we have used the p_0–fatness property of $X \setminus \Omega$. Notice that we need $\operatorname{diam}(\sigma B_j)$ to be less than r_0, the radius in the definition q–fatness (see Section 2.9). Now lemma 3.2 gives

$$\operatorname{cap}_{p_0}(S_j, 2\sigma B_j) \geq c r^{-p_0} \mu(\sigma B_j)$$

and inserting this in (3.6) we get

$$\begin{aligned}\int_\Omega \Big|\frac{u}{\delta}\Big|^p d\mu &\leq c \sum_{j=1}^\infty r_j^{-p} \mu(B_j) \Big(\frac{c}{r^{-p_0}\mu(\sigma B_j)} \int_{2\tau'\sigma B_j} g_u^{p_0} d\mu\Big)^{p/p_0}\\ &= c \sum_{j=1}^\infty \mu(B_j) \Big(\frac{1}{\mu(\sigma B_j)} \int_{2\tau'\sigma B_j} g_u^{p_0} d\mu\Big)^{p/p_0}\\ &= c \sum_{j=1}^\infty \mu(B_j) \Big(\frac{\mu(2\tau\sigma B_j)}{\mu(\sigma B_j)} \fint_{2\tau'\sigma B_j} g_u^{p_0} d\mu\Big)^{p/p_0}\\ &\leq c \sum_{j=1}^\infty \mu(B_j) \Big(\fint_{2\tau'\sigma B_j} g_u^{p_0} d\mu\Big)^{p/p_0}.\end{aligned}$$

In the last step we used the doubling property. Let M denote the Hardy–Littlewood maximal function on X. Then

$$\Big(\fint_{2\tau'\sigma B_j} g_u^{p_0} d\mu\Big)^{p/p_0} \leq \big(M(g_u^{p_0})\big)^{p/p_0}$$

and

$$\begin{aligned}\int_\Omega \Big|\frac{u}{\delta}\Big|^p d\mu &\leq c \sum_{j=1}^\infty \mu(B_j)\big(M(g_u^{p_0})\big)^{p/p_0} \leq c \sum_{j=1}^\infty \int_{B_j} \big(M(g_u^{p_0})\big)^{p/p_0} d\mu\\ &\leq c \int_X \big(M(g_u^{p_0})\big)^{p/p_0} d\mu \leq c \int_X (g_u^{p_0})^{p/p_0} d\mu\end{aligned}$$

by Theorem 3.10. Since the minimal upper gradient of u is zero μ–almost everywhere u is zero, we get (3.5). $\square$

Lemma 3.12 is formulated in [17] for (X, d, μ) with μ Borel regular outer measure and for Ω open such that $X \setminus \Omega$ satisfies a measure thickness assumption. In general a measure thickness condition is stronger than a fatness assumption. However, the lemma follows also from Theorem 3.9 and Proposition 3.11. See [6] and [17].

Lemma 3.12. *In the setting of Proposition 3.11, let* $(u_i)_{i=1}^{\infty}$ *be a bounded sequence in* $N_0^{1,p}(\Omega)$. *If* $u_i \to u$ μ*–a.e., then* $u \in N_0^{1,p}(\Omega)$.

Proof. It follows from Proposition 3.11 that

$$\int_\Omega \left(\frac{|u_i(x)|}{\operatorname{dist}(x, X \setminus \Omega)} \right)^p \mathrm{d}\mu \leq c(p) \int_\Omega g_{u_i}(x)^p \, \mathrm{d}\mu \leq c, \tag{3.7}$$

since (u_i) is a bounded sequence in $N_0^{1,p}(\Omega)$. On the other hand $u_i \to u$ almost everywhere, so that

$$\int_\Omega \left(\frac{|u(x)|}{\operatorname{dist}(x, X \setminus \Omega)} \right)^p \mathrm{d}\mu \leq \liminf_{i \to \infty} \int_\Omega \left(\frac{|u_i(x)|}{\operatorname{dist}(x, X \setminus \Omega)} \right)^p \mathrm{d}\mu \leq c.$$

Theorem 3.9 finishes the proof. $\square$

We point out that the assertion of the next proposition is not trivial. Even in $\mathbb{R}^n$ some type of thickness assumption on the complement of the domain is needed, see [12]. Li and Martio show in [27] that for example p–fatness of $\mathbb{R}^n \setminus \Omega$ suffices to (3.8) to hold. The same result holds in the metric case and the proof follows from Proposition 3.11.

Proposition 3.13. *Let X be a proper, doubling, LLC metric measure space supporting a weak $(1, q)$–Poincaré inequality for some $1 < q < p$ and suppose that Ω is a bounded domain in X such that $X \setminus \Omega$ is uniformly p–fat. Then*

$$N_0^{1,p}(\Omega) = N^{1,p}(\Omega) \cap \bigcap_{s<p} N_0^{1,s}(\Omega). \tag{3.8}$$

Proof. The inclusion "$\supset$" in (3.8) in clear as Ω is bounded. It remains to prove the case "$\subset$".

Since $X \setminus \Omega$ is uniformly p–fat, it is also $(p-\varepsilon)$–fat for all $\varepsilon > 0$ small enough as discussed in Section 2.9. Consequently,

$$\int_\Omega \left(\frac{|u|}{\operatorname{dist}(z, X \setminus \Omega)} \right)^{p-\varepsilon} \mathrm{d}\mu \le c \int_\Omega g_u^{p-\varepsilon} \, \mathrm{d}\mu \tag{3.9}$$

for all $\varepsilon > 0$ small enough, by Proposition 3.11. We show now that (3.5) holds also for p. Indeed,

$$\begin{aligned} \int_\Omega \left(\frac{|u|}{\operatorname{dist}(z, X \setminus \Omega)} \right)^{p} \mathrm{d}\mu &= \lim_{\varepsilon \to 0} \int_\Omega \left(\frac{|u|}{\operatorname{dist}(z, X \setminus \Omega)} \right)^{p-\varepsilon} \mathrm{d}\mu \\ &\le \lim_{\varepsilon \to 0} c \int_\Omega g_u^{p-\varepsilon} \, \mathrm{d}\mu = c \int_\Omega g_u^{p} \, \mathrm{d}\mu, \end{aligned}$$

and the assertion follows by Proposition 3.11. $\square$

Chapter 4

Higher integrability of quasiminimizers

After all the work we have done in Chapters 2 and 3 we are now ready to prove the higher integrability result that plays an important role in the proof of our main theorem 5.1.

