

Liability-driven investment in longevity risk management

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Abstract

This paper studies optimal investment from the point of view of an investor with longevity-linked liabilities. The relevant optimization problems are rarely analytically tractable, but we are able to show numerically that liability driven investment can significantly outperform common strategies that do not take the liabilities into account. In problems without liabilities the advantage disappears, which suggests that the superiority of the proposed strategies is indeed based on connections between liabilities and asset returns.

Keywords: Longevity risk, Mortality risk, Stochastic mortality, Stochastic optimization, Hedging.

1 Introduction

Longevity risk, the uncertainty in future mortality developments, affects pension providers, life insurers, and governments. The population structure of developed countries is increasingly leaning towards the old, and the effects of medical advances and lifestyle choices on mortality remain unpredictable, which creates an increasingly acute need for life insurance and pension plans to hedge themselves against longevity risk.

Various longevity-linked instruments have been proposed for the management of longevity risk; see e.g. [10, 13, 6, 19, 26]. It has been shown how such instruments, once in existence, can be used to hedge mortality risk exposures in pensions or life insurance liabilities [10, 12, 11, 15, 25, 18]. Indeed, demand for longevity-linked instruments appears to exist, and some longevity transactions have already taken place. However, a major challenge facing the development of longevity markets is the hedging of the risk that stems from issuing longevity-linked securities. The supply for mortality-linked instruments might increase if their cash-flows could be hedged by appropriately trading in assets for which liquid markets already exist. Such development has been seen e.g. in options markets which flourished after the publication of the seminal

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Black–Scholes–Merton model. Even though the cash flows of mortality-linked instruments cannot be replicated, it may be possible to diminish the residual risk by an appropriate choice of an investment strategy.

This work addresses the above issues by studying optimal investment from the point of view of an insurer with longevity-linked liabilities. We demonstrate how the hedge of a longevity-linked cash flow can be improved by taking the liabilities into account in investment decisions. This is achieved by optimally diversifying a given initial capital amongst several investment strategies, some of which employ statistical connections between assets and liabilities. The approach can be applied to the pricing and hedging of longevity-linked instruments, as well as the asset-liability management of pension plans and life insurers.

The most straightforward approach to hedging of longevity-linked instruments is *natural hedging*, where an insurer hedges longevity risk by taking positions with opposite exposures to longevity developments [17, 28]. Such an approach is obviously limited by the demand on the relevant insurance products. Another popular approach builds on *risk neutral valuation* which is based on the no-arbitrage principle from financial economics; see e.g. [12, 20, 3], Cairns et al. [16] as well as Section ?? of Wilkie et al. [29]. In analogy with the Black–Scholes–Merton theory, it has been suggested that longevity-linked instruments could then be hedged using *delta hedging* by determining price sensitivities with respect to traded securities. This approach is, however, invalidated by the fact that the payouts of longevity-linked instruments cannot be replicated by liquidly traded assets as assumed by the risk neutral valuation theory; see the discussion in [29, 4, 5].

We approach the hedging problem by studying optimal investment in the presence of longevity-linked liabilities. We employ a computational procedure that constructs diversified strategies from parametric *basis strategies*, and suggest trading strategies that are motivated by connections between mortality and financial markets observed in [2]. We notice that the risk associated with the diversified strategy diminishes as these liability-driven basis strategies are included. To assess to which extent the reduction in risk was due to the liability connection, we performed the same computations without liabilities. In this case, the inclusion of the longevity-linked liability-driven strategies had a negligible effect on the aggregate risk, suggesting that the reduction in the risk of the original problem resulted from taking the longevity-linked liabilities into consideration in the investment decisions.

Section 2 formulates the asset-liability management problem of a longevity-linked cash flow. Section 3 introduces investment strategies that serve as basis strategies for the computational procedure described in Section 4. Section 5 presents results from a simulation study, and Section 6 concludes.

