# Inverse problems and counter examples

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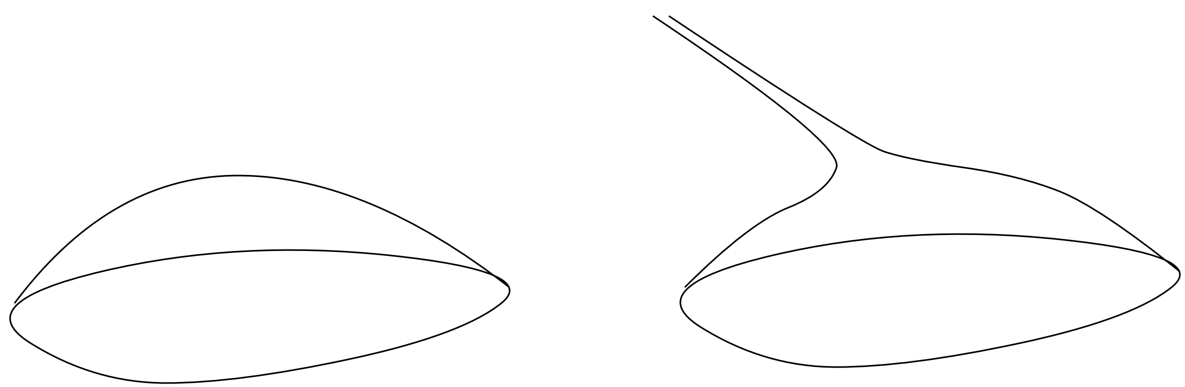
**Non-uniqueness results** (L.-Taylor-Uhlmann 2003)
Let $(M, g)$ be a compact 2-dimensional manifold. Let $x_0 \in M$, and consider manifold

$$\widetilde{M} = M \setminus \{x_0\}$$

with metric

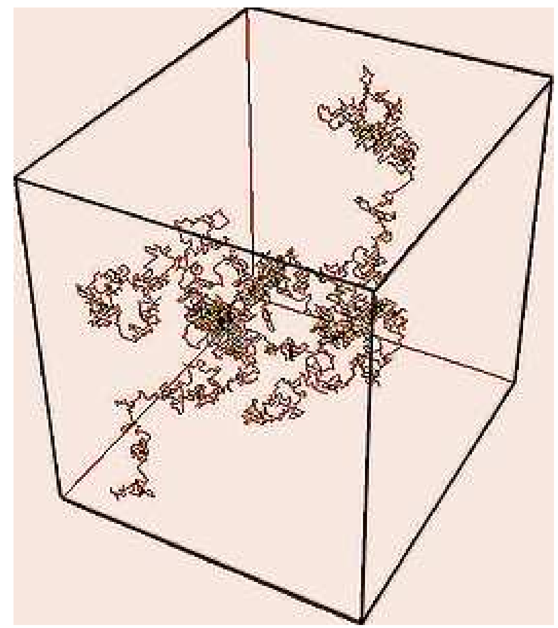
$$\widetilde{g}_{ij}(x) = \frac{1}{d_M(x, x_0)^2} g_{ij}(x).$$



## Random walk on a Riemannian manifold

**Random walk problem.** Let $y \in \partial M$ and let $\nu$ be the unit normal vector. Consider a random walk process $B^x(t)$ starting at the point $\gamma_{y,\nu}(\varepsilon)$, where $\varepsilon > 0$ is small. Let us measure the probability distribution of the points where $B^x(t)$ hits first time to the boundary. Does these measurement determine the metric on $M$?

Inverse conductivity problem
is equivalent
to a random walk problem.