Theorem 4.1. *Let Ω be an open and bounded subset of X such that $X \setminus \Omega$ is of positive p–capacity and uniformly p–fat. Let $w \in N^{1,s}(\Omega)$ for some $s > p$.*

If $u \in N^{1,p}(\Omega)$ is a quasiminimizer of the p–energy integral in Ω with boundary data w, then there exists $0 < \delta_0 = \delta_0(p) \leq s - p$ such that $g_u \in L^{p+\delta}(\Omega)$ for all $0 < \delta < \delta_0$ and

$$\left(\int_\Omega g_u^{p+\delta} \, \mathrm{d}\mu \right)^{1/(p+\delta)} \leq c \left[\left(\int_\Omega g_u^p \, \mathrm{d}\mu \right)^{1/p} + \left(\int_\Omega g_w^{p+\delta} \, \mathrm{d}\mu \right)^{1/(p+\delta)} \right],$$

where c depends only on p and on the constants related to the space and to the domain Ω.

The local regularity of quasiminimizers, i.e. Hölder continuity, was studied by Kinnunen and Shanmugalingam in [23]. In particular they proved that a quasiminimizer belongs to the so called De Giorgi class. The following Caccioppoli type inequality is a corollary of this result, but for the completeness we present here a proof. We follow the reasoning of Kinnunen and Shanmugalingam.

Theorem 4.2 (Caccioppoli inequality). *Let Ω be an open subset of X. If $u \in N^{1,p}(\Omega)$ is a quasiminimizer of the p–energy integral in Ω,* *there exists $c > 0$ such that*

$$\int_{B(x,r)} g_u^p \,\mathrm{d}\mu \le \frac{c}{(R-r)^p} \int_{B(x,R)} |u - u_{B(x,R)}|^p \,\mathrm{d}\mu \tag{4.1}$$

holds for all $x \in \Omega$ and $0 < r < R$ so that $B(x,R) \subset\subset \Omega$.

Proof. Denote $u_{B(x,R)} = u_B$ for short. Let η be a Lipschitz function such that $0 \le \eta \le 1$, $\eta = 1$ on $B(x,r)$ and η vanishes outside $B(x,R)$. In addition, suppose that $g_\eta \le c/(R-r)$. Set $v = u - \eta(u - u_B)$, in which case $u - v \in N_0^{1,p}(B(x,R))$. Since u is a quasiminimizer, we have

$$\int_{B(x,r)} g_u^p \,\mathrm{d}\mu \le \int_{B(x,R)} g_u^p \,\mathrm{d}\mu \le K \int_{B(x,R)} g_v^p \,\mathrm{d}\mu\,.$$

Writing $v = (1-\eta)(u - u_B) + u_B$, we get by Lemma 3.2

$$g_v \le (1-\eta) g_{u-u_B} + |u - u_B| g_{(1-\eta)} = (1-\eta) g_u + |u - u_B| g_\eta$$

on Ω. Hence,

$$\begin{aligned}
\int_{B(x,r)} g_u^p \,\mathrm{d}\mu &\le K \int_{B(x,R)} \big[(1-\eta) g_u + |u - u_B| g_\eta\big]^p \,\mathrm{d}\mu \\
&\le c \int_{B(x,R)} (1-\eta)^p g_u^p \,\mathrm{d}\mu + \frac{c}{(R-r)^p} \int_{B(x,R)} |u - u_B|^p \,\mathrm{d}\mu \qquad (4.2) \\
&\le c \int_{B(x,R)\setminus B(x,r)} g_u^p \,\mathrm{d}\mu + \frac{c}{(R-r)^p} \int_{B(x,R)} |u - u_B|^p \,\mathrm{d}\mu,
\end{aligned}$$

because $\eta = 1$ on $B(x,r)$. Now add $c\int_{B(x,r)} g_u^p \,\mathrm{d}\mu$ to both sides of (4.2), divide by c and obtain

$$\int_{B(x,r)} g_u^p \,\mathrm{d}\mu \;\le\; \theta \int_{B(x,r)} g_u^p \,\mathrm{d}\mu + \frac{c}{(R-r)^p} \int_{B(x,R)} |u - u_B|^p \,\mathrm{d}\mu, \tag{4.3}$$

where $\theta = c/(1+c) < 1$. The following standard iteration lemma finishes the proof. $\square$

Lemma 4.3. *Let $f : [r_1, r_2] \subset \mathbb{R} \to [0,\infty)$ be a bounded non–negative function. Suppose that for all ρ, r such that $r_1 \le \rho < r \le r_2$*

$$f(\rho) \le \theta f(r) + \big(A(r-\rho)^{-\alpha} + B(r-\rho)^{-\beta} + C\big) \tag{4.4}$$

holds true for some constants $A, B, C \ge 0$, $\alpha > \beta > 0$ and $0 \le \theta < 1$. Then

$$f(r_1) \le c(\alpha,\theta)\big(A(r_2-r_1)^{-\alpha} + B(r_2-r_1)^{-\beta} + C\big). \tag{4.5}$$

For a proof of Lemma 4.3 we refer to [8].

We prove the global higher intergrability of upper gradients of quasiminimizers. The proof follows similar way as the Euclidean proof of Kilpeläinen and Koskela in [18] for solutions of p–harmonic type equations. The growth of integrability is achieved in a standard way by application of the Gehring lemma (its proof in the metric setting may be found for example in [31] or [39]). Notice that the lemma holds in all doubling metric measure spaces.

Theorem 4.4 (Gehring lemma). *Let $s \in [s_0, s_1]$, where $s_0, s_1 > 1$ are fixed. Let $g \in L^s_{loc}(X)$ and $f \in L^{s_1}_{loc}(X)$ be non–negative functions. Assume that there exists constant $b > 1$ such that for every ball $B \subset \sigma B \subset X$ the following inequality*

$$\fint_B g^s \,\mathrm{d}\mu \le b\left[\left(\fint_{\sigma B} g \,\mathrm{d}\mu\right)^s + \fint_{\sigma B} f^s \,\mathrm{d}\mu\right]$$

holds for some $\sigma > 1$. Then there exists $\varepsilon_0 = \varepsilon_0(s_0, s_1, c_d, \sigma, b) > 0$ such that $g \in L^{\tilde{s}}_{loc}(X, \mu)$ for $\tilde{s} \in [s, s + \varepsilon_0)$ and moreover

$$\left(\fint_B g^{\tilde{s}} \,\mathrm{d}\mu\right)^{1/\tilde{s}} \le c\left[\left(\fint_{\sigma B} g^s \,\mathrm{d}\mu\right)^{1/s} + \left(\fint_{\sigma B} f^{\tilde{s}} \,\mathrm{d}\mu\right)^{1/\tilde{s}}\right]$$

for $c = c(s_0, s_1, c_d, \sigma, b)$.