2 The asset-liability management problem

Consider an insurer with given initial capital w_0 and longevity-linked liabilities with claims c_t over time $t = 1, 2, \dots, T$. After paying out c_t at time t , the insurer invests the remaining wealth in financial markets. We look for investment strategies whose proceeds fit the liabilities as well as possible in the sense of a given risk measure ρ on the residual wealth at time T .

We assume that a finite set J of liquid assets (bonds, equities, ...) can

be traded at $t = 0, \dots, T$. The return on asset j over period $[t - 1, t]$ will be denoted $R_{t,j}$. The amount of cash invested in asset j over period $(t, t + 1]$ will be denoted by $h_{t,j}$. The asset-liability management problem of the insurer can then be written as

$$\begin{aligned}
& \text{minimize} && \rho\left(\sum_{j \in J} h_{T,j}\right) \quad \text{over } h \in \mathcal{N} \\
& \text{subject to} && \sum_{j \in J} h_{0,j} \leq w_0 \\
& && \sum_{j \in J} h_{t,j} \leq \sum_{j \in J} R_{t,j} h_{t-1,j} - c_t \quad t = 1, \dots, T \\
& && h_t \in D_t, \quad t = 1, \dots, T
\end{aligned} \tag{ALM}$$

The liabilities $(c_t)_{t=0}^T$ and the investment returns $(R_{t,j})_{t=0}^T$ will be modeled as stochastic processes on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t=1}^T, P)$. The set \mathcal{N} denotes the \mathbb{R}^J -valued $(\mathcal{F}_t)_{t=1}^T$ -adapted investment strategies $(h_t)_{t=0}^T$. Being adapted means that the portfolio h_t chosen at time t may only depend on information observed by time t . The last constraint describes portfolio constraints. The set D_t is allowed to be random but known at time t^1 . The risk measure ρ is a convex function on the space of real-valued random variables. It describes the insurer's preferences over random terminal wealth distributions. We refer the reader to [21, Chapter 4] for a general treatment of risk measures.

Liability-driven investment refers to the general principle that optimal investment strategies depend on the liabilities. The same idea is behind the famous Black–Scholes–Merton option pricing model where the price of an option is defined as the least amount of initial capital that allows for the implementation of an investment strategy whose proceeds match the option payout exactly. In the case of longevity-linked liabilities, exact replication is not possible so one has to evoke the risk preferences as is done in problem (ALM) in terms of the risk measure ρ .

Problems of the form (ALM) arise naturally in reserving for existing insurance liabilities as well as in underwriting new insurance contracts. Optimal risk adjusted reserves are obtained as the least initial wealth w_0 that allow for acceptable levels of risk in (ALM). In underwriting, one looks for a premium that would allow the insurer to take on the additional liabilities without worsening the optimal value of (ALM); see [27] for a general study of risk management-based valuation of uncertain cash-flows.

3 Investment strategies

In this section we present investment strategies that are used in subsequent numerical illustrations. We recall some well-known trading strategies recommended for long-term investment, and also introduce new strategies that try to employ the connections between the longevity-linked liabilities and asset returns.

We will describe the proportions of wealth invested in each asset $j \in J$ at time t by vector $\pi_t = [\pi_t^1, \pi_t^2, \dots, \pi_t^J]$ whose components sum up to one. The

¹More precisely, D_t is assumed \mathcal{F}_t -measurable, i.e. $\{\omega \in \Omega \mid D_t(\omega) \cap U \neq \emptyset\} \in \mathcal{F}_t$ for every open U .

amount of wealth invested in each asset can be expressed as $h_t = \pi_t w_t$, where for $t = 1, \dots, T$

$$w_t = \sum_{j \in J} R_{t,j} h_{t-1,j} - c_t$$

is the *net wealth* of the investor at time t .

3.1 Non-liability-driven investment strategies

In *buy and hold* (B&H) strategies the initial asset allocation π_0 is held over the subsequent time periods. To cover a nonzero claim process c_t , each asset is liquidated in proportion to the the initial investment. In other words, one invests

$$h_{t,j} = \begin{cases} \pi_0^j w_0 & t = 0, \\ R_{t,j} h_{t-1,j} - \pi_0^j c_t & t = 1, \dots, T, \end{cases}$$

units of cash in asset $j \in J$ at the beginning of the holding period starting at time t .