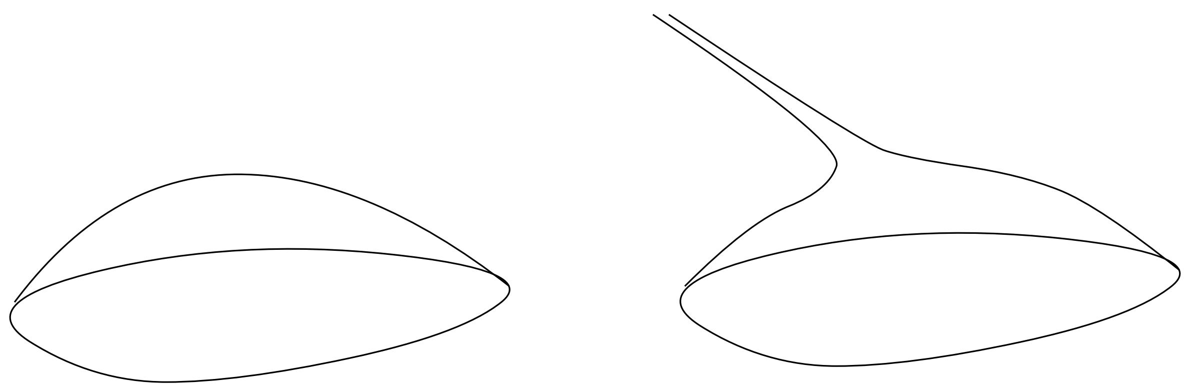


Assume $\Delta_g u = 0$, $u|_{\partial M} = h$. Then by the Feynman-Kac formula

$$u(x) = \mathbb{E}(h(B^x(\tau))), \quad h = u|_{\partial M},$$

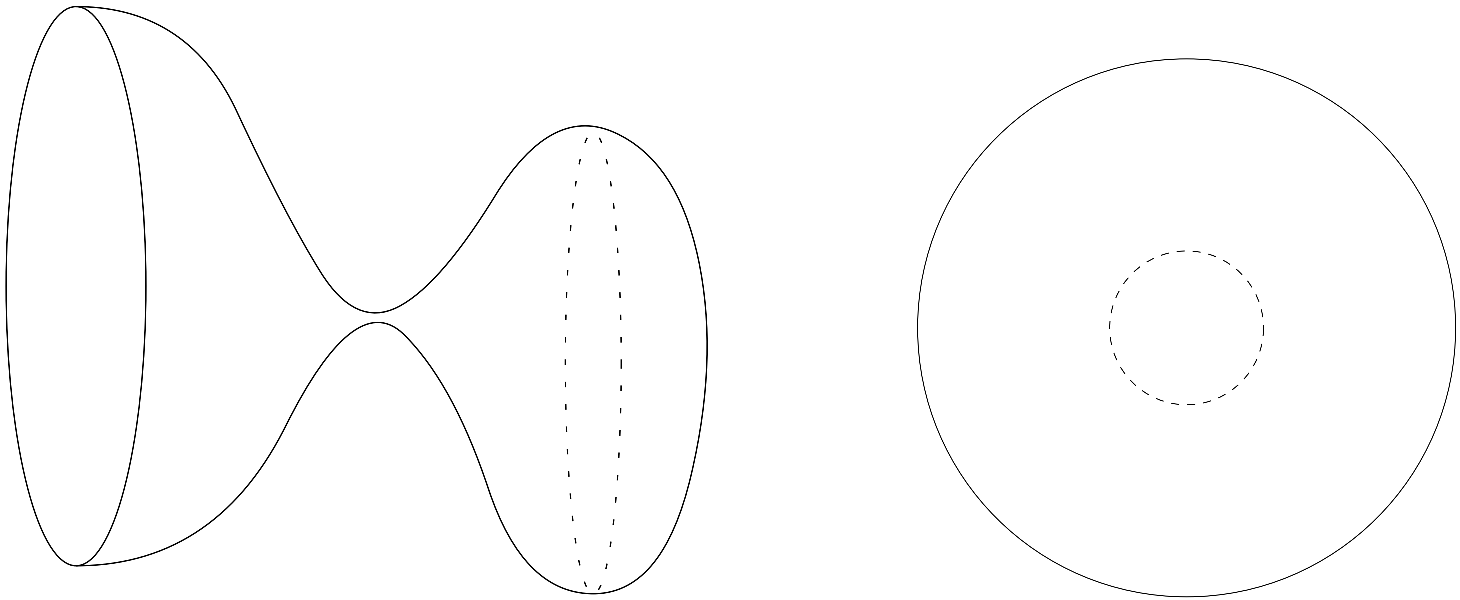
where $B^x(t)$ is Brownian motion starting at $x$ and $\tau$ is hitting time to $\partial M$.
Normal derivative of $u$ can be found by moving the starting point $x$ to boundary. This yields