Proof of Theorem 4.1. Recall that X is a locally linearly convex space that supports a weak $(1, q)$–Poincaré inequality for some $1 < q < p$. Since $p \in (q, q^*)$, the space supports also a weak (p, q)–Poincaré inequality (see 2.3). Remember also that $X \setminus \Omega$ is uniformly p–fat.

We start by adjusting the fatness and the Poincaré inequality exponents o that they are compatible, as we did in the proof of Proposition 3.11. We know that $X \setminus \Omega$ is also uniformly p_0–fat for some $p_0 < p$ by a remark in Section 2.9. If $p_0 < q$, we can increase it in order to have $q = p_0$ and to be able to use the (p, p_0)–Poincaré inequality. If $p_0 \ge q$ then the (p, p_0)–Poincaré inequality follows from the Hölder inequality.

Choose a ball B_0 in X such that $\Omega \subset\subset B_0 \subset 2B_0$. Fix $r > 0$ and let $B = B(x_0, r)$ be a ball such that $4\lambda B \subset 2B_0$, where λ is the multiplicative coefficient of radius in the (p, p_0)–Poincaré inequality.

If $2\lambda B \subset \Omega$ then by the Caccioppoli estimate (4.1), doubling condition and (p,p_0)–Poincaré inequality we have

$$\begin{aligned}\left(\fint_B g_u^p \,\mathrm{d}\mu\right)^{1/p} &\le \frac{c}{r}\left(\fint_{2B} |u-u_{2B}|^p \,\mathrm{d}\mu\right)^{1/p} \\ &\le c\left(\fint_{2\lambda B} g_u^{p_0} \,\mathrm{d}\mu\right)^{1/p_0}.\end{aligned} \tag{4.6}$$

Assume thus that $2\lambda B \setminus \Omega \neq \emptyset$. Choose a Lipschitz cut–off function η such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B, $\operatorname{supp}\eta \subset 2B$ and $g_\eta \le \frac{c}{r}$. Then $\eta(u-w) \in N_0^{1,p}(2B\cap\Omega)$ and we may use it as a test function in (2.3). Hence

$$\int_{2B\cap\Omega} g_u^p \,\mathrm{d}\mu \le K \int_{2B\cap\Omega} g_v^p \,\mathrm{d}\mu,$$

where $v = u - \eta(u-w)$ and $g_v \le |u-w| g_\eta + (1-\eta)(g_u+g_w) + g_w$. Therefore we obtain

$$\begin{aligned}\int_{B\cap\Omega} g_u^p \,\mathrm{d}\mu \le c \int_{(2B\setminus B)\cap\Omega} &(1-\eta)^p g_u^p \,\mathrm{d}\mu \\ &+ c\int_{2B\cap\Omega} |u-w|^p g_\eta^p \,\mathrm{d}\mu + c\int_{2B\cap\Omega} (2-\eta)^p g_w^p \,\mathrm{d}\mu\,.\end{aligned}$$

Adding $c\int_{B\cap\Omega} g_u^p \,\mathrm{d}\mu$ to both sides of the inequality and dividing by $(1+c)$ implies

$$\int_{B\cap\Omega} g_u^p \,\mathrm{d}\mu \;\le\; \theta \int_{2B\cap\Omega} g_u^p \,\mathrm{d}\mu + \frac{\theta}{r^p}\int_{2B\cap\Omega} |u-w|^p \,\mathrm{d}\mu + \theta\int_{2B\cap\Omega} g_w^p \,\mathrm{d}\mu,$$

where $\theta = c/(1+c) < 1$. As in the proof of the Caccioppoli inequality, applying Lemma 4.3 we obtain

$$\int_{B\cap\Omega} g_u^p \,\mathrm{d}\mu \le \frac{c}{r^p}\int_{2B\cap\Omega} |u-w|^p \,\mathrm{d}\mu + c\int_{2B\cap\Omega} g_w^p \,\mathrm{d}\mu\,. \tag{4.7}$$

We will consider the integrals on the right–hand side on the larger ball $4B$. We estimate the first integral on the right–hand side using Proposition 3.4 with $q = p_0$. This gives

$$\begin{aligned}\left(\frac{c}{r^p}\fint_{4B} |u-w|^p \,\mathrm{d}\mu\right)^{1/p} &\le \frac{c}{r}\left(\frac{1}{\operatorname{cap}_{p_0}(S,4B)}\int_{4\lambda B} g_{u-w}^{p_0} \,\mathrm{d}\mu\right)^{1/p_0} \\ &\le c\left(\frac{\mu(2B) r^{-p_0}}{\operatorname{cap}_{p_0}(S,4B)}\fint_{4\lambda B} g_{u-w}^{p_0} \,\mathrm{d}\mu\right)^{1/p_0}\end{aligned}$$

by the doubling condition. Here $S = \{x \in 2B\colon u(x) = w(x)\}$. Since $u = w$ p–q.e. and thus p_0–q.e. in $X \setminus \Omega$ and the set $X \setminus \Omega$ is uniformly p_0–fat, we have

$$\operatorname{cap}_{p_0}(S, 4B) \geq \operatorname{cap}_{p_0}(2B \setminus \Omega; 4B) \geq c_f \operatorname{cap}_{p_0}(2B; 4B) \geq c\mu(2B) r^{-p_0}$$

(a similar reasoning is in the proof of Theorem 3.11.) Hence,

$$\begin{aligned}\Big(\frac{c}{r^p} \fint_{4B} |u-w|^p \,\mathrm{d}\mu\Big)^{1/p} &\leq c\Big(\fint_{4\lambda B} g_{u-w}^{p_0} \,\mathrm{d}\mu\Big)^{1/p_0}\\ &= c\Big(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_{u-w}^{p_0} \,\mathrm{d}\mu\Big)^{1/p_0},\end{aligned}$$

because $u - w = 0$ p–q.e. and thus μ–a.e. in $X \setminus \Omega$ and therefore $g_{u-w} = 0$ μ–a.e. in $X \setminus \Omega$. A simple estimation gives now

$$\begin{aligned}\Big(\frac{c}{r^p} \fint_{4B} |u-w|^p \,\mathrm{d}\mu\Big)^{1/p} &\leq c\Big(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_u^{p_0} \,\mathrm{d}\mu\Big)^{1/p_0}\\ &\quad + c\Big(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_w^{p_0} \,\mathrm{d}\mu\Big)^{1/p_0}. \qquad (4.8)\end{aligned}$$