Fixed proportions (FP) is a strategy where the allocation is rebalanced at the beginning of each holding period into fixed proportions given by a constant vector $\pi \in \mathbb{R}^J$ as

$$h_t = \pi w_t.$$

A *target date fund* (TDF) is an well-known strategy in the pension industry ([14]). In a TDF, the proportion invested in risky assets diminishes as a pre-determined target date approaches. We implement TDFs by adjusting the allocation between two complementary subsets J^r and J^s of the set of all assets J . Here J^s consists of “safe” assets and J^r consists of the remaining “risky” assets. In the simulations presented in Section 5 safe assets consist of government bonds, and risky assets comprise equities and corporate bonds. The *proportional exposure* to J^r at time t is given by

$$e_t = a - bt.$$

The parameter a determines the initial proportion invested in J^r and b defines how fast the proportion decreases in time. Choices of such a and b that

$$a \geq 0 \quad \text{and} \quad a - bT \geq 0.$$

guarantees that the exposure e_t in the risky assets remains nonnegative. A TDF can be written as

$$h_t = \pi_t w_t$$

where the vector π_t is adjusted to give the specified proportional exposure:

$$\sum_{j \in J^r} \pi_t^j = e_t.$$

Within J^r and J^s the wealth is allocated using FP rules.

3.2 Liability-driven investment strategies

This subsection presents strategies in which the proportions invested in different assets are connected to the development of the longevity-linked liabilities. Some of the strategies utilize the connections between mortality and financial markets observed in [2], while others employ, directly or indirectly, the current and forecast future cash flows of longevity-linked liabilities to determine the asset allocation.

A well-known liability-driven strategy is the *constant proportion portfolio insurance* (CPPI) strategy; see e.g. [7, 8, 9]. The proportion of wealth invested in risky assets is given by

$$\begin{aligned} e_t &= \frac{m}{w_t} \max\{w_t - F_t, 0\} \\ &= m \max\{1 - \frac{F_t}{w_t}, 0\}, \end{aligned}$$

where the *floor* F_t represents the value of outstanding claims at time t and the parameter $m \geq 0$ determines the fraction of the cushion (wealth over the floor) invested in risky assets. Risky and safe assets are here the same as in TDF strategies. We define the floor through

$$\begin{aligned} F_T &= 0, \\ F_t &= (1 + r)F_{t-1} - \bar{c}_t \quad t = 0, \dots, T, \end{aligned}$$

where r is a deterministic discount factor and \bar{c}_t is the median of claim amount at time t . In this type of strategies, the liabilities are taken into account not only in the projected claim amounts \bar{c}_t but also in the remaining wealth w_t , which for a given w_0 depends on the realized values of the claims c_t .

The idea behind *spread strategies* is to capture the connections between mortality and asset returns described in [2]. Statistical analysis suggests that term spread is connected to the survival probabilities of over 50-year-olds. When term spread is high, the survival probabilities v^3 tend to be high. On the other hand, the yields of long-term treasury bonds are then relatively high, compared to short-term treasury bonds. Hence, relatively high yields on long-term bonds are connected with high survival probabilities of the old. *Term spread strategies* aim to utilize this notion. The proportion of wealth invested in long-term treasury bonds is determined as a function of the term spread s_t^T by

$$\pi_t^L = \sigma^{a,b}(s_t^T),$$

where

$$\sigma^{a,b}(s) = \frac{1}{1 + e^{-b(s+a)}},$$

and $b > 0$ and $a \in \mathbb{R}$ are user-defined parameters. The remaining wealth is invested in short-term government bonds.

Correspondingly, the analysis in [2] shows that the low credit spreads are connected to high survival probabilities of over 50-year-olds, and simultaneously the returns on riskier corporate bonds are relatively low compared to less risky bonds. Hence, in the case of *credit spread strategies* the proportional exposure to riskier corporate bonds is

$$\pi_t^C = \sigma(s_t^C),$$

The remaining wealth is invested in less risky bonds.