$$\Lambda_{M,g} = \Lambda_{\widetilde{M},\widetilde{g}}$$



Two manifolds having same boundary measurements

**A ball with an invisible pocket.**
(Greenleaf-L.-Uhlmann 2003)
Motivation: Random walk and a bag.

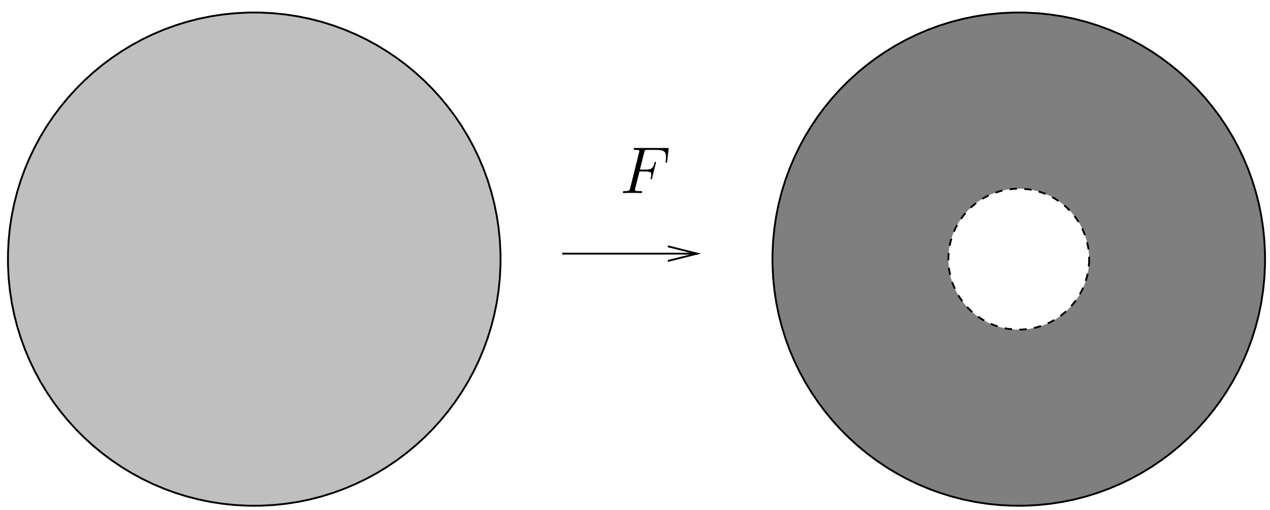


1. The probability that the random walk sent from the boundary enters the bag is very small. Thus the interior of the bag has a small influence on the boundary measurements.

2. The boundary measurements depend only on the intrinsic Riemannian metric of the surface. They do not depend on the way how the surface lies in $\mathbb{R}^3$.

Let $\Omega = B(0,2) \subset \mathbb{R}^3$ be a ball and $D = B(0,1)$. Consider the map

$$F : \Omega \setminus \{0\} \to \Omega \setminus \overline{D}$$
$$F(x) = (\frac{|x|}{2} + 1)\frac{x}{|x|}.$$



Let $\gamma = 1$ be the homogeneous conductivity in $B(0,2)$ and define $\sigma = F_*\gamma$. Then in the spherical coordinates $(r,\phi,\theta) \mapsto (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ we have

$$\gamma = \begin{pmatrix} r^2\sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & (\sin\theta)^{-1} \end{pmatrix},$$

$$\sigma = \begin{pmatrix} 2(r-1)^2\sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}.$$

Then $\sigma$ is defined on $B(0,2)\setminus B(0,1)$. Let $\widehat{\sigma}$ be conductivity in $B(0,2)$ such that $\hat{\sigma} = \sigma$ in $B(0,2)\setminus B(0,1)$. Then

$$\Lambda_{\widehat{\sigma}} = \Lambda_\gamma.$$

All boundary measurements for the homogeneous conductivity $\gamma = 1$ and the degenerated conductivity $\widehat{\sigma}$ are the same.