By the Hölder inequality

$$\begin{aligned}\Big(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_w^{p_0} \,\mathrm{d}\mu\Big)^{1/p_0} &= \Big(\fint_{4\lambda B} g_w^{p_0} \chi_{4\lambda B \cap \Omega} \,\mathrm{d}\mu\Big)^{1/p_0}\\ \leq \Big(\fint_{4\lambda B} g_w^{p} \chi_{4\lambda B \cap \Omega} \,\mathrm{d}\mu\Big)^{1/p} &= \Big(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_w^{p} \,\mathrm{d}\mu\Big)^{1/p}, \qquad (4.9)\end{aligned}$$

so that combining (4.7), (4.9), (4.8) and using the doubling property we obtain the inequality

$$\begin{aligned}\Big(\frac{1}{\mu(B)} \int_{B \cap \Omega} g_u^{p} \,\mathrm{d}\mu\Big)^{1/p} &\leq b\Big(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_u^{p_0} \,\mathrm{d}\mu\Big)^{1/p_0}\\ &\quad + c\Big(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_w^{p} \,\mathrm{d}\mu\Big)^{1/p}. \qquad (4.10)\end{aligned}$$

Here the constants b and c depend only on p, Ω and on the constants associated with the structure of the space.

Set now

$$g(x) = \begin{cases} g_u^{p_0} & \text{if } \quad x \in \Omega,\\ 0 & \text{otherwise,}\end{cases}$$

$$f(x)=\begin{cases} g_w^{p_0} & \text{if} \quad x\in\Omega,\\ 0 & \text{otherwise}\end{cases}$$

and $s = p/p_0$. The inequalities (4.6) and (4.10) imply that whenever $4\lambda B\subset 2B_0$, the following reverse Hölder inequality

$$\fint_B g^s\,\mathrm{d}\mu \le b\Big(\fint_{4\lambda B} g\,\mathrm{d}\mu\Big)^s + c\fint_{4\lambda B} f^s\,\mathrm{d}\mu$$

holds for $s>1$ (p is strictly greater than p_0) and with $b=b(p)$. Applying now the Gehring lemma we obtain better integrability of g and the inequality

$$\Big(\fint_B g^{\tilde s}\,\mathrm{d}\mu\Big)^{1/\tilde s} \le c\Big[\Big(\fint_{4\lambda B} g^s\,\mathrm{d}\mu\Big)^{1/s} + \Big(\fint_{4\lambda B} f^{\tilde s}\,\mathrm{d}\mu\Big)^{1/\tilde s}\Big] \tag{4.11}$$

with $c=c(b,c_d,\lambda)$ and $\tilde s\in[s,s+\varepsilon_0)$, where $\varepsilon_0=\varepsilon_0(b,c_d,\lambda)$.

Since the diameter of Ω is finite we may choose a finite number of balls B_j, $j=1,2,\dots,N$, such that

$$2\lambda B_j\subset B_0 \quad \text{and} \quad \Omega\subset\bigcup_{j=1}^N B_j$$

with fixed λ. A multiplication of (4.11) by $\mu(B)^{1/\tilde s}$ and raising it to the power $\tilde s$ gives us

$$\int_B g^{\tilde s}\,\mathrm{d}\mu \le c\Big(\frac{\mu(B)^{s/\tilde s}}{\mu(4\lambda B)}\int_{4\lambda B} g^s\,\mathrm{d}\mu\Big)^{\tilde s/s} + c\frac{\mu(B)}{\mu(4\lambda B)}\int_{4\lambda B} f^{\tilde s}\,\mathrm{d}\mu\,. \tag{4.12}$$

We proceed by summing (4.12) over j and obtain

$$\begin{aligned}
\sum_{j=1}^N\int_{B_j} g^{\tilde s}\,\mathrm{d}\mu &\le c\sum_{j=1}^N\Big(\frac{\mu(B_j)^{s/\tilde s}}{\mu(4\lambda B_j)}\int_{4\lambda B_j} g^s\,\mathrm{d}\mu\Big)^{\tilde s/s} + c\sum_{j=1}^N\int_{4\lambda B_j} f^{\tilde s}\,\mathrm{d}\mu\\
&\le c\max_{1\le j\le N}\frac{\mu(B_j)}{\mu(4\lambda B_j)^{\tilde s/s}}\sum_{j=1}^N\Big(\int_{4\lambda B_j} g^s\,\mathrm{d}\mu\Big)^{\tilde s/s} + c\sum_{j=1}^N\int_{4\lambda B_j} f^{\tilde s}\,\mathrm{d}\mu\\
&\le c\sum_{j=1}^N\Big(\int_{3B_0} g^s\,\mathrm{d}\mu\Big)^{\tilde s/s} + c\sum_{j=1}^N\int_{3B_0} f^{\tilde s}\,\mathrm{d}\mu\\
&\le c\Big(\int_{3B_0} g^s\,\mathrm{d}\mu\Big)^{\tilde s/s} + c\int_{3B_0} f^{\tilde s}\,\mathrm{d}\mu,
\end{aligned}$$

that implies

$$\left(\int_\Omega g^{\tilde{s}}\,\mathrm{d}\mu\right)^{1/\tilde{s}} \le c\left(\int_{3B_0} g^s\,\mathrm{d}\mu\right)^{1/s} + c\left(\int_{3B_0} f^{\tilde{s}}\,\mathrm{d}\mu\right)^{1/\tilde{s}}. \tag{4.13}$$

If we now substitute g and f by g_u and g_w in (4.13), respectively, and set $\tilde{p} = \tilde{s}p_0$ we get

$$\left(\int_\Omega g_u^{\tilde{p}}\,\mathrm{d}\mu\right)^{1/\tilde{s}} \le c\left(\int_\Omega g_u^{p}\,\mathrm{d}\mu\right)^{1/s} + c\left(\int_\Omega g_w^{\tilde{p}}\,\mathrm{d}\mu\right)^{1/\tilde{s}}. \tag{4.14}$$

Finally, we raise (4.14) to the power $1/p_0$ and this finishes the proof. □

Chapter 5

Stability of quasiminimizers

We now turn to the main theorem of the thesis.

Theorem 5.1. *Let Ω be an open and bounded subset of X such that $X \setminus \Omega$ is of positive p–capacity and uniformly p–fat. Let $w \in N^{1,s}(\Omega)$ for some $s > p$. Assume $p = \lim_{i\to\infty} p_i$ and let $(u_i)_{i=1}^{\infty}$ be a sequence of K–quasiminimizers of the p_i–energy in Ω with boundary data w. If*

$$u_i \to u \quad \mu\text{–a.e. in } \Omega$$

then u is a K–quasiminimizer of the p–energy integral in Ω with boundary data w.