In *survival index strategies* the wealth between assets depends on the *survival index* S of a given population. The value of the survival index S_t time t is defined as the fraction of the population that survives until time t . The value of S_t gives indication on future liabilities: the smaller the remaining number of survivors, the smaller the future cash flows are likely to be. Hence, the proportional exposure in asset j at time t is given by

$$\pi_t^j = g^a(S_t),$$

where

$$g^a(s) = \min\{as, 1\},$$

and $a \in \mathbb{R}$. The rest is invested in other assets using fix-mix strategies.

In *wealth strategies* the proportion invested in asset j depends on the proportion of initial wealth w_t/w_0 remaining at time t . The proportional exposure at time t is given by

$$\pi_t^j = g^a(w_t/w_0),$$

where $a \in \mathbb{R}$. The rest is invested in other assets. Wealth-dependent strategies resemble CPPI strategies in the sense that both define the proportions of wealth invested in various assets in terms of the present wealth. However, the wealth strategies do not depend on median liabilities like CPPI, but the liabilities are reflected only in the present level of wealth.

4 Diversification procedure

We now briefly recall the numerical procedure presented in [22, 24]. It is a computational method for diversifying the initial wealth w_0 amongst a set of simple parametric strategies called *basis strategies*. The convex combination of feasible basis strategies is always feasible, since the optimization problem is convex. The investment strategies presented in the previous section serve as basis strategies in the numerical illustrations in Section 5.

Consider a finite set $\{h^i \mid i \in I\}$ of basis strategies that invest the amount $h_{t,j}^i$ in asset j at time t . The problem of finding an optimal diversification amongst the basis strategies can be written as

$$\underset{\alpha \in X}{\text{minimize}} \quad \rho \left(\sum_{i \in I} \alpha^i w_T^i \right),$$

where $w_T^i = \sum_{j \in J} h_{T,j}^i$ is the terminal wealth obtained by following strategy h^i when starting with initial capital w_0 , and

$$X = \{ \alpha \in \mathbb{R}_+^I \mid \sum_{i \in I} \alpha^i = 1 \}.$$

are the weights in the convex combination. In this work we employ the entropic risk measure

$$\rho(X) = \frac{1}{\gamma} \log E[e^{-\gamma X}],$$

in which case the minimization problem becomes

$$\underset{\alpha \in X}{\text{minimize}} \quad E[e^{-\gamma(\sum_{i \in I} \alpha^i w_T^i)}].$$

Because of the convexity of D_t , $\sum_{i \in I} \alpha^i h_t^i \in D_t$ for $t = 0, \dots, T$. In addition, the budget constraint of the aggregate strategy $\sum_{j \in J} h_{t,j} \leq \sum_{j \in J} R_{t,j} h_{t-1,j} - c_t$ holds, if it holds for individual strategies. This is a finite-dimensional convex optimization problem, but the objective involves high-dimensional integration.

In order to solve (4), we form a quadrature approximation of the objective as e.g. in [22, 24]. A finite number N of return and claim scenarios (R^k, c^k) , $k = 1, \dots, N$ is generated over time $t = 0, \dots, T$. Here R^k denotes a realization of the $|J|$ -dimensional process $(R_t)_{t=1}^T$ where $R_t = (R_{t,j})_{j \in J}$. The expectation is then approximated by

$$\frac{1}{N} \sum_{k=1}^N e^{-\gamma(\sum_{i \in I} \alpha^i w_T^{i,k})},$$

where $w_T^{i,k}$ is the terminal wealth in scenario k , obtained by following strategy h^i . Given a realization (R^k, c^k) and a strategy h^i , the corresponding wealth process $w^{i,k} = (w_t^{i,k})_{t=0}^T$ is given recursively by

$$w_t^{i,k} = \begin{cases} w_0 & \text{for } t = 0, \\ \sum_{j \in J} R_{t,j}^k h_{t-1,j}^{i,k} - c_t^k & \text{for } t > 0. \end{cases}$$

The resulting minimization problem is of a form that is, in principle, straightforward to solve using numerical optimization algorithms.