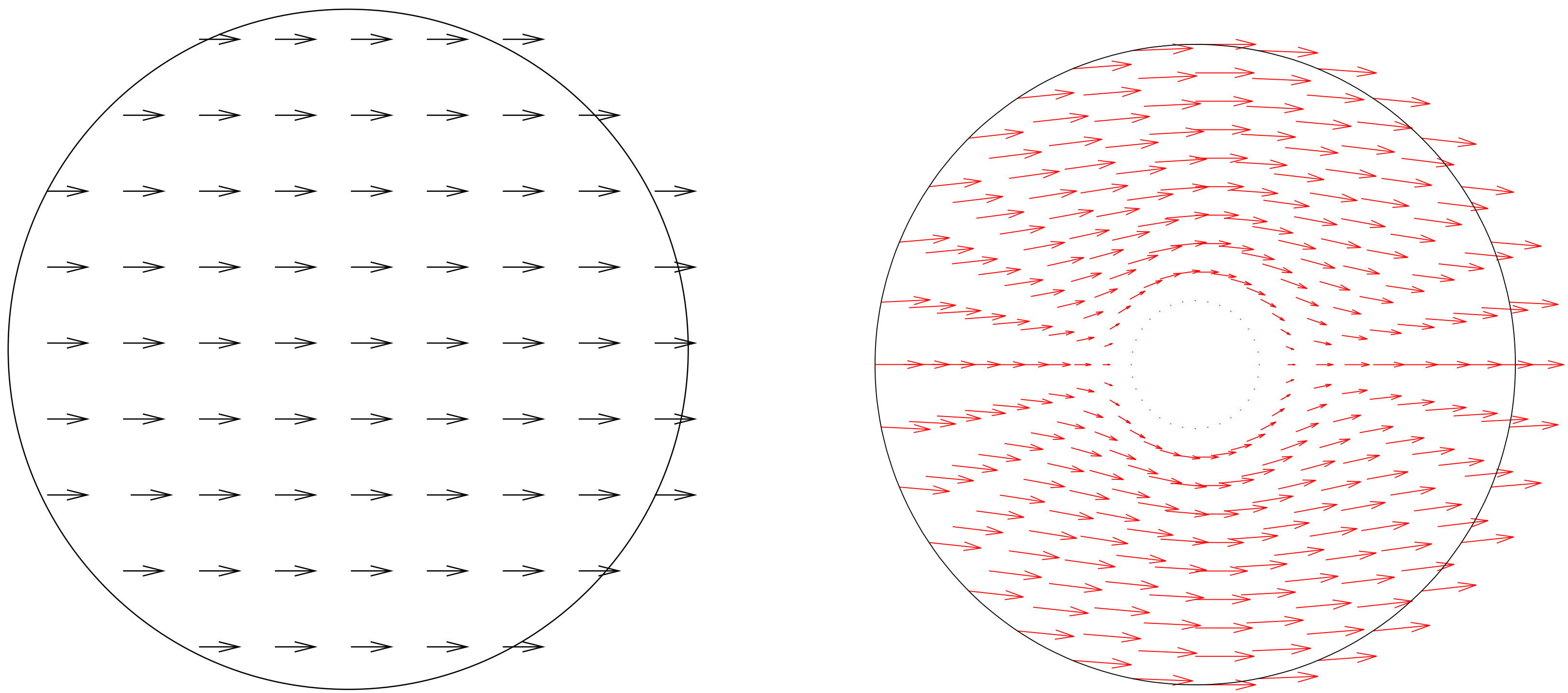
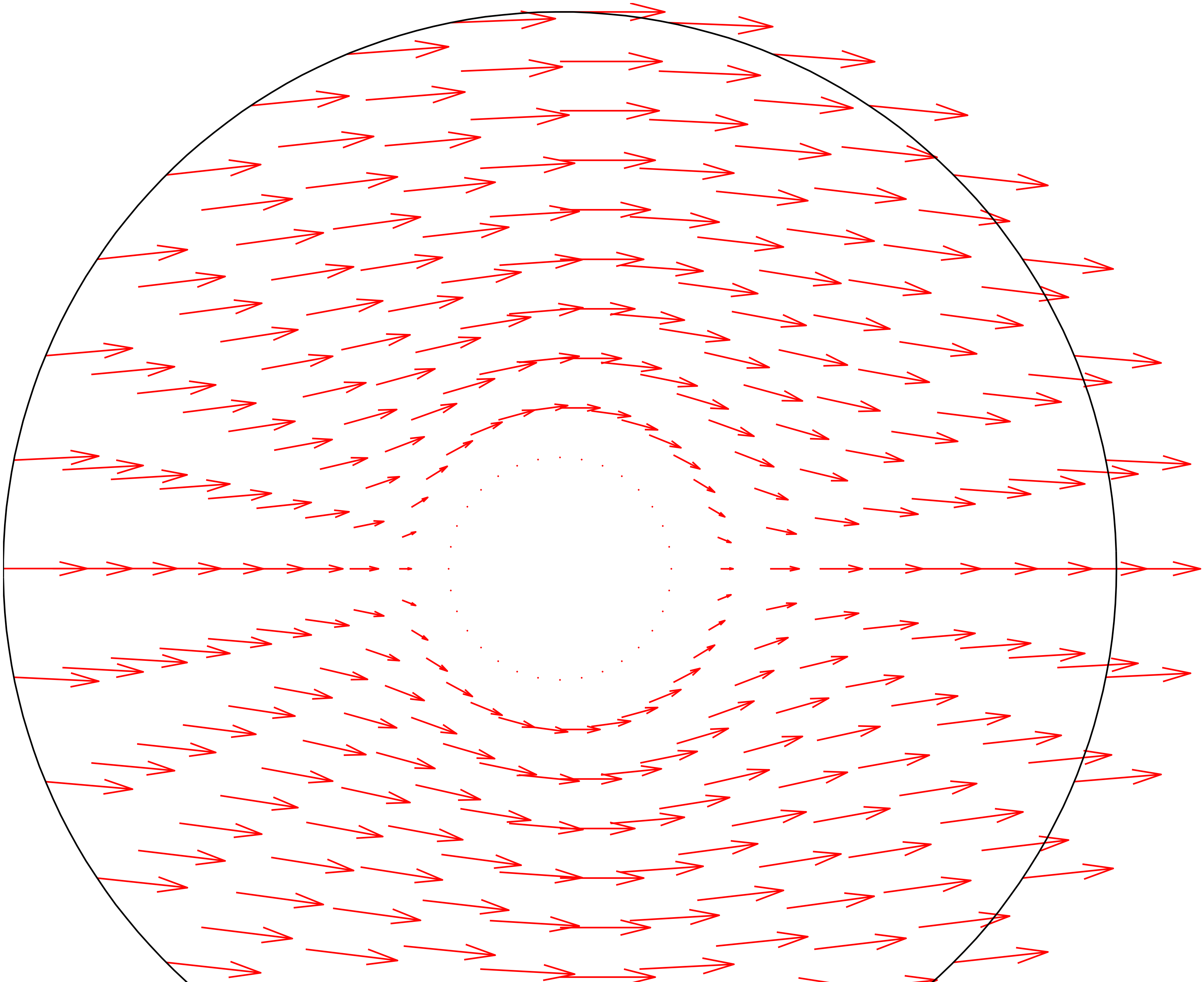


Figure: Analytic solutions for the currents.



The invisibility construction can be generalized for Helmholtz and Maxwell’s equations.

These will be considered in the Symposium on Inverse Problems Honoring Alberto Calderón.