The proof of the main theorem is considerably long, so we have devided it into three parts. Before the last part, that is the actual proof, we present two lemmas 5.3 and 5.4. The former makes use of the metric version Rellich-Kondrachov theorem, see [11] or [25]. Recall that the lower decay of order of the measure is defined as

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \geq c_d \left(\frac{r}{r_0}\right)^s,$$

whenever $x \in B(x_0, r_0)$ and $r \leq r_0$. If μ is doubling, then this is satisfied with the exponent $s = \log_2 c_d$. Hence, in our case $s = Q$.

Theorem 5.2 (Rellich-Kondrachov). *Let (X, d, μ) be a doubling metric measure space and let s be the lower decay order of the measure. Suppose that all pairs $\{u_i, g_i\}$ satisfy the $(1,p)$–Poincaré inequality with fixed constants c and τ. Fix a ball B and assume that the sequence $||u_i||_{L^1(B)} + ||g_i||_{L^p(5\tau B)}$ is bounded. Then there is a subsequence of (u_i) that converges in $L^q(B)$ for each $1 \leq q < ps/(s-p)$, when $p < s$ and for each $q \geq 1$, when $p \geq s$.*

By a remark in section 2.9 we can assume that $X \setminus \Omega$ is uniformly p_0–fat. Since $p = \lim_{i\to\infty} p_i$ we can assume also that $p_i \in (q, q^*)$.

Functions u_i are supposed to be not equal to the boundary data w, i.e. we assume that there is a set of positive measure where $u_i \neq w$ μ–a.e., otherwise the result is trivial.

We start with a lemma concerning uniform higher integrability of u_i and u.

Lemma 5.3. *Let u_i and u be as in Theorem 5.1. Then there exists $\varepsilon_0 > 0$ such that*

$$\begin{aligned} u_i, u &\in L^{p+\varepsilon_0}(\Omega), \\ g_{u_i}, g_u &\in L^{p+\varepsilon_0}(\Omega) \end{aligned}$$

and there is a subsequence such that

$$\begin{array}{lcl} u_i \to u & \textit{in} & L^{p+\varepsilon_0}(\Omega) \\ g_{u_i} \rightharpoonup g & \textit{weakly in} & L^{p+\varepsilon_0}(\Omega), \end{array}$$

where g is a weak upper gradient of u.

It is not known whether the sequence of minimal upper gradients converges weakly to the minimal upper gradient of u or not. Yet the lemma implies that

$$\|g_u\|_{L^p(\Omega)} \leq \liminf_{i\to\infty} \|g_{u_i}\|_{L^p(\Omega)},$$

which is extremely useful in applications.

Proof. By Theorem 4.1 for every p_i there exists $\delta_i = \delta_i(p_i)$ such that the minimal p_i–weak upper gradient g_{u_i} belongs to the space $L^{p_i+\delta_i}(\Omega)$ and

$$\begin{aligned} &\left(\int_\Omega g_{u_i}^{p_i+\delta_i} \, \mathrm{d}\mu \right)^{1/(p_i+\delta_i)} \\ &\qquad \leq c_i \left(\int_\Omega g_{u_i}^{p_i} \, \mathrm{d}\mu \right)^{1/p_i} + c_i \left(\int_\Omega g_w^{p_i+\delta_i} \, \mathrm{d}\mu \right)^{1/(p_i+\delta_i)}. \end{aligned} \tag{5.1}$$

Since u_i is a quasiminimizer of the p_i–energy functional in Ω with boundary data w and thus $u_i - w \in N_0^{1,p_i}(\Omega)$, we have

$$\begin{aligned} \int_\Omega g_{u_i}^{p_i} \, \mathrm{d}\mu &\leq K \int_\Omega g_w^{p_i} \, \mathrm{d}\mu \\ &\leq K(\mu(\Omega))^{\delta_i/(p_i+\delta_i)} \left(\int_\Omega g_w^{p_i+\delta_i} \, \mathrm{d}\mu \right)^{p_i/(p_i+\delta_i)}, \end{aligned}$$

and therefore

$$\left(\int_\Omega g_{u_i}^{p_i+\delta_i}\,\mathrm{d}\mu\right)^{1/(p_i+\delta_i)} \le c_i\left(\int_\Omega g_w^{p_i+\delta_i}\,\mathrm{d}\mu\right)^{1/(p_i+\delta_i)}. \tag{5.2}$$

Now remark that when $p_i \in (q, q^*)$ and $p_i \to p$ we have

$$\delta_i \ge \delta_0 = \delta_0(p) \quad \text{and} \quad c_i \le c = c(p).$$

Indeed, in order to prove (5.1) we first show that a reverse Hölder inequality

$$\left(\frac{1}{\mu(B)}\int_{B\cap\Omega} g_{u_i}^{p_i}\,\mathrm{d}\mu\right)^{1/p_i} \le b_i\left(\frac{1}{\mu(\sigma B)}\int_{\sigma B\cap\Omega} g_{u_i}^{p_0}\,\mathrm{d}\mu\right)^{1/p_0} + c_i\left(\frac{1}{\mu(\sigma B)}\int_{\sigma B\cap\Omega} g_w^{p_i}\,\mathrm{d}\mu\right)^{1/p_i} \tag{5.3}$$

holds for some $\sigma > 1$ and then we apply the Gehring lemma. The constant b_i in (5.3) depends on p_i. However, when $p_i \in (q, q^*)$, it may be chosen independently on p_i i.e. $b_i \le b$ for some $b = b(p)$. The bound will depend on p due to the fact, that we apply the $(1, p_0)$–Poincaré inequality and p_0 is chosen to be sufficiently close to p. The assertion follows because of the fact that in (5.1) δ_i is inverse proportional to b_i and c_i is comparable to b_i (see e.g. [39]).