5 Numerical results

In the following numerical illustrations, the termination date was set to $T = 30$, and the cash flows c_t were defined as the survival index S_t of a cohort of US females aged 65 at time $t = 0$. The structure of this instrument is the same as in the first longevity bond issued in 2004 by the European Investment Bank (for a more detailed description see e.g. [6]). The asset returns R_t and liabilities c_t were modelled as a multivariate stochastic process as described in Appendix A. We constructed $N = 10^6$ scenarios for the numerical procedure described above.

Our aim was to investigate if liability-driven investment strategies can lead to reductions in the risk associated with a cash flow of longevity-linked liabilities. To this end, we used two sets of basis strategies. The first set consisted of non-liability-driven basis strategies, namely 30 FP strategies, 24 TDF strategies, and four buy and hold strategies. The second set encompassed both the above non-liability-driven and additional liability-driven basis strategies, including 15 term spread strategies, 15 credit spread strategies, 50 survival index strategies and 50 wealth strategies. We computed the optimal aggregate investment strategy and the corresponding value of the risk measure function ρ for each set, using the numerical procedure of the previous section. We then proceeded to compare the optimal values of the objective ρ associated with each set. In order to discern to which extent a possible reduction in risk can be attributed to considering the liabilities, as opposed to merely having a larger number of strategies, we also

considered a portfolio optimization problem without liabilities for both sets of basis strategies. The optimal allocations were computed for different values of risk aversion parameters γ . The larger the parameter, the more risk averse the investor.

Table 1 summarizes the resulting values of the objective function. We observe that as the risk aversion grows, so does the reduction in risk of the ALM problem with liabilities as the liability-driven strategies are included. This is plausible since the higher the risk aversion, the more the risk measure places importance to the fact that the asset returns conform to the liabilities. As for the optimization problem with zero liabilities, the effect of adding the liability-driven strategies was negligible and independent of the level of risk aversion.

Table 1: Values of objective function ρ .

Basis strategies	$\gamma = 0.05$		$\gamma = 0.1$		$\gamma = 0.3$		$\gamma = 0.5$	
	$c_t = S_t$	$c_t = 0$	$c_t = S_t$	$c_t = 0$	$c_t = S_t$	$c_t = 0$	$c_t = S_t$	$c_t = 0$
Non-LDI	-27.46	-75.14	-18.64	-60.82	-11.16	-46.73	-9.17	-41.81
All	-27.90	-75.14	-19.84	-60.84	-12.40	-46.87	-10.16	-42.14
reduction (%)	1.6	0.006	6.47	0.04	11.14	0.3	10.71	0.8

Tables 2 and 3 show the optimal allocations to each set of the basic investment strategies and both problems for risk aversion parameter $\gamma = 0.3$. After the liability-driven strategies were included in the optimization procedure, none of the non-liability driven strategies were included in the optimal allocation of the problem with $c_t = S_t$, whereas in the optimal allocation of the portfolio optimization problem a non-liability driven fixed proportions strategy still had the highest weight.

Table 2: Diversified strategy, non-liability-driven strategies, $w_0 = 15$, $\gamma = 0.3$

$c_t = S_t$	Weight (%)	Type	π
	97.7	FP	$\pi^2 = 1 - 0.25$ $\pi^3 = 0.25$
	2.3	FP	$\pi^2 = 1 - 0.35$ $\pi^3 = 0.35$
$c_t = 0$	Weight	Type	π
	59.8	FP	$\pi^2 = 1 - 0.25$ $\pi^3 = 0.25$
	40.2	FP	$\pi^2 = 1 - 0.15$ $\pi^3 = 0.15$