For i sufficiently large we may assume

$$p + \varepsilon_0 \le p_i + \delta_0 \le p_i + \delta_i \le s,$$

where $\varepsilon_0 = \delta_0/2$. By this assumption and the uniform bound for c_i, applying the Hölder inequality and (5.2) we obtain

$$\begin{aligned}\left(\int_\Omega g_{u_i}^{p+\varepsilon_0}\,\mathrm{d}\mu\right)^{1/p+\varepsilon_0} &\le c\left(\int_\Omega g_{u_i}^{p_i+\delta_i}\,\mathrm{d}\mu\right)^{1/(p_i+\delta_i)} \\ &\le c\left(\int_\Omega g_w^{s}\,\mathrm{d}\mu\right)^{1/s} < \infty.\end{aligned}$$

Since

$$\begin{aligned}\left(\int_\Omega g_{u_i-w}^{p+\varepsilon_0}\,\mathrm{d}\mu\right)^{1/p+\varepsilon_0} &\le \left(\int_\Omega g_{u_i}^{p+\varepsilon_0}\,\mathrm{d}\mu\right)^{1/p+\varepsilon_0} + \left(\int_\Omega g_w^{p+\varepsilon_0}\,\mathrm{d}\mu\right)^{1/p+\varepsilon_0} \\ &\le c\left(\int_\Omega g_w^{s}\,\mathrm{d}\mu\right)^{1/s},\end{aligned}$$

it follows that

$$\sup_i \|g_{u_i-w}\|_{L^{p+\varepsilon_0}(\Omega)} < \infty. \tag{5.4}$$

Using Proposition 3.4 we are able to find a uniform $L^{p+\varepsilon_0}$–bound for the sequence (u_i-w) as well. Observe, that decreasing ε_0 if necessary, we may additionally assume that $p+\varepsilon_0 < q^*$. So choose $t = p+\varepsilon_0$, $q = p+\varepsilon_0$ in Proposition 3.4 and fix $B_0 = B(x_0, r_0)$ such that $\Omega \subset B_0$. We note again that the minimal p_i–weak upper gradient of $u_i - w$ satisfies $g_{u_i-w} = 0$ μ-a.e. on the set $S = \{x \in B_0 \colon u(x) = w(x)\}$. On the other hand $u_i - w$ is zero p_i–quasieverywhere on $X \setminus \Omega$ and thus μ–almost everywhere on $X \setminus \Omega$. In addition, observe that p–fatness always implies $p+\varepsilon_0$–fatness, so that $\operatorname{cap}_{p+\varepsilon_0}(S, 2B_0) \geq c\mu(B_0) r^{p+\varepsilon_0}$. It follows that

$$\begin{aligned}
\left(\int_\Omega |u_i - w|^{p+\varepsilon_0}\,\mathrm{d}\mu\right)^{1/p+\varepsilon_0} &\leq c\left(\fint_{2B_0} |u_i - w|^{p+\varepsilon_0}\,\mathrm{d}\mu\right)^{1/p+\varepsilon_0}\\
&\leq \left(\frac{c}{\operatorname{cap}_{p+\varepsilon_0}(S, 2B_0)}\int_{2\tau' B_0} g_{u_i-w}^{p+\varepsilon_0}\,\mathrm{d}\mu\right)^{1/p+\varepsilon_0}\\
&\leq c r_0 \left(\int_\Omega g_{u_i-w}^{p+\varepsilon_0}\,\mathrm{d}\mu\right)^{1/p+\varepsilon_0}\\
&\leq c\left(\int_\Omega g_w^s\,\mathrm{d}\mu\right)^{1/s},
\end{aligned}$$

by the Hölder inequality. Together with (5.4) this implies

$$\sup_i \|u_i - w\|_{N^{1,p+\varepsilon_0}(\Omega)} < \infty.$$

Hence the sequence $(u_i - w)$ uniformly bounded in $N^{1,p+\varepsilon_0}(\Omega)$. In particular, for i large enough the pairs $\{u_i, g_{u_i}\}$ satisfy the $(1, p+\varepsilon_0)$–Poincaré inequality with fixed constants c and τ.

Then choose a ball B_0 such that $\Omega \subset B_0$, and extend $u_i - w$ to be zero in $X \setminus \Omega$. The sequence

$$\|u_i - w\|_{L^1(B_0)} + \|g_{u_i-w}\|_{L^{p+\varepsilon_0}(5\tau B_0)}$$

is certainly bounded. By the Rellich–Kondrachov theorem 5.2 there exists $\tilde{u} \in L^{p+\varepsilon_0}(B_0)$ and a subsequence such that

$$u_{i_k} - w \to \tilde{u} - w \text{ in } L^{p+\varepsilon_0}(B_0).$$

Since $u_i \to u$ μ–a.e. in Ω, it follows that $\tilde{u} = u$ μ–a.e. in Ω and $u \in L^{p+\varepsilon_0}(\Omega)$. As the norms of $g_{u_{i_k}}$ are uniformly bounded in $L^{p+\varepsilon_0}(\Omega)$, there exist $g \in L^{p+\varepsilon_0}(\Omega)$ a subsequence, also denoted $g_{u_{i_k}}$ for brevity, such that

$$g_{u_{i_k}} \rightharpoonup g \text{ weakly in } L^{p+\varepsilon_0}(\Omega).$$

By Lemma 3.3 g is a weak upper gradient of u and $u \in N^{p+\varepsilon_0}(\Omega)$. $\square$

As the proof of Lemma 5.3 shows, it is not necessary to suppose the existence of a limit u a priori. Hence, if desired, the statement of Theorem 5.1 could be reformulated.

Let $D \subset \Omega$ be a compact set and for $t > 0$ write

$$D(t) = \{x \in \Omega \colon \operatorname{dist}(x, D) < t\}.$$

Then $D(t) \subset\subset \Omega$ for $t \in (0, t_0)$ where $t_0 = \operatorname{dist}(D, X \setminus \Omega)$. We reformulate a lemma by Kinnunen and Martio [22] so that it applies to the present case.

Lemma 5.4. *Let u_i, u be as in Theorem 5.1. Then for almost every $t \in (0, t_0)$ we have*

$$\limsup_{i \to \infty} \int_{D(t)} g_{u_i}^{p_i} \, d\mu \leq c \int_{D(t)} g_u^p \, d\mu,$$

where the constant c depends only on K and p.

Proof. Let $0 < t' < t < t_0$. Choose a Lipschitz cut–off function η such that $0 \leq \eta \leq 1$ and

$$\begin{aligned} \eta = 1 &\quad \text{on} \quad D(t'), \\ \eta = 0 &\quad \text{on} \quad \Omega \setminus D(t). \end{aligned}$$

Define a function

$$\phi_i = \eta(u - u_i).$$

For i large enough $p_i < p + \varepsilon_0$. Then, since u_i and u belong to $N^{1,p+\varepsilon_0}(\Omega)$ it follows that $\phi_i \in N_0^{1,p_i}(D(t))$. Therefore by the quasiminimizing property of u_i we have

$$\int_{D(t')} g_{u_i}^{p_i} \, d\mu \leq \int_{D(t)} g_{u_i}^{p_i} \, d\mu \leq K \int_{D(t)} g_{u_i+\phi_i}^{p_i} \, d\mu \, .$$

Lemma 3.2 implies

$$g_{u_i+\phi_i} \leq (1 - \eta) g_{u_i} + g_\eta |u - u_i| + \eta g_u,$$

and hence

$$\int_{D(t')} g_{u_i}^{p_i}\,\mathrm{d}\mu \le c\Big(\int_{D(t)} (1-\eta)^{p_i} g_{u_i}^{p_i}\,\mathrm{d}\mu + \int_{D(t)} g_\eta^{p_i}|u-u_i|^{p_i}\,\mathrm{d}\mu + \int_{D(t)} \eta^{p_i} g_u^{p_i}\,\mathrm{d}\mu\Big)$$

with c depending only on D and p_i. Observing that $\eta \equiv 1$ on $D(t')$ we add $c\int_{D(t')} g_{u_i}^{p_i}\,\mathrm{d}\mu$ to the both sides of the inequality and obtain

$$(1+c)\int_{D(t')} g_{u_i}^{p_i}\,\mathrm{d}\mu \le c\Big(\int_{D(t)} (1-\eta)^{p_i} g_{u_i}^{p_i}\,\mathrm{d}\mu + \int_{D(t)} g_\eta^{p_i}|u-u_i|^{p_i}\,\mathrm{d}\mu + \int_{D(t)} \eta^{p_i} g_u^{p_i}\,\mathrm{d}\mu\Big).$$

Define now on $(0,t_0)$ a function

$$\Psi(t) = \limsup_{i\to\infty}\int_{D(t)} g_{u_i}^{p_i}\,\mathrm{d}\mu\,.$$

By definition Ψ is a nondecreasing function of t and by the uniform higher integrability of u_i it is finite for every $t \in (0,t_0)$. Therefore its set of points of discontinuity is at most countable. Let t be a point of continuity of Ψ. Taking limes superior on both sides of the last inequality we obtain

$$(1+c)\Psi(t') \le c\Psi(t) + c\limsup_{i\to\infty}\int_{D(t)} |u-u_i|^{p_i}\,\mathrm{d}\mu + c\int_{D(t)} g_u^p\,\mathrm{d}\mu\,.$$

The second term on the right hand side tends to zero. To see this, apply first the Hölder inequality and then Lemma 5.3. Hence, since t is a point of continuity of Ψ we obtain

$$(1+c)\Psi(t) \le c\Psi(t) + c\int_{D(t)} g_u^p\,\mathrm{d}\mu,$$

and furthermore

$$\Psi(t) \le c\int_{D(t)} g_u^p\,\mathrm{d}\mu\,.$$

$\square$

Proof of Theorem 5.1 . In order to show that u is a quasiminimizer of the p–energy integral with boundary data w, we need to show first that $u-w \in N_0^{1,p}(\Omega)$. This does not follow immediately from the compactness argument used to extract the convergent subsequence.

We proceed as follows. For every $\varepsilon > 0$ and for i sufficiently large $p_i > p - \varepsilon$ so that $u_i - w \in N_0^{1,p-\varepsilon}(\Omega)$. By the Sobolev inequality (Lemma 3.5) we get

$$\begin{aligned} ||u_i - w||_{N_0^{1,p-\varepsilon}(\Omega)} &\leq c||g_{u_i - w}||_{L^{p-\varepsilon}(\Omega)} \\ &\leq c||g_{u_i - w}||_{L^p(\Omega)}, \end{aligned}$$

i.e. the norms of $(u_i - w)$ are uniformly bounded in $N_0^{1,p-\varepsilon}(\Omega)$.

As $X \setminus \Omega$ is uniformly p_0–fat, it is also uniformly $(p-\varepsilon)$–fat for ε small enough. In addition, $u_i \to u$ μ–a.e., so by Lemma 3.12

$$u - w \in N_0^{1,p-\varepsilon}(\Omega)$$

for all $\varepsilon > 0$ such that $p_0 < p - \varepsilon$. Hence, by Proposition 3.13, the p–fatness of $X \setminus \Omega$ implies also

$$u - w \in N_0^{1,p}(\Omega).$$

It remains to show that for every open $\Omega' \subset\subset \Omega$ and every $\phi \in N_0^{1,p}(\Omega')$

$$\int_{\overline{\Omega'}} g_u^p \, \mathrm{d}\mu \leq K \int_{\overline{\Omega'}} g_{u+\phi}^p \, \mathrm{d}\mu \,. \tag{5.5}$$

Let $\varepsilon > 0$ be arbitrary. For i sufficiently large $p_i > p - \varepsilon$. Since (g_{u_i}) converges weakly to a weak upper gradient g of u by Lemma 5.3, for every μ–measurable subset E of Ω we have

$$\begin{aligned} \int_E g_u^{p-\varepsilon} \, \mathrm{d}\mu \leq \int_E g^{p-\varepsilon} \, \mathrm{d}\mu &\leq \liminf_i \int_E g_{u_i}^{p-\varepsilon} \, \mathrm{d}\mu \\ &\leq \liminf_i \left(\int_E g_{u_i}^{p_i} \, \mathrm{d}\mu \right)^{(p-\varepsilon)/p_i} \mu(E)^{1-(p-\varepsilon)/p_i} \\ &\leq \liminf_i \left(\int_E g_{u_i}^{p_i} \, \mathrm{d}\mu \right)^{(p-\varepsilon)/p} \mu(E)^{\varepsilon/p}, \end{aligned}$$

where we have used the Hölder inequality. Notice that g is the weak limit of g_{u_i} and g_u is the minimal weak upper gradient of u. Passing to zero with ε we conclude that

$$\int_E g_u^p \, \mathrm{d}\mu \leq \liminf_i \int_E g_{u_i}^{p_i} \, \mathrm{d}\mu \,. \tag{5.6}$$

We will first show that the inequality (5.5) holds for every Lipschitz function compactly supported in Ω', i.e. for $\phi \in \mathrm{Lip}_C(\Omega')$. Fix $\varepsilon > 0$ and choose open sets Ω'' and Ω_0 such that

$$\Omega' \subset\subset \Omega'' \subset\subset \Omega_0 \subset\subset \Omega$$

and

$$\int_{\Omega_0 \setminus \overline{\Omega'}} g_u^p \, \mathrm{d}\mu < \varepsilon.$$

Let η be a Lipschitz cut–off function such that $0 \leq \eta \leq 1$, $\eta = 1$ in a neighbourhood of $\overline{\Omega'}$, and $\eta = 0$ in $\Omega \setminus \Omega''$. Define a function ϕ_i as

$$\phi_i = \phi + \eta(u - u_i).$$

Since $\phi \in \mathrm{Lip}_C(\Omega')$ and both u_i, $u \in N^{1,p+\varepsilon_0}(\Omega)$ it follows that $\phi_i \in N_0^{1,p_i}(\Omega'')$. Hence by the quasiminimizing property of u_i we get

$$\begin{aligned} \int_{\Omega''} g_{u_i}^{p_i} \, \mathrm{d}\mu &\leq K \int_{\Omega''} g_{u_i+\phi_i}^{p_i} \, \mathrm{d}\mu \\ &= K \int_{\overline{\Omega'}} g_{u_i+\phi_i}^{p_i} \, \mathrm{d}\mu + \int_{\Omega'' \setminus \overline{\Omega'}} g_{u_i+\phi_i}^{p_i} \, \mathrm{d}\mu \, . \end{aligned} \tag{5.7}$$