Table 3: Diversified strategy, all strategies, $w_0 = 15$, $\gamma = 0.3$

$c_t = S_t$	Weight (%)	Type	π
	52.7	Survival index	$\pi^2 = g^a(S_t)$ $\pi^3 = 1 - g^a(S_t)$ $a = 1$
	19.0	Wealth	$\pi^2 = g^a(w_t/w_0)$ $\pi^3 = 1 - g^a(w_t/w_0)$ $a = 0.5$
	13.8	Survival index	$\pi^2 = g^a(S_t)$ $\pi^3 = 1 - g^a(S_t)$ $a = 0.75$
	7.4	Wealth	$\pi^2 = g^a(w_t/w_0)$ $\pi^3 = 1 - g^a(w_t/w_0)$ $a = 0.75$
	7.1	Term spread	$\pi^1 = 1 - \sigma(s_t^T)^{a,b}$ $\pi^2 = \sigma(s_t^T)^{a,b}$ $a = -0.5, b = 5$
$c_t = 0$	Weight	Type	π
	44.3	FP	$\pi^2 = 1 - 0.35$ $\pi^3 = 0.35$ -
	37.6	Term spread	$\pi^1 = 1 - \sigma(s_t^T)^{a,b}$ $\pi^2 = \sigma(s_t^T)^{a,b}$ $a = -0.5, b = 5$
	9.6	Survival index	$\pi^2 = g^a(S_t)$, $\pi^3 = 1 - g^a(S_t)$, $a = 1$
	8.4	Wealth	$\pi^2 = g^a(w_t/w_0)$ $\pi^3 = 1 - g^a(w_t/w_0)$ $a = 0.5$

Tables 4 and 5 show the best five individual strategies with the smallest risks for both problems. While all the best strategies of the problem with liabilities were liability-driven, all the best ones for the problem without liabilities were non-liability driven strategies. Note that when $c_t = 0$, CPPI reduces to a fixed proportions strategy.

Table 4: Five best basis strategies, with liabilities, $w_0 = 15$, $\gamma = 0.3$

Type	Parameters	π	ρ
Survival index	$a = 0.75$	$\pi^2 = g^a(S_t)$ $\pi^3 = 1 - g^a(S_t)$	-11.80
Survival index	$a = 1$	$\pi^2 = g^a(S_t)$ $\pi^3 = 1 - g^a(S_t)$	-11.23
Wealth	$a = 1$	$\pi^2 = g^a(w_t/w_0)$ $\pi^3 = 1 - g^a(w_t/w_0)$	-11.22
FP	-	$\pi^2 = 1 - 0.25$ $\pi^3 = 0.25$	-11.13
CPPI	$m = 0.2, r = 0.04$	$\pi^2 = 1 - e_t$ $\pi^3 = e_t$	-10.89

Table 5: Five best basis strategies, without liabilities, $w_0 = 15$, $\gamma = 0.3$

Type	Parameters	π	ρ
CPPI/FP	$m = 0.2, r = 0.03$	$\pi^2 = 1 - e_t$ $\pi^3 = e_t$	-46.62
FP	-	$\pi^2 = 1 - 0.25$ $\pi^3 = 0.25$	-46.46
FP	-	$\pi^2 = 1 - 0.15$ $\pi^3 = 0.15$	-46.13
TDF	$a = 0.2, b = 0.003$	$\pi^2 = 1 - e_t$ $\pi^3 = e_t$	-45.98
TDF	$a = 0.25, b = 0.005$	$\pi^2 = 1 - e_t$ $\pi^3 = e_t$	-45.08

Figure 1 illustrates the effect of the liability link by plotting the proportions π^2 of wealth invested in 5-year government bonds as a function of remaining wealth w_t at $t = 15$ in different scenarios. In the case of the ALM problem of the longevity-linked cash flow, π^2 is higher when w_t is higher. For the portfolio optimization problem, however, the connection is much less clear.