Since $\eta \equiv 1$ in a neighbourhood of $\overline{\Omega'}$ it follows that

$$u_i + \phi_i = u + \phi \quad \text{in a neighbourhood of } \overline{\Omega'}. \tag{5.8}$$

On the other hand, in $\Omega'' \setminus \overline{\Omega'}$ we have $\phi \equiv 0$, and therefore

$$u_i + \phi_i = u_i + \eta(u - u_i).$$

Now Lemma 3.2 implies that

$$g_{u_i+\phi_i} \leq (1-\eta) g_{u_i} + \eta g_u + g_\eta |u - u_i|.$$

Therefore

$$\begin{aligned} \int_{\Omega'' \setminus \overline{\Omega'}} g_{u_i+\phi_i}^{p_i} \, \mathrm{d}\mu &\leq c \int_{\Omega'' \setminus \overline{\Omega'}} (1-\eta)^{p_i} g_{u_i}^{p_i} \, \mathrm{d}\mu \\ &\quad + c \int_{\Omega'' \setminus \overline{\Omega'}} \eta^{p_i} g_u^{p_i} \, \mathrm{d}\mu + c \int_{\Omega'' \setminus \overline{\Omega'}} g_\eta^{p_i} |u - u_i|^{p_i} \, \mathrm{d}\mu \, . \end{aligned} \tag{5.9}$$

We estimate the integrals on the right–hand side separately.

Since $\eta \equiv 1$ on a neighbourhood of $\overline{\Omega'}$, there exists a compact set $D \subset \overline{\Omega''}$ such that $D \cap \overline{\Omega'} = \emptyset$ and

$$\int_{\Omega'' \setminus \overline{\Omega'}} (1-\eta)^{p_i} g_{u_i}^{p_i} \, \mathrm{d}\mu \leq \int_D g_{u_i}^{p_i} \, \mathrm{d}\mu \, .$$

For t sufficiently small we have $D(t) \subset \Omega_0 \setminus \overline{\Omega'}$. So we choose t such that we may apply Lemma 5.4, in other words

$$\limsup_{i\to\infty} \int_{D(t)} g_{u_i}^{p_i} \,\mathrm{d}\mu \leq c \int_{D(t)} g_u^p \,\mathrm{d}\mu \,.$$

Consequently

$$\begin{aligned} \limsup_{i\to\infty} \int_{\Omega''\setminus\overline{\Omega'}} (1-\eta)^{p_i} g_{u_i}^{p_i} \,\mathrm{d}\mu &\leq \limsup_{i\to\infty} \int_D g_{u_i}^{p_i} \,\mathrm{d}\mu \\ &\leq \limsup_{i\to\infty} \int_{D(t)} g_{u_i}^{p_i} \,\mathrm{d}\mu \leq c \int_{D(t)} g_u^p \,\mathrm{d}\mu \leq c\varepsilon, \end{aligned} \tag{5.10}$$

by the choice of Ω_0.

Also the second integral is arbitrarily small. Again by the choice of Ω_0 we have

$$\limsup_{i\to\infty} \int_{\Omega''\setminus\overline{\Omega'}} \eta^{p_i} g_u^{p_i} \,\mathrm{d}\mu \leq \int_{\Omega''\setminus\overline{\Omega'}} g_u^p \,\mathrm{d}\mu \leq \int_{\Omega_0\setminus\overline{\Omega'}} g_u^p \,\mathrm{d}\mu \leq \varepsilon. \tag{5.11}$$

Observe, that for a Lipschitz function its minimal p–weak upper gradient is bounded by its Lipschitz constant μ–almost everywhere. This allows us to conclude that

$$\begin{aligned} \limsup_{i\to\infty} \int_{\Omega''\setminus\overline{\Omega'}} g_\eta^{p_i} |u-u_i|^{p_i} \,\mathrm{d}\mu &\leq c \limsup_{i\to\infty} \int_{\Omega''\setminus\overline{\Omega'}} |u-u_i|^{p_i} \,\mathrm{d}\mu \\ &\leq c \limsup_{i\to\infty} \left(\int_{\Omega''\setminus\overline{\Omega'}} |u-u_i|^{p+\varepsilon_0} \,\mathrm{d}\mu \right)^{\frac{p_i}{p+\varepsilon_0}} = 0, \end{aligned} \tag{5.12}$$

by the Lebesgue dominated convergence theorem.

By the estimates (5.10), (5.11) and (5.12) we obtain from (5.9)

$$\limsup_{i\to\infty} \int_{\Omega''\setminus\overline{\Omega'}} g_{u_i+\phi_i}^{p_i} \,\mathrm{d}\mu \leq c\varepsilon. \tag{5.13}$$

Finally by (5.6), (5.7), (5.8) and (5.13) we have

$$\begin{aligned} \int_{\overline{\Omega'}} g_u^p \,\mathrm{d}\mu &\leq \liminf_{i\to\infty} \int_{\Omega''} g_{u_i}^{p_i} \,\mathrm{d}\mu \\ &\leq K \liminf_{i\to\infty} \int_{\Omega''} g_{u_i+\phi_i}^{p_i} \,\mathrm{d}\mu \\ &\leq K \liminf_{i\to\infty} \int_{\overline{\Omega'}} g_{u+\phi}^{p_i} \,\mathrm{d}\mu + K \liminf_{i\to\infty} \int_{\Omega''\setminus\overline{\Omega'}} g_{u_i+\phi_i}^{p_i} \,\mathrm{d}\mu \\ &\leq K \int_{\overline{\Omega'}} g_{u+\phi}^p \,\mathrm{d}\mu + c\varepsilon. \end{aligned} \tag{5.14}$$

Passing to zero with ε we obtain the desired inequality for any $\phi \in \mathrm{Lip}_C(\Omega')$.

The result for $\phi \in N_0^{1,p}(\Omega')$ follows by approximation, i.e. if $\phi \in N_0^{1,p}(\Omega')$ then for any $\varepsilon > 0$ we may find a function $\phi_\varepsilon \in \mathrm{Lip}_C(\Omega')$ such that

$$\|\phi_\varepsilon - \phi\|_{N^{1,p}(\Omega')} < \varepsilon.$$

$\square$

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