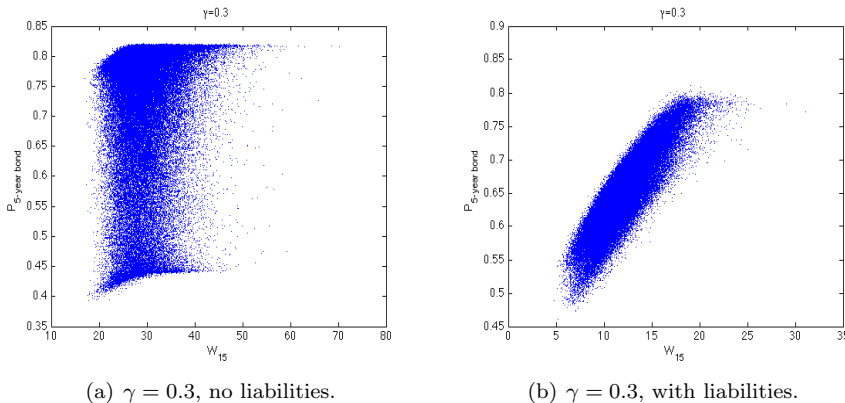


Figure 1: Proportion of wealth invested at time $t = 15$ in 5-year bonds as a function of W_{15} . All strategies.

6 Conclusions

This paper presented several liability-driven investment strategies for longevity-linked liabilities. We were able to show numerically that liability-driven investment can significantly outperform common strategies that do not take into account the liabilities. These strategies may help pension insurers and issuers of longevity-linked instruments in asset-liability management, reserving, and in underwriting new insurance contracts.

While encouraging, the results still leave substantial room for improvement. The basis strategies employed in the simulations are only an example of liability-

driven strategies. Discovering and utilizing new connections between longevity-linked cash flows and asset returns would further improve the overall hedging strategy.

A Assets and liabilities

We consider a set J of assets consisting of

1. US Treasury bills (1-year rate)
2. US Treasury bonds (5 -year rate)
3. US Corporate bonds
4. US equity (S&P total return index).

These are the asset classes between which the investment strategies distribute the existing wealth, and the above numbers are the indices with which the strategies are referred to. Returns on government bonds are given by the formula

$$R_t^i = \exp(Y_{t-1}^i \Delta t - D \Delta Y_t^i),$$

where Y_t^i is the yield to maturity of each bond $i \in \{1, 2\}$ at time t , and D is the duration. Following [23], corporate bond returns are computed by

$$R_t^3 = \exp(c + (Y_t^3 - \alpha S_t^C) \Delta t - D \Delta Y_t^3),$$

where Y_t^3 is the yield to maturity of the bond, S_t^C is here the credit spread between the yields of corporate bonds and longer-term government bonds Y_t^2 , and D is again the duration. Setting $c = 1$ and $\alpha = 1$ yields

$$R_t^3 = \exp(Y_t^2 \Delta t - D \Delta Y_t^3).$$

The total return of the equity is calculated in terms of its total return index S_t^E ,

$$R_t^4 = \frac{S_t^E}{S_{t-1}^E}.$$

The value of the liabilities depends on the survival index of cohort of US females aged 65 at the beginning of the observation period. The population dynamics are governed by three mortality risk factors of the mortality model presented in [1].

We briefly recall the stochastic model proposed in [1]. Let $E_{x,t}$ be the number of individuals aged $[x, x + 1)$ years at the beginning of year t in a given population. The number of survivors $E_{x+1,t+1}$ among the $E_{x,t}$ individuals during year $[t, t + 1)$ can be described by the binomial distribution:

$$E_{x+1,t+1} \sim \text{Bin}(E_{x,t}, p_{x,t}), \tag{1}$$

where $p_{x,t}$ is the probability that an x year-old individual randomly selected at the beginning of year t survives until $t + 1$.

The future values of E_{t+1} are obtained by sampling from $\text{Bin}(E_t, p_{x,t})$. However, as the population grows, the fraction $E_{t+1}/[E_t p_{x+t,t}]$ converges in distribution to constant 1. For large populations, the population dynamics are well

described by $E(x+1, t+1) = E_{x,t} p_{x,t}$, when the main uncertainty comes from unpredictable variations in the future values of $p_{x,t}$. In this work, we employ the latter approach.

As in [1], we model the *survival probabilities* $p_{x,t}$ with the formula

$$p_{x,t} = \frac{\exp(\sum_{i=1}^n v_t^i \phi^i(x))}{1 + \exp(\sum_{i=1}^n v_t^i \phi^i(x))}, \quad (2)$$

where ϕ^i are user-defined *basis functions* and v_t^i are stochastic *risk factors* that may vary over time.

As in [1], we will use the three piecewise linear basis functions given by

$$\begin{aligned} \phi^1(x) &= \begin{cases} 1 - \frac{x-18}{32} & \text{for } x \leq 50 \\ 0 & \text{for } x \geq 50, \end{cases} \\ \phi^2(x) &= \begin{cases} \frac{1}{32}(x-18) & \text{for } x \leq 50 \\ 2 - \frac{x}{50} & \text{for } x \geq 50, \end{cases} \\ \phi^3(x) &= \begin{cases} 0 & \text{for } x \leq 50 \\ \frac{x}{50} - 1 & \text{for } x \geq 50. \end{cases} \end{aligned}$$

The linear combination $\sum_{i=1}^3 v_t^i \phi^i(x)$ will then be piecewise linear and continuous as a function of the age x . The risk factors v_t^i now represent points on logistic survival probability curve:

$$v_t^1 = \text{logit } p_{18,t}, \quad v_t^2 = \text{logit } p_{50,t}, \quad v_t^3 = \text{logit } p_{100,t}.$$

Once the basis functions ϕ^i are fixed, the realized values of the corresponding risk factors v_t^i can be easily calculated from historical data using standard maximum likelihood estimation.

As in [2], we model the future development of and connections between mortality risk factors and spreads with the following system of equations

$$\begin{aligned} \Delta v_t^1 &= a^{11} v_{t-1}^1 + b^1 + \varepsilon_t^1 \\ \Delta v_t^2 &= b^2 + \varepsilon_t^2 \\ \Delta v_t^3 &= a^{33} v_{t-1}^3 + a^{34} g_{t-1} + b^3 + \varepsilon_t^3 \\ \Delta g_t &= a^{45} s_{t-1}^T + a^{46} s_{t-1}^C + b^4 + \varepsilon_t^4 \\ \Delta s_t^T &= a^{55} s_{t-1}^T + b_5 + \varepsilon_t^5 \\ \Delta s_t^C &= a^{66} s_{t-1}^C + b_6 + \varepsilon_t^6 \\ \Delta y_t^1 &= a^{77} y_{t-1}^1 + b_7 + \varepsilon_t^7 \\ \Delta s_t^E &= b_8 + \varepsilon_t^8. \end{aligned}$$

where v_t^i are mortality risk factors, g_t is the natural logarithm of per capita real GDP, s_t^T is the term spread between the logarithms of yields to maturity for 5-year and 1-year government bonds, and s_t^C is the logarithm of the credit spread between the logarithmic yields to maturity of BAA and AAA rated corporate bonds. In addition to the risk factors included in the original model, the 1-year government bond yield $y_t^1 = \log(Y_t^1)$ was added to enable computation of

bond returns, as well as the S&P total return index $s_t = \log(S_t^E)$ as pension plans typically invest in the stock market. The terms ε_t^i are random variables describing the unexpected development in the risk factors.

Once the 1-year government bond yield is known, the 5-year government bond yield can be computed by means of the term spread. Due to lack of data, we approximate the credit spread between government bonds and corporate bonds with the spread s_t^C between corporate bonds of varying riskiness, obtaining the corporate bond yield.

Final year of available mortality data was 2007. Parameters of the time series model were estimated as in [2], with the exception that the mean reversion yields of 1-year and 5-year government bonds and corporate bonds were set to 2.5%, 3.5% and 4.5%, respectively, and expected return on equity was set to 8%. Durations D for the 1-year and 5-year Treasury bonds were 1 and 5 years, respectively, and 5 years for the corporate bonds. In the case of negative wealth, required funds were borrowed from the money market at the 1-year rate adjusted by a loan margin of 1%.